Only one of generalized gradients can be elliptic

by Jerzy Kalina, Antoni Pierzchalski and Paweł Walczak (Łódź)

Abstract. Decomposing the space of $k$-tensors on a manifold $M$ into the components invariant and irreducible under the action of $\text{GL}(n)$ (or $\text{O}(n)$ when $M$ carries a Riemannian structure) one can define generalized gradients as differential operators obtained from a linear connection $\nabla$ on $M$ by restriction and projection to such components. We study the ellipticity of gradients defined in this way.

Introduction. We decompose a connection $\nabla$ on an $n$-dimensional $C^\infty$-manifold $M$ (in particular, a Riemannian connection on a Riemannian manifold $(M, g)$) into the sum of first order differential operators $\nabla_{\alpha\beta}$ acting on covariant $k$-tensors, $k = 1, 2, \ldots$, and arising from the decomposition of the space $T^k$ of $k$-tensors into the direct sum of irreducible $\text{GL}(n)$-invariant (or, in the Riemannian case, $\text{O}(n)$-invariant) subspaces. Following [SW] we shall call them $\text{GL}(n)$- and $\text{O}(n)$-gradients, respectively.

Some of the gradients $\nabla_{\alpha\beta}$ have important geometric meaning. The best known is the exterior derivative $d$ corresponding to skew-symmetric tensors. Its role in geometry and topology of manifolds cannot be overestimated. Another one, known as the Ahlfors operator $S : T^1 \to S^0_2$, is defined for 1-forms $\omega$ by the splitting

$$\nabla \omega = \frac{1}{2} d\omega + S\omega - \frac{1}{n} \delta \omega \cdot g$$

and corresponds to the subbundle of traceless symmetric 2-tensors. It appears to play an important role in conformal and quasi-conformal geometry (see the recent papers [ØP], [P], etc.).

In Section 1, we recall (after H. Weyl [We]) the theory of Young diagrams and schemes and define our operators $\nabla_{\alpha\beta}$. In Section 2, we consider

1991 Mathematics Subject Classification: 53C05, 20G05.

Key words and phrases: connection, group representation, Young diagram, elliptic operator.

The authors were supported by the KBN grant 2 P301 036 04.
the ellipticity of operators corresponding to GL($n$)-invariant subspaces. We
distinguish a suitable extension of a Young diagram $\alpha$ and show that $\nabla^{\alpha\beta}$
is elliptic if and only if $\beta$ is a distinguished extension of $\alpha$. In Section 3, we
get some particular ellipticity results for operators corresponding to O($n$)-
invariant subspaces. We end with some remarks.

Similar problems could be considered for any connection $\nabla$,
$$\nabla : C^\infty(\xi) \to C^\infty(T^*M \otimes \xi),$$
in any vector bundle $\xi$ over a manifold $M$ and any Lie group $G$ acting
simultaneously in $T^*M$ and $\xi$. Splitting $\xi$ and $\tilde{\xi} = T^*M \otimes \xi$ into the direct
sums of irreducible $G$-invariant subbundles, $\xi = \bigoplus_\alpha \xi_\alpha$ and $\tilde{\xi} = \bigoplus_\beta \tilde{\xi}_\beta$,
$G$-gradients could be defined as
$$\nabla^{\alpha\beta} = \tilde{\pi}_\beta \circ \nabla \circ \iota_\alpha,$$
where $\iota_\alpha : \xi_\alpha \to \xi$ and $\tilde{\pi}_\beta : \tilde{\xi} \to \tilde{\xi}_\beta$ are the canonical maps. One of interesting
examples of this sort is the classical Dirac operator $D$ which could be
considered as an elliptic Spin($n$)-gradient in a spinor bundle over a manifold
equipped with a spinor structure. Ellipticity of general $G$-gradients will be
studied elsewhere.

1. Young diagrams. Let $W$ be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) of dimension
$n$. Fix $k \in \mathbb{N}$ and take a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\alpha_1 \geq \ldots \geq \alpha_r \geq 1$, $\alpha_1 + \ldots + \alpha_r = k$. Such an $\alpha$ is called a Young scheme of length
$k$. In some references a Young scheme is called a decomposition. It can be
represented by the figure consisting of $r$ rows of squares and such that the
number of squares in the $j$th row is $\alpha_j$.

A Young scheme can be filled with numbers $1, \ldots, k$ distributed in any
order. A scheme filled with numbers is called a Young diagram. Without
loss of generality we can assume that the numbers grow both in rows and
columns.

