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An energy estimate for the complex Monge–Ampère operator

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Abstract. We prove an energy estimate for the complex Monge–Ampère operator, and a comparison theorem for the corresponding capacity and energy. The results are pluricomplex counterparts to results in classical potential theory.

Introduction. Recall that in classical potential theory, a positive measure μ is said to have *finite energy* if

$$\int -G_{\Omega}(x,y)\,d\mu(x)\,d\mu(y)<\infty,$$

where G_{Ω} is the Green function for the domain Ω. It is shown that

$$\int -G_{\Omega}(x,y)\,d\mu(x)\,d\nu(y)$$

defines an inner product on the linear space of measures spanned by the measures of finite energy. In particular, we have the Cauchy–Schwarz inequality

$$\Big(\int -G_{\Omega}\,d\mu\,d\nu\Big)\le\Big(\int -G_{\Omega}\,d\mu\,d\mu\Big)^{1/2}\Big(\int -G_{\Omega}\,d\nu\,d\nu\Big)^{1/2}.$$

In this paper, we prove the following analogue of this inequality for the complex Monge–Ampère operator:

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$, $n\ge 2$. Suppose $u,v\in \mathrm{PSH}\cap L^{\infty}(\Omega)$ with $\lim_{z\to\xi}u(z)=\lim_{z\to\xi}v(z)=0$, $\forall\xi\in\partial\Omega$. If $p\ge1$, $0\le j\le n$, then*

$$\int(-u)^p(dd^cu)^j\wedge(dd^cv)^{n-j}$$
$$\le D_{p,j}\Big(\int(-u)^p(dd^cu)^n\Big)^{(p+j)/(n+p)}\Big(\int(-v)^p(dd^cv)^n\Big)^{(n-j)/(n+p)}$$

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where $D_{p,j} = p^{(p+j)(n-j)/(p-1)}$ *for* $p > 1$ *and* $D_{p,j} = \exp(1+j)(n-j)$ *for* $p = 1$.

For the classical notation of energy and Green potentials we refer to Landkof [6], and for the pluripotential theory to the survey article by Bedford [1].

2. Proof of the theorem. In order to be able to integrate by parts, we first assume that

$$\int_{\Omega} ((dd^c u)^n + (dd^c v)^n) < \infty. \tag{2.1}$$

Then for the mixed terms we have

$$\int_{\Omega} (dd^c u)^j \wedge (dd^c v)^{n-j} \le \int_{\Omega} (dd^c(u+v))^n < \infty, \quad 0 \le j \le n,$$

where the last inequality is obtained from the comparison principle and the assumption above (cf. [5]). For let $\mu = (dd^c(u+v))^n$ and choose $1 < \alpha < 2$ such that $\mu\{u = \alpha v\} = 0$. Then $\mu\Omega = \mu\{(1+\alpha)u/\alpha < u+v\} + \mu\{(1+\alpha)v < u+v\}$, and thus $\mu\Omega \le 3^n \int_{\Omega}((dd^c u)^n + (dd^c v)^n)$ by the comparison principle, which proves the boundedness of the mixed terms.

Since $d^c u \wedge (dd^c u)^{j-1} \wedge (dd^c v)^{n-j}$ is a positive measure on $\{u = -\varepsilon\}$ (cf. [4]), we have

$$\begin{aligned} 0 &\le \int_{\{u=-\varepsilon\}} (-v)^p d^c u \wedge (dd^c u)^{j-1} \wedge (dd^c v)^{n-j} \\ &\le \sup\{(-v(z))^p \mid u(z) = -\varepsilon\} \cdot \int_{\Omega} (dd^c u)^j \wedge (dd^c v)^{n-j} \to 0, \quad \varepsilon \searrow 0. \end{aligned}$$

Therefore, we can integrate by parts in this case. Define

$$\begin{aligned} x_j &= \log \int (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j}, \\ y_j &= \log \int (-v)^p (dd^c v)^j \wedge (dd^c u)^{n-j}. \end{aligned}$$

Then integration by parts and Hölder's inequality give

$$\begin{aligned} &\int (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j} \\ &\quad = -\int dv \wedge d^c(-u)^p \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &\quad = \int v dd^c(-u)^p \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &\quad = p(p-1) \int v(-u)^{p-2} du \wedge d^c u \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &\quad\quad + p \int (-v)(-u)^{p-1} (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1} \\ &\quad\quad \le p \int (-v)(-u)^{p-1} (dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1} \end{aligned}$$

$$\leq \Big(p\int(-v)^p(dd^cu)^{j+1}\wedge(dd^cv)^{n-j-1}\Big)^{1/p}$$
$$\times\Big(p\int(-u)^p(dd^cu)^{j+1}\wedge(dd^cv)^{n-j-1}\Big)^{(p-1)/p}.$$

