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L^p-decay of solutions to dissipative-dispersive perturbations of conservation laws

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Abstract. We study the decay in time of the spatial L^p-norm ($1 \le p \le \infty$) of solutions to parabolic conservation laws with dispersive and dissipative terms added

$$u_t - u_{xxt} - \nu u_{xx} + bu_x = f(u)_x \quad \text{or} \quad u_t + u_{xxx} - \nu u_{xx} + bu_x = f(u)_x,$$

and we show that under general assumptions about the nonlinearity, solutions of the nonlinear equations have the same long time behavior as their linearizations at the zero solution.

1. Introduction. It is well known that for $u_0 \in L^1(\mathbb{R})$, the solution to the one-dimensional heat equation

$$u_t = u_{xx}, \quad t \ge 0,\ x \in \mathbb{R},$$

with the initial condition $u(x,0) = u_0(x)$ satisfies the estimate

$$|u(\cdot,t)|_{L^p} \le Ct^{(1/p-1)/2}|u_0|_{L^1}.$$

This time decay is valid for more general parabolic equations, including nonlinear ones. For example, M. E. Schonbek [17, 18] considered n-dimensional parabolic conservation laws $u_t - \Delta u = \operatorname{div} f(u)$ with initial conditions $u_0 \in L^1(\mathbb{R}^n)$, and she has shown that for sufficiently regular u_0 the L^p-norm of the solution in the spatial variable decays like $t^{n(1/p-1)/2}$. Her results are generalized by M. Escobedo and E. Zuazua in [13] for less regular initial data. They also show that the long time behavior of solutions is given by a one-parameter family of self-similar solutions.

The aim of this paper is to examine similar decay properties of solutions to more general one-dimensional evolution equations. We consider the

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Cauchy problem for two perturbed nonlinear conservation laws

$$u_t - u_{xxt} - \nu u_{xx} + bu_x = f(u)_x \tag{1.1}$$

and

$$u_t + u_{xxx} - \nu u_{xx} + bu_x = f(u)_x, \tag{1.2}$$

where $\nu \geq 0$ and b are fixed constants, $t \geq 0$, $x \in \mathbb{R}$, supplemented with the initial condition

$$u(x, 0) = u_0(x). \tag{1.3}$$

Both the equations are obtained from the hyperbolic conservation law $u_t - f(u)_x = 0$ by adding the simplest terms u_{xxx} or $-u_{xxt}$, and $-\nu u_{xx}$ modeling dispersive and dissipative phenomena. In the case of $b = 1$ and $f(u) = -u^2/2$, (1.1) and (1.2) represent a marriage of the Benjamin–Bona–Mahony [5] equation $u_t + u_x - u_{xxt} + uu_x = 0$, or the Korteweg–de Vries equation $u_t + u_x + u_{xxx} + uu_x = 0$, with the classical Burgers equation $u_t + uu_x = \nu u_{xx}$. These equations arise as mathematical models for the unidirectional propagation of nonlinear, dispersive, long waves. Here $u(x, t)$ describes the displacement of the medium from the equilibrium position, x and t are proportional to the distance and elapsed time.

The long time behavior of solutions to (1.1) and (1.2) has already been investigated. For example, Biler [6] and Dix [11] deal with equations more general than (1.2), but they assume that $f(u) = \mathcal{O}(|u|^p)$ as $u \to 0$ for sufficiently large p. Similar results concerning (1.1) are proved by Amick, Bona, and Schonbek [4] for $f(u) = -u^2/2$, and by Bona and Luo [7] and Zhang [20], [21] for $f(u) = -u^r/r$ and $r \geq 3$. In these works the assumption that f is sufficiently flat at 0 (that is, $f(u) = \mathcal{O}(|u|^p)$ as $u \to 0$ for p sufficiently large) implies that solutions to the nonlinear equations (1.1) or (1.2) have the identical decay properties as their linearizations (i.e. when $f \equiv 0$). Another approach to study the decay of solutions to (1.1) and (1.2) with small (in some sense) initial data, based on nonlinear scattering theory, can be found in [3], [8], [11], [12], [15].

This work is devoted to investigating the asymptotic behavior of solutions to (1.1) and (1.2) as $t \to \infty$ in the case of general nonlinearities, but we restrict our attention mainly to solutions of (1.1). We obtain new estimates of the L^p-norms of solutions to the linearized problem (i.e. with $f(u) \equiv 0$), and this allows us to get the decay of solutions to the nonlinear equation. Our argument involves an integral equation called the Duhamel formula (see (2.4)) giving the solution in implicit form, and estimates of the L^p-norms of oscillatory integrals.