Take a Young diagram $\alpha$ and denote by $H_\alpha$ and $V_\alpha$ the subgroups of
the symmetric group $S_k$ consisting of all permutations preserving rows and
columns, respectively. $\alpha$ determines the linear operator (called the Young
symmetrizer) $P_\alpha : W^k \to W^k$, $W^k = \bigotimes_k W$, given by
$$P_\alpha = \sum_{\tau \in H_\alpha, \sigma \in V_\alpha} \text{sgn } \sigma \cdot \tau \sigma,$$
where the action of any permutation $\varrho \in S_k$ on simple tensors is given by
$$\varrho(v_1 \otimes \ldots \otimes v_k) = v_{\varrho^{-1}(1)} \otimes \ldots \otimes v_{\varrho^{-1}(k)}$$
for all $v_1, \ldots, v_k \in W$. It is well known that
$$P_\alpha^2 = m_\alpha P_\alpha$$
Generalized gradients

for some \( m_\alpha \in \mathbb{N} \) and that \( W_\alpha = \text{im} P_\alpha \) is an invariant subspace of \( W^k \) for the standard representation of \( \text{GL}(n) \) in \( W^k \). This representation is irreducible on \( W_\alpha \). Moreover,

\[
W^k = \bigoplus_\alpha W_\alpha.
\]

If \( W \) is equipped with a scalar product \( g = \langle \cdot, \cdot \rangle \), then \( g \) allows defining contractions in \( W^k \). An element \( w \) of \( W^k \) is said to be traceless if \( C(w) = 0 \) for any contraction \( C : W^k \to W^{k-2} \). (In particular, all 1-tensors are traceless.) Traceless tensors form a linear subspace \( W^k_0 \) of \( W^k \). Its orthogonal complement consists of all the tensors of the form

\[
\sum_{\sigma \in S_k} \sigma(g \otimes w_\sigma),
\]

where \( w_\sigma \in W^{k-2} \). For simplicity, denote the space of tensors of the form (4) by \( g \otimes W^{k-2} \) so that

\[
W^k = W^k_0 \oplus (g \otimes W^{k-2}).
\]

The intersection \( W^k_0 = W_\alpha \cap W^k_0 \) is non-trivial if and only if the sum of lengths of the first two columns of a Young diagram \( \alpha \) is \( \leq n \). A diagram like this is called admissible and the corresponding space \( W^k_0_\alpha \) is invariant and irreducible under the \( O(n) \)-action. Moreover,

\[
W^k_0 = \bigoplus_\alpha W^k_0_\alpha,
\]

where \( \alpha \) ranges over the set of all admissible Young diagrams with numbers growing both in rows and columns. Comparing (5) and (6), and proceeding with the analogous decompositions of \( W^{k-2}, W^{k-4}, \) etc., one gets the decomposition of \( W^k \) into the direct (in fact, orthogonal) sum of irreducible \( O(n) \)-invariant subspaces.

2. \( \text{GL}(n) \)-gradients. Let \( \beta = (\beta_1, \ldots, \beta_s) \) be a Young scheme of length \( k+1 \) obtained from \( \alpha \) by an extension by a single square. The corresponding diagram should have \( k+1 \) in the added square, while the ordering in the other part of the diagram is the same as in \( \alpha \). We call \( \beta \) a distinguished extension of \( \alpha \) if

\[
s = r, \ \beta_1 = \alpha_1 + 1, \ \beta_2 = \alpha_2, \ldots, \beta_s = \alpha_s.
\]

In other words, \( \beta \) is distinguished when the added square is situated at the end of the first row.

Take an arbitrary \( v \in W \) and consider a linear mapping \( \otimes_v : W^k \to W^{k+1} \) defined by

\[
\otimes_v(v_1 \otimes \ldots \otimes v_k) = v_1 \otimes \ldots \otimes v_k \otimes v.
\]
Theorem 1. For \( v \neq 0 \) the mapping
\[
P_\beta \circ \otimes_v|_{W_\alpha} : W_\alpha \to W_\beta
\]
is injective if and only if \( \beta \) is the distinguished extension of \( \alpha \).

Before the proof we make the following observations.

Lemma 1. Assume that \( i, j, i \neq j \), are in the same column of a Young diagram \( \alpha \). Then
\[
P_\alpha(v) = 0,
\]
whenever \( v = v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_{k+1} \) and \( v_j = v_i \).