Taking logarithms, we get

$$x_j \leq \frac{p-1}{p}x_{j+1}+\frac{1}{p}y_{n-j-1}+\log p$$

and

$$y_j \leq \frac{p-1}{p}y_{j+1}+\frac{1}{p}x_{n-j-1}+\log p.$$

In matrix notation,

$$S\begin{pmatrix} x_0\\ y_0\\ \vdots\\ x_n\\ y_n\end{pmatrix} \leq \log p\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \tag{2.2}$$

where S is the $2n\times(2n+2)$ matrix

$$S=\begin{pmatrix}
1 & 0 & \frac{1-p}{p} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{p} & 0 & 0\\
0 & 1 & 0 & \frac{1-p}{p} & 0 & \cdots & 0 & -\frac{1}{p} & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \frac{1-p}{p} & \cdots & -\frac{1}{p} & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & -\frac{1}{p} & 0 & 0 & \cdots & 1 & 0 & \frac{1-p}{p} & 0 & 0\\
0 & -\frac{1}{p} & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \frac{1-p}{p} & 0\\
-\frac{1}{p} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \frac{1-p}{p}
\end{pmatrix}.$$

Let A denote the left $2n\times 2n$ submatrix of S. We will find that A is invertible and that A^{-1} has nonnegative elements. So multiplication of the system (2.2) with A^{-1} will preserve the inequality and give a reduced row-echelon form. To this end consider the system of equations

$$A\begin{pmatrix} x_0\\ y_0\\ \vdots\\ x_{n-1}\\ y_{n-1}\end{pmatrix}=\begin{pmatrix} c_0\\ d_0\\ \vdots\\ c_{n-1}\\ d_{n-1}\end{pmatrix}.$$

A calculation shows that then

$$x_j = \frac{n-j}{(p-1)(p+n)} \sum_{k=0}^{j-1} (k+1) c_k + \frac{p+j}{(p-1)(p+n)} \sum_{k=j}^{n-1} (p-1+n-k) c_k + \frac{n-j}{(p-1)(p+n)} \sum_{k=n-j}^{n-1} (p-1+n-k) d_k + \frac{p+j}{(p-1)(p+n)} \sum_{k=0}^{n-j-1} (k+1) d_k, \tag{2.3}$$

and similarly for y_j. This shows that A^{-1} exists and has nonnegative elements. It follows from (2.3) that

$$A^{-1}S = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 & A_0 \\ 0 & I & 0 & \cdots & 0 & 0 & A_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 & A_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & I & A_{n-1} \end{pmatrix}, \tag{2.4}$$

where I is the 2×2 identity matrix and

$$A_j = -\begin{pmatrix} \frac{p+j}{p+n} & \frac{n-j}{p+n} \\ \frac{n-j}{p+n} & \frac{p+j}{p+n} \end{pmatrix}.$$

Then (2.2) implies that

$$A^{-1}S \begin{pmatrix} x_0 \\ y_0 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \leq \log p \, A^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.5}$$

To compute the right hand side of (2.5), we have to find

$$A^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x'_0 \\ y'_0 \\ \vdots \\ x'_{n-1} \\ y'_{n-1} \end{pmatrix}. \tag{2.6}$$

Thus we put $c_k = d_k = 1$ in (2.3) and get

$$x'_j = y'_j = \frac{(p+j)(n-j)}{p-1}. \tag{2.7}$$

We substitute (2.7) and (2.6) in (2.5) and obtain

$$\begin{aligned} x_j - \frac{p+j}{p+n}x_n - \frac{n-j}{p+n}y_n &\le \frac{(p+j)(n-j)}{p-1}\log p, \\ y_j - \frac{n-j}{p+n}x_n - \frac{p+j}{p+n}y_n &\le \frac{(p+j)(n-j)}{p-1}\log p. \end{aligned} \tag{2.8}$$