Remark 1.1. Of course, we can assume $f(0) = 0$, and replacing $f(u)$ by $f(u) - f'(0)u$ and b by $b - f'(0)$ allows also postulating $f'(0) = 0$. Moreover, if $f \in C^2(\mathbb{R})$, then $|f(u)| \le C|u|^2$, with C locally bounded in u. This will be the main hypothesis on the nonlinearity in further considerations.

This paper is organized as follows. In Section 2 we discuss our main results concerning (1.1). Section 3 presents simple asymptotic properties of solutions to (1.1), a maximum principle in the case when $\nu > |b|$ in (1.1), and other technical tools. Section 4 contains proofs of Theorems 2.1 and 2.2 and Corollary 2.1. The decay in the pure dispersive case $\nu = 0$ in (1.1) is examined in Section 5. In Section 6 we summarize the analogous theory related to the equation (1.2).

NOTATION. We denote the $L^p(\mathbb{R})$-norms by $|\cdot|_p$, the Sobolev space $W^{m,p}(\mathbb{R})$ norms by $\|\cdot\|_{m,p}$, and the case $p = 2$ deserves the special notation: $W^{m,2}(\mathbb{R}) = H^m(\mathbb{R})$ and $\|\cdot\|_{m,2} = \|\cdot\|_m$. We shall also use the Banach space

$$L_1^2(\mathbb{R}) = \Big\{ g \in L^2(\mathbb{R}) : \int |g(x)|^2 (1+|x|)^2\,dx < \infty \Big\}$$

endowed with the weighted norm $|g|_{2,1} = (\int |g(x)|^2(1+|x|)^2\,dx)^{1/2}$. Clearly, $g \in L_1^2(\mathbb{R})$ if and only if $\widehat{g} \in H^1(\mathbb{R})$. For simplicity of notation we write $L^p, H^m, \ldots$ instead of $L^p(\mathbb{R}), H^m(\mathbb{R}), \ldots$ The Fourier transforms will be denoted by $\widehat{v}(\xi) = (2\pi)^{-1} \int \exp(-ix\xi) v(x)\,dx$ with the variable ξ dual to x. All the integrals with no integration limits are meant to be calculated over $\mathbb{R}$. Generic positive constants are denoted by C; they do not depend on x and t, they may depend sometimes on u_0, and they may vary from line to line.

2. Main results. In this section we present results concerning solutions to (1.1). Their counterparts for the equation (1.2) are formulated in Section 6.

For technical reasons, we change variables in (1.1) introducing the new function $v(x,t) = u(x+bt,t)$, which solves the problem

$$\begin{aligned} v_t - v_{xxt} - \nu v_{xx} + b v_{xxx} &= f(u)_x, \\ v(x,0) &= u_0(x). \end{aligned} \tag{2.1}$$

Note that $|v(\cdot,t)|_p = |u(\cdot,t)|_p$ for all $t \ge 0$ and $p \in [1,\infty]$.

Our first step in the study of properties of solutions to the equation (2.1) is to consider its linearization around the trivial solution $u \equiv 0$, namely

$$v_t - v_{xxt} - \nu v_{xx} + b v_{xxx} = 0. \tag{2.2}$$

Taking the Fourier transform with respect to the variable x we can solve (2.2) and write this solution as the action of a semigroup of linear operators $S(t)$ on the initial condition u_0:

$$v(x,t) = S(t)u_0(x) = \int \exp(t\Phi(\xi) + ix\xi)\widehat{u}_0(\xi)\,d\xi, \tag{2.3}$$

where $\Phi(\xi) = (-\nu\xi^2 + ib\xi^3)(1+\xi^2)^{-1}$. The next proposition describes the decay properties of solutions to the linearized equation (2.2).

PROPOSITION 2.1. *Let* $1 \le p \le \infty$ *and* $S(t)$ *be the semigroup of linear operators* (2.3), $u_0 \in H^1 \cap L^2_1$. *Then there exist positive constants* C, α *independent of* t, u_0, *and* $A = A(p, \|u_0\|_1, |u_0|_{2,1})$ *such that*

$$|S(t)u_0(\cdot)|_p \le C(1+t)^{(1/p-1)/2}|u_0|_1 + A(p, \|u_0\|_1, |u_0|_{2,1})e^{-\alpha t}.$$

A (sufficiently regular) solution of (1.2) satisfies the integral equation obtained from the variation of parameter formula or the Duhamel formula, which can be checked using the Fourier transform with respect to x:

$$v(x,t) = S(t)u_0(x) + \int_0^t S(t-\tau)K' * f(v(x,\tau))\,d\tau. \tag{2.4}$$

Here $K(x) = \exp(-|x|)/2$ is the fundamental solution of the operator $I - \partial_x^2$.