Proof. Denote by \( V^+_\alpha \) and \( V^-_\alpha \) the subsets of \( V_\alpha \) consisting of odd and even permutations \( \sigma \in V_\alpha \), respectively, \( V^+_\alpha \cup V^-_\alpha = V_\alpha \). The mapping
\[
\sigma \mapsto \tilde{\sigma} = \sigma \circ t_{ij},
\]
where \( t_{ij} \) is the transposition, is a one-to-one map of \( V^+_\alpha \) onto \( V^-_\alpha \). If \( v_i = v_j \), then
\[
\sum_{\sigma \in V_\alpha} \sigma(v) = \sum_{\sigma \in V^+_\alpha} \sigma(v) - \sum_{\sigma \in V^-_\alpha} \sigma(v) = 0,
\]
because the terms corresponding to \( \sigma \) and \( \tilde{\sigma} \) are the same. Now, the statement follows from formulae (1) and (12). \( \blacksquare \)

Lemma 2. If \( \beta \) is the distinguished extension of \( \alpha \), then
\[
P_\beta = m_\alpha \left[ \text{id} + \sum_{t \in T_\alpha} t \circ \text{id} \right]
\]
on \( W_\alpha \otimes W \), where \( T_\alpha \) denotes the set of all transpositions of \( k+1 \) with the numbers from the first row.

Proof. Since \( V_\beta = V_\alpha \) up to the canonical isomorphism and \( H_\beta = H_\alpha \cup \bigcup_{t \in T_\alpha} tH_\alpha \), we have
\[
P_\beta = \sum_{\tau \in H_\beta, \sigma \in V_\alpha} \text{sgn} \sigma \cdot \tau \sigma.
\]
Consequently,
\[
P_\beta(P_\alpha v \otimes w) = \sum_{\tau \in H_\beta} \tau \left( \sum_{\sigma \in V_\alpha} \text{sgn} \sigma \cdot \sigma(P_\alpha v) \otimes w \right)
= \sum_{\sigma \in V_\alpha, \tau \in H_\alpha} \text{sgn} \sigma \cdot \tau \sigma(P_\alpha v) \otimes w
\[
+ \sum_{t \in T_\alpha} t \left( \sum_{\sigma \in V_\alpha, \tau \in H_\alpha} \text{sgn} \sigma \cdot \tau \sigma(P_\alpha v) \otimes w \right) 
= P_\alpha^2 v \otimes w + \sum_{t \in T_\alpha} t(P_\alpha^2 v \otimes w),
\]
for any \( v \in W^k \) and \( w \in W \). Now, the proof is completed by applying (2).

**Lemma 3.** If \( v_1, \ldots, v_l \in W \) are linearly independent, \( \varrho \) is a permutation mapping the numbers \( 1, \ldots, \alpha_1 \) onto the numbers of the first row of the diagram \( \alpha, \alpha_1 + 1, \ldots, \alpha_1 + \alpha_2 \) onto the numbers of the second row etc., and
\[
\omega = \varrho^{-1}(\otimes^{\alpha_1} v_1 \otimes \ldots \otimes^{\alpha_l} v_l),
\]
then \( P_\alpha \omega \neq 0 \).

**Proof.** The statement follows from (1) and the following:

(i) Any two permutations \( \sigma_1 \) and \( \sigma_2 \) of \( V_\alpha \) satisfying \( \tau \sigma_1 \omega = \tau \sigma_2 \omega \) for some \( \tau \in H_\alpha \) have the same sign.

(ii) Any two products obtained from \( \omega \) by permuting factors are linearly dependent if and only if they are equal.

**Proof of Theorem 1.** Assume first that \( \beta \) is the distinguished extension of \( \alpha \). If \( \eta \in W_\alpha \) and \( P_\beta(\eta \otimes w) = 0 \), then, by Lemma 2,
\[
\eta \otimes w + \sum_{t} t(\eta \otimes w) = 0.
\]
Take \( w = e_1 \), \( \eta = \sum \eta_{i_1 \ldots i_k} e_{i_1} \otimes \ldots \otimes e_{i_k} \), where \( \{e_1, \ldots, e_k\} \) is a basis of \( W \). Then the last equality is equivalent to
\[
\sum \eta_{i_1 \ldots i_k} (e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e_1 + e_1 \otimes e_{i_2} \otimes \ldots \otimes e_{i_k} \otimes e_1 + \ldots + e_{i_1} \otimes \ldots \otimes e_{i_{k-1}} \otimes e_1 \otimes e_{i_k}) = 0.
\]
Now, if \( i_1, \ldots, i_k > 1 \), then \( \eta_{i_1 \ldots i_k} = 0 \) because all the terms are linearly independent. If \( i_1 = 1, i_2, \ldots, i_k > 1 \), then
\[
2\eta_{i_2 \ldots i_k} e_1 \otimes e_{i_2} \otimes \ldots \otimes e_{i_k} \otimes e_1 + \text{(terms linearly independent of the first one)} = 0,
\]
so \( \eta_{i_2 \ldots i_k} = 0 \).