This concludes the proof for the case $p > 1$ and the extra assumption (2.1). Since the integrals are continuous in p, and since

$$\lim_{p\to 1}\frac{\log p}{p-1} = 1,$$

the inequality also holds for $p = 1$. To complete the proof of the theorem, we have to remove the assumption (2.1). We can assume that

$$\int((-u)^p(dd^c u)^n + (-v)^p(dd^c v)^n) < \infty,$$

otherwise there is nothing to prove. Let $\varepsilon > 0$ be given an let u_r denote the usual regularization

$$u_r(z) = \int u(z - r\xi)\phi(\xi)\,dV(\xi),$$

where V is the Lebesgue measure on $\mathbb{C}^n$, and ϕ is a fixed radial, nonnegative, smooth and compactly supported function in the unit ball of $\mathbb{C}^n$ with $\int \phi\,dV = 1$. Let $\omega \Subset \Omega$ be a strictly pseudoconvex domain containing $\{u < -\varepsilon/4\}$. Then $u_r \in \mathrm{PSH}(\omega) \cap C^\infty(\overline{\omega})$ if $r < d(\omega, {}^c\Omega)$, and we define

$$u^\omega_{r,\varepsilon} = \begin{cases} u_r & \text{if } u_r < -\varepsilon, \\ \varepsilon h^\omega_{\{u_r<-\varepsilon\}} & \text{if } u_r \ge -\varepsilon, \end{cases}$$

where h^ω_E is the relative extremal function

$$h^\omega_E(z) = \sup\{\phi(z) \mid \phi \in \mathrm{PSH}(\omega),\ \phi \le 0,\ \phi|_E \le -1\} \tag{2.9}$$

with respect to ω. By Sard's theorem, the boundary of $\{u_r < -\varepsilon\}$ is a smooth manifold for all ε outside a set of Lebesgue measure zero. We consider only those ε's. Then $\lim_{\{u_r\le-\varepsilon\}\ni\xi\to z} h^\omega_{\{u_r<-\varepsilon\}}(\xi) = -1$ for all $z \in \overline{\{u_r < -\varepsilon\}}$, so $u^\omega_{r,\varepsilon}$ is plurisubharmonic on ω. Now,

$$\begin{aligned} \int_\omega (-u^\omega_{r,\varepsilon})^p(dd^c u^\omega_{r,\varepsilon})^n &= \int_{\{u_r<-\varepsilon\}} \ldots + \int_{\{u_r\ge-\varepsilon\}} \ldots \\ &\le \int_K (-u_r)^p(dd^c u_r)^n + \varepsilon^p \int_{\{u_r=-\varepsilon\}} (dd^c u^\omega_{r,\varepsilon})^n \end{aligned}$$

for all compact sets K in ω containing $\{u<-\varepsilon\}$. Furthermore,

$$\begin{aligned}\int_\omega (dd^c u^\omega_{r,\varepsilon})^n &= \int_\omega (dd^c \varepsilon h^\omega_{\{u_r<-\varepsilon\}})^n = \int_{\{u_r=-\varepsilon\}} (dd^c \varepsilon h^\omega_{\{u_r<-\varepsilon\}})^n \\ &\le \int_{\{u<(\varepsilon/4)h^\omega_{\{u_r<-\varepsilon\}}-\varepsilon/4\}} (dd^c \varepsilon h^\omega_{\{u_r<-\varepsilon\}})^n \\ &= 4^n \int_{\{u<(\varepsilon/4)h^\omega_{\{u_r<-\varepsilon\}}-\varepsilon/4\}} \left(dd^c\left(\frac{\varepsilon}{4}h^\omega_{\{u_r<-\varepsilon\}}-\frac{\varepsilon}{4}\right)\right)^n \\ &\le 4^n \int_{\{u<-\varepsilon/4\}} (dd^c u)^n\end{aligned}$$

by the comparison principle. Combining these two inequalities, we get

$$\int_\omega (-u^\omega_{r,\varepsilon})^p (dd^c u^\omega_{r,\varepsilon})^n \le \int_K (-u_r)^p (dd^c u_r)^n + \varepsilon^p \int_{\{u<-\varepsilon/4\}} (dd^c u)^n.$$