A solution $v \in C([0,T);\ \mathcal{X})$ of (2.4) for some $T > 0$ and a Banach space $\mathcal{X}$ is meant to be a generalized solution to (2.1) and is called a *mild solution*. All results from this work concern mild solutions to (2.1), and (2.4) will be the main tool in our proofs.

If $u_0 \in H^s$ for $s \ge 1$, then the Cauchy problem (2.1) with $\nu \ge 0$ and $f \in C^1(\mathbb{R})$ has a unique global mild solution $v \in C^1([0,\infty)$. The local existence can be shown by a standard argument using the Banach fixed point theorem. This local solution can be extended to $[0,\infty)$, because $\|v(\cdot,t)\|_1 \le \|u_0\|_1$ (see Proposition 3.1). If in addition $u_0 \in L^1$, then $v \in C([0,\infty);\ L^1)$. We refer the reader to [4, Theorem 2.2] for more details.

The local existence of mild solutions to (1.2) with $\nu > 0$ in the space $C^1([0,T);\ H^1)$ can be justified in a similar way. Moreover, if $u_0 \in L^1$, then $u \in C([0,T);\ L^1)$. If $\|u_0\|_1$ is sufficiently small or if $|f'(u)| \le C(|u|^p + 1)$ with $0 \le p < 2$ (see [6]), then the problem (1.2), (1.3) is globally well-posed. We refer to [2] for a discussion of other conditions guaranteeing the global existence. The analysis is much more subtle when $\nu = 0$ in (1.2), and a review of the recent theory can be found in [15].

An important tool in proving the decay of solutions to parabolic conservation laws is the property that the nonnegative initial data u_0 produce nonnegative solutions $u(x,t)$ for all $t \ge 0$. A direct consequence of this is: for $u_0 \in L^1(\mathbb{R}^n)$, $|u(\cdot,t)|_1 \le |u_0|_1$ and this estimate gives the decay of other L^p-norms of u (cf. [13], [17], [18]). In the case of the equations (1.1) and (1.2), the maximum principle mentioned above usually fails, hence it is necessary to use different techniques in order to get boundedness of the norm $|u(\cdot,t)|_1$ for $t \ge 0$. This bound seems to be crucial in the proof of the decay of solutions to (1.1) and (1.2), as observed in [4, Lemma 5.1]. That result says that if $f(u) = -u^2/2$ and $u_0 \in L^1 \cap H^1$, then the estimates

$\sup_{t>0}|u(\cdot,t)|_1 < \infty$ and $\sup_{t>0} t^{1/2}|u(\cdot,t)|_2^2 < \infty$ of solutions to (1.1) or (1.2) are equivalent. Similar considerations for other norms are used in the proofs of Corollary 5.2 of [4] and Corollary 5.2 of [7]. We extend those results to general f and other L^p-norms of u. Our first theorem says that for general nonlinearities L^p-decay properties of solutions to (1.1) for each p are equivalent, and this *extrapolation principle* improves the results cited above.

THEOREM 2.1. *Let $v(x,t)$ be a solution of* (2.1) *with $f \in C^2(\mathbb{R})$ and $\nu > 0$ corresponding to the initial data $u_0 \in L_1^2 \cap H^1$. Assume that for some $p_0 \in [1,\infty)$ and a constant $C > 0$,*

$$|v(\cdot,t)|_{p_0} \le C(1+t)^{(1/p_0-1)/2} \quad \text{for all } t \ge 0. \tag{2.5}$$

Then for every $p \in [1,\infty]$ there exists a constant $C = C(p, |u_0|_{2,1}, \|u_0\|_1)$ such that the inequality (2.5) *holds with p_0 replaced by p.*

COROLLARY 2.1. *Suppose that $S(t)u_0(x)$ is defined by* (2.3). *Let $2 \le p \le \infty$ and $u_0 \in H^2$. Under the assumptions of Theorem* 2.1 *if $f''(0) = 0$, then*

$$|v(\cdot,t) - S(t)u_0(\cdot)|_p = o(t^{(1/p-1)/2}) \quad \text{as } t \to \infty.$$

Remark 2.1. If $f''(0) = 0$, then $\lim_{t\to\infty} t^{1/4}|v(\cdot,t)|_2 = (8\nu\pi)^{-1/4} \times |\int u_0(x)\,dx|$. This is a consequence of Corollary 2.1 and Lemma 3.3 of [4], where this limit was computed for the linearized equation. This shows that if $\int u_0(x)\,dx \ne 0$, then the decay rate $t^{-1/4}$ of the norm $|\cdot|_2$ is optimal.

Theorem 2.1 suggests the question when the assumed estimate (2.5) does hold for some $p_0 \in [1,\infty)$. The next theorem gives three sufficient conditions guaranteeing the validity of (2.5).