We can repeat the reasoning for the other coefficients. Consequently, \( \eta = 0 \) and the mapping (9) is injective.

Assume now that \( \beta \) is a non-distinguished extension of \( \alpha \). Then, by Lemma 1,
\[
P_\beta(P_\alpha \omega \otimes v_1) = 0,
\]
where \( \omega \) is of the form (15), while, by Lemma 3, \( P_\alpha \omega \neq 0 \).
Now, consider any connection $\nabla$ on a manifold $M$ and extend it to covariant $k$-tensor fields, $k = 1, 2, \ldots$, in the standard way:

$$\nabla \omega(X_1, \ldots, X_{k+1}) = (\nabla_{X_{k+1}} \omega)(X_1, \ldots, X_k).$$

For any two diagrams $\alpha$ and $\beta$ of length $k$ and $k + 1$, respectively, denote by $\nabla^{\alpha\beta}$ the differential operator given by

$$\nabla^{\alpha\beta} = P_\beta \circ \nabla|_{T_\alpha},$$

where $T_\alpha$ denotes the space of all $k$-tensor fields $\omega$ such that $\omega(x) \in (T^*_x M)_\alpha$ for any $x \in M$. Since $P_\beta$ is linear the symbol of the operator $\nabla^{\alpha\beta}$ is given by

$$\sigma(\nabla^{\alpha\beta}, w^*)(\omega) = P_\beta(\omega \otimes w^*)$$

for any covector $w^* \in T^*_x M$, any $\omega \in (T^*_x M)_\alpha$ and $x \in M$. Theorem 1 together with (18) yields

**Corollary.** The operator $\nabla^{\alpha\beta}$ is elliptic if and only if $\beta$ is the distinguished extension of $\alpha$. ■

### 3. $O(n)$-gradients.

Given two admissible Young diagrams $\alpha$ and $\beta$ of length $k$ and $k + 1$, respectively, and a Riemannian connection $\nabla$ on a Riemannian manifold $(M, g)$ one can consider the differential operator $\nabla^{\alpha\beta}$ given by

$$\nabla^{\alpha\beta} = \pi \circ P_\beta \circ \nabla|_{W^0_\alpha},$$

where $W^0_\alpha$ denotes the subspace of $W_\alpha$ consisting of all the traceless tensor fields and $\pi$ is the projection of $k$-tensors to traceless $k$-tensors defined by the decomposition (5). The operator (19) differs from $\nabla^{\alpha\beta}$ of Section 2 but this should lead to no misunderstandings. Again, since $\pi$ is a linear map, the symbol of $\nabla^{\alpha\beta}$ is given by the formula analogous to (18):

$$\sigma(\nabla^{\alpha\beta}, w^*)(\omega) = \pi(P_\beta(\omega \otimes w))$$

for any traceless $\omega$ and $w \in TM$. (Hereafter, vectors and covectors are identified by the Riemannian structure.)

Note that since $\nabla$ is Riemannian, $\nabla_X \omega$ is traceless for any vector field $X$ and any traceless $k$-tensor $\omega$ while $\nabla \omega$ itself can have non-vanishing contractions of the form $C^{i+1}_{i} \nabla \omega$, where $i \leq k$. Note also, that, in general, the distinguished extension of an admissible Young diagram is admissible again. The only exception is that of a one-column diagram of length $n$. These observations together with results of Section 2 motivate the following

**Conjecture.** $\nabla^{\alpha\beta}$ is elliptic if and only if $\beta$ is the distinguished extension of $\alpha$, both $\alpha$ and $\beta$ being admissible.

An elementary proof of the conjecture seems unlikely, because there is no algorithm providing the traceless component of $k$-tensors, even of the
form $ω \otimes v$ with $ω$ being traceless and $v$ a single vector. However, we can prove, in an elementary way, ellipticity of $∇^{αβ}$ in some particular cases and the “if” part completely.