We now let $r \searrow 0$; then $u^\omega_{r,\varepsilon}$ decreases to

$$u^\omega_\varepsilon = \begin{cases} u & \text{if } u<-\varepsilon, \\ \varepsilon h^\omega_{\{u<-\varepsilon\}} & \text{if } u\ge -\varepsilon,\end{cases}$$

and

$$\int_\omega (-u^\omega_\varepsilon)^p (dd^c u^\omega_\varepsilon)^n \le \int_K (-u)^p (dd^c u)^n + \varepsilon^p \int_{\{u<\varepsilon/4\}} (dd^c u)^n$$

so if we let ω and K increase to Ω, then u^ω_ε decreases to u^Ω_ε and

$$\int (-u^\Omega_\varepsilon)^p (dd^c u^\Omega_\varepsilon)^n \le \int (-u)^p (dd^c u)^n + \varepsilon^p \int_{\{u<\varepsilon/4\}} (dd^c u)^n.$$

If we now let $\varepsilon \searrow 0$ then

$$\lim_{\varepsilon\to 0} \int (-u^\Omega_\varepsilon)^p (dd^c u^\Omega_\varepsilon)^n \le \int (-u)^p (dd^c u)^n$$

and similarly for v. Also, by semicontinuity we have

$$\liminf_{\varepsilon\to 0} \int (-u^\Omega_\varepsilon)^p (dd^c u^\Omega_\varepsilon)^j \wedge (dd^c u^\Omega_\varepsilon)^{n-j} \ge \int (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j}.$$

We have already proved the inequalities for u^Ω_ε and v^Ω_ε so the above inequalities complete the proof of the theorem.

Remark. The theorem can be generalized to more than two functions. Also, it can be proved that $D_{1,j}=1$ (see [7]).

3. An application. Let Ω be a strictly pseudoconvex set in $\mathbb{C}^n$, $n\ge 2$, and denote by P the class of bounded plurisubharmonic functions ϕ on Ω such that $\lim_{z\to\xi}\phi(z)=0, \forall \xi\in\partial\Omega$ and $\int_\Omega (dd^c\phi)^n<\infty$. In analogy with the notation of capacity and energy in classical potential theory, we consider the pluricomplex capacity, defined by Bedford and Taylor in [2],

$$d(F)=\sup\Big\{\int\limits_F (dd^c u)^n \,\Big|\, u\in P,\ -1\le u\le 0\Big\},$$

and the pluricomplex energy,

$$I(F)=\inf\Big\{\int -u(dd^c u)^n \,\Big|\, u\in P,\ \int\limits_F (dd^c u)^n \ge 1\Big\},$$

of a compact subset F of Ω. If $\int_F (dd^c u)^n = 0, \forall u\in P$, we say that F has *infinite energy*; this happens exactly when F is pluripolar.

THEOREM 3.1. *Suppose that F is not pluripolar. Then*

$$D_{1,0}^{-(n+1)/n} \le d(F)^{1/n} I(F) \le 1. \tag{3.1}$$

Proof. Let $\psi = h_F^*/d(F)^{1/n}\in P$, where h_F^* denotes the smallest upper semicontinuous majorant of the relative extremal function $h_F = h_F^{\Omega}$ defined by (2.9). Then $\operatorname{supp}(dd^c\psi)^n \subset F$ and $\int_F (dd^c\psi)^n = 1$ by [2]. Therefore,

$$I(F)\le \frac{1}{d(E)}\int -\frac{h_F^*}{d(F)^{1/n}}(dd^c h_F^*)^n = \frac{1}{d(F)^{1/n}}$$

since $h_F^* = -1$ on F outside a pluripolar set. This proves the last inequality in (3.1).

To prove the first inequality we use Theorem 1.1. If $u\in P$ with $\int_F (dd^c u)^n \ge 1$, then

$$\begin{aligned}1 \le \int -h_F(dd^c u)^n &\le D_{1,0}\Big(\int -h_F(dd^c h_F)^n\Big)^{1/(n+1)}\Big(\int -u(dd^c u)^n\Big)^{n/(n+1)}\\ &= D_{1,0}\, d(F)^{1/(n+1)}\Big(\int -u(dd^c u)^n\Big)^{n/(n+1)}\end{aligned}$$

so

$$D_{1,0}^{-(n+1)/n} \le d(F)^{1/n}\int -u(dd^c u)^n.$$

Taking infimum with respect to u we get the first inequality in (3.1), and the proof of the theorem is complete. ■

Remark. By [7], $D_{1,0}=1$, so we have in fact

$$d(F)^{1/n} I(F) = 1.$$

This is the pluricomplex counterpart of the classical fact that capacity times energy equals 1 (cf. [3], p. 20). For further results on pluricomplex energy, see [5].

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