**Theorem 2.** (i) If $α$ is trivial, i.e. consists of a single row or of a single column, $β$ is the distinguished extension of $α$ and both $α$ and $β$ are admissible, then the operator $∇^{αβ}$ is elliptic.

(ii) If $β$ is a non-distinguished extension of $α$, then $∇^{αβ}$ is not elliptic.

**Proof.** (i) Assume first that $α$ is a single row. Then so is $β$ and the spaces $T_α$ and $T_β$ consist of symmetric tensors. From (13) and (20) it follows that the ellipticity of $∇^{αβ}$ is equivalent to the following statement:

(*) If $ω$ is traceless and symmetric, $v$ is a non-vanishing vector and

$$\omega \otimes v \in g \otimes W^{k-1},$$

then $ω = 0$.

Since $β$ is admissible, $n > 1$. To prove (*) take an orthonormal frame $e_1, \ldots, e_n$ and assume, without loss of generality, that $v = e_1$. Since the symmetric algebra is isomorphic to the algebra of polynomials and the tensors in (21) are symmetric, we can replace (21) by the equality

$$x_1 \cdot P(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} x_i^2 \right) \cdot Q(x_1, \ldots, x_n),$$

where $P$ and $Q$ are polynomials. From (22) it follows that $Q$ is of the form $x_1 \cdot Q'$ for another polynomial $Q'$ and therefore, $P = \sum x_i^2 \cdot Q'$. Since $P$ corresponds to $ω$, the last equality shows that $ω \in (g \otimes W^{k-2}) \cap W_k = \{0\}$.

Assume now that $α$ is a single column. The space $W_α$ consists of skew-symmetric tensors and $β$ is admissible if and only if $k < n$. Assume that $ω \in W_α$ and

$$ω \otimes v + (-1)^{k-1} v \otimes ω \in g \otimes W^{k-1}$$

for some $v \neq 0$. (Note that, by Lemma 2, the tensor in (23) coincides with $P_βω$.) From (23) it follows that

$$ω = v \wedge η$$

for some $(k-1)$-form $η$. In fact, otherwise $ω \otimes v \pm v \otimes ω$, when decomposed into a sum of simple tensors, would contain a term $w_1 \otimes \ldots \otimes w_{k+1}$ with all the factors $w_i$ linearly independent while tensors of $g \otimes W^{k-1}$ do not admit terms of this sort. Moreover, one could choose $η$ in (24) to be a $(k-1)$-form on the orthogonal complement $\{v\}^⊥$ of the one-dimensional space spanned by $v$. If so, $ω \otimes v \pm v \otimes ω$ would contain no non-trivial terms of the form

$$g(w \otimes w \otimes w_1 \otimes \ldots \otimes w_{k-1})$$
with $g \in S_{k-1}$ and $w \in \{v\}^+$ while all the non-zero tensors of $g \otimes W^{k-1}$ do. Consequently, $\omega = 0$.

(ii) Assume that $\alpha$ is admissible and put $m = \min\{\delta_1, n/2\}$, where $\delta_i$ is the length of the $i$th column of $\alpha$. Since $\delta_1 + \delta_2 \leq n$, it follows that $\beta_2 \leq m$.

Split the set $\{1, 2, \ldots, k\}$ into the sum $A \cup B \cup C$ of pairwise disjoint subsets such that $|A| = |B| = m$. Set $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{2m + 1, \ldots, n\}$.

Fix an orthonormal frame $(e_1, \ldots, e_n)$ of $W$ and denote by $\omega$ the sum of all the terms of the form
\begin{equation}
(-1)^l \cdot e_{i_1} \otimes \ldots \otimes e_{i_k},
\end{equation}
where $i_r \in \{a_s, b_s\}$ when $r$ belongs to the $s$th row of the Young diagram $\alpha$ and $s \leq m$, $i_r = c_s$ when $r$ belongs to the $s$th row of $\alpha$ and $s > m$, and
\begin{equation}
l = \left[\frac{1}{2} \# \{r : i_r \in B\}\right].
\end{equation}

It is easy to see that both tensors $\omega$ and $P_\alpha \omega$ are traceless while $P_\alpha(\omega)$ is 0.

Take any non-distinguished extension $\beta$ of $\alpha$ and denote by $s$ the number of the column of $\beta$ which contains $k + 1$. Write $\omega$ in the form
\begin{equation}
\omega = \omega_A + \omega_B,
\end{equation}
where $\omega_A$ (resp., $\omega_B$) is the sum of all the terms of the form (26) for which $i_r \in A$ (resp., $i_r \in B$) for the $r$ which appears in the first row and $s$th column of $\alpha$. Let $v = e_{a_1} + e_{b_1}$. Then
\begin{equation}
\sum_{\sigma \in H_\beta} \text{sgn} \sigma \cdot \sigma(\omega_A \otimes e_{a_1}) = \sum_{\sigma \in H_\beta} \text{sgn} \sigma \cdot \sigma(\omega_B \otimes e_{b_1}) = 0
\end{equation}
by Lemma 1. Also,
\begin{equation}
\sum_{\sigma \in H_\beta} \text{sgn} \sigma \cdot \sigma(\omega_A \otimes e_{b_1}) = \sum_{\sigma \in H_\beta} \text{sgn} \sigma \cdot \sigma(\omega_B \otimes e_{a_1})
\end{equation}
because for any term in the first sum there exists a unique term in the second sum with $e_{a_1}$ and $e_{b_1}$ interchanged. Equalities (27)–(29) together with (1) and the definition of $v$ imply that $P_\beta(\omega \otimes v) = 0$.

Finally, following the proof of Lemma 2 one can show that
\begin{equation}
P_\beta = m_\alpha \sum_{t \in T^k} \sum_{t' \in T^k} \text{sgn} t \cdot t' \circ (P_\alpha \otimes \text{id}) \circ t,
\end{equation}
where $T^k$ (resp., $T^n$) consists of the identity and all the transpositions of $k + 1$ with the elements of the row (resp., column) containing it. It follows that
\begin{equation}
P_\beta(P_\alpha \omega \otimes v) = m_\alpha P_\beta(\omega \otimes v) = 0. \quad \square
4. Final remarks. (i) Denote by $N(k)$ the number of components in the decomposition (3). It is easy to observe that $N(1) = 1$, $N(2) = 2$, $N(3) = 4$, $N(4) = 10$, $N(5) = 26$, etc. The above observation motivates the recurrent formula

$$N(k) = N(k - 1) + (k - 1) \cdot N(k - 2).$$

The authors could not find anything like this in the literature. A numerical experiment showed that (32) holds for small $k$, say $k \leq 20$.

(ii) As we said in Section 3, there is no explicit formula for the traceless part of a tensor. In some sense, a formula of this sort could be obtained in the following way. Put

$$E = \bigoplus_{\{k\}} T^{k-2}$$

and define an endomorphism $K : E \to E$ by the formula

$$K(\omega_{ij}) = \left(C^l_j \left( \sum_{r,s} t_r \circ t_s (g \otimes \omega_{rs}) \right) \right),$$

where $t_r$ (resp. $t_s$) is the transposition of the terms 1 and $r$ (resp., 1 and $s$).

$K$ is an isomorphism. In fact, if $K(\Omega) = 0$, $\Omega = (\omega_{ij})$, then the tensor

$$\Theta = \sum_{r,s} t_r \circ t_s (g \otimes \omega_{rs})$$

is traceless and—because of its form—orthogonal to the space of traceless tensors, and therefore, it vanishes. Decomposing tensors $\omega_{ij}$ according to (6) and proceeding inductively one would get $\omega_{ij} = 0$ for all $i$ and $j$, i.e. $\Omega = 0$.

The traceless part $\omega_0$ of any $k$-tensor $\omega$ is given by the formula

$$\omega_0 = \omega - \Theta,$$

where $\Theta$ is given by (35) with $(\omega_{ij}) = K^{-1}((C^l_j \omega))$. In fact, from the definition of $K$ it follows immediately that $C^l_j \Theta = C^l_j \omega$ for all $i$ and $j$.

After submitting the paper, the authors, working jointly with B. Ørsted and G. Zhang, proved the Conjecture from Section 3 as well as formula (32). See Elliptic gradients and highest weights, Bull. Polish Acad. Sci. Math. 44 (1996), 527–535.

References


Institute of Mathematics
Technical University of Łódź
Al. Politechniki 11
93-590 Łódź, Poland
E-mail: jkalina@imul.uni.lodz.pl

University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: antoni@imul.uni.lodz.pl
pawelwal@imul.uni.lodz.pl

Reçu par la Rédaction le 10.2.1995