THEOREM 2.2. *Let v denote the solution of* (2.1) *corresponding to the initial condition $u_0 \in H^1 \cap L_1^2$, $f \in C^2(\mathbb{R})$, and $\nu > 0$. Suppose in addition that one of the following three conditions is satisfied*:

(i) *$\nu > |b|$ and $u_0 \in H^2$,*
(ii) *$|u_0|_1$ is sufficiently small,*
(iii) *$|f'(v)| \le C|v|^2$ for some $C > 0$.*

Then for every $1 \le p \le \infty$ there exists $C_p = C(p,u_0)$ such that

$$|v(\cdot,t)|_p \le C_p(1+t)^{(1/p-1)/2} \quad \text{for all } t \ge 0.$$

For $\nu > |b|$ the dissipative term dominates dispersive effects and then the maximum principle mentioned above begins to be valid after finite time and the inequality $|v(\cdot,t)|_1 \le |v(\cdot,T)|$ holds for some $T \ge 0$ and all $t > T$ (cf. Proposition 3.2 and Corollary 3.1). Hence we have (2.5) with $p_0 = 1$.

The decay result for (2.1) under the assumption (ii) is new, while an analogous fact for (1.2) is known (see [11]).

The decay of solutions to (1.1) and (1.2) with f satisfying (iii) is proved in [7] and [21]. Here we present a shorter, direct argument.

We also examine the decay in time when $\nu = 0$ in (1.1), that is, in the absence of dissipative effects in our mathematical model. We consider the L^∞-decay of solutions to the initial value problem

$$u_t - u_{xxt} + bu_x = f(u)_x \tag{2.6}$$

with the initial condition $u(x,0) = u_0(x)$. Here we shall assume $b \neq 0$. This assumption seems to be essential, because for $f(u) \equiv 0$ the equation $u_t - u_{xxt} = 0$ has no remarkable asymptotic properties. We can check using the Fourier transform that in this case every sufficiently regular solution is constant in time.

The solution of (2.6) satisfies the integral formula (2.4) with $S(t)$ replaced by $T(t)$, the semigroup of linear operators associated with the linearization of (2.6) at 0. In the Fourier variables we can write

$$T(t)u_0(x) = \int \exp(it\Psi(\xi) + ix\xi)\widehat{u}_0(\xi)\, d\xi, \tag{2.7}$$

where $\Psi(\xi) = -b\xi(1+\xi^2)^{-1}$.

The next theorem improves results from [3] and from [12] for $f(u) = -u^{p+1}/(p+1)$. Our proof allows us to consider more general nonlinearities and we assume that one of the norms $\|u_0\|_{2,1}$ and $\|u_0\|_1$ of the initial condition is small enough, instead of assuming this either for $|u_0|_1 + \|u_0\|_5$ as in [3], or for $|u_0|_1 + \|u_0\|_{7/2}$ as in [12]. The van der Corput lemma (Lemma 3.2) used in the proof simplifies several technical computations.

THEOREM 2.3. *Let u be the solution of* (1.1) *with the initial condition $u_0 \in W^{2,1} \cap H^1$. Assume $\nu = 0$, $b \neq 0$, and $|f'(u)| \le C|u|^p$ for some $p > 4$. Then there exists $\delta > 0$ such that if either $\|u_0\|_{2,1} < \delta$ or $\|u_0\|_1 < \delta$, then*

$$|u(\cdot,t)|_\infty \le C(1+t)^{-1/3} \quad \textit{for all } t \ge 0,$$

where the constant $C > 0$ depends only on u_0. Moreover, there exist u^-, u^+ such that $\|u(\cdot,t) - u^\mp(\cdot,t)\|_1$ tends to 0 as t tends to $\mp\infty$, where $u^\mp(x,t) = T(t)u^\mp(x)$ is the solution of the linearized equation.

3. Basic lemmas and preliminary estimates. Our goal here is to gather basic facts needed to prove the theorems from Section 2. The first proposition collects some elementary properties of solutions to (1.1).

PROPOSITION 3.1. *Assume that u is the solution of* (1.1) *corresponding to the initial data $u_0 \in H^2$. Then*

(i) $\|u(\cdot,t)\|_1 \le \|u_0\|_1$,
(ii) $u_x, u_t, u_{xt} \in L^2(\mathbb{R} \times \mathbb{R}^+)$,
(iii) $|u_x(\cdot,t)|_2 \to 0$ *and* $|u(\cdot,t)|_\infty \to 0$ *as* $t \to \infty$.

The proof follows the arguments for [4, Lemma 3.1] and [7, Lemma 4.1 and Corollary 4.2], and consequently we skip it.

If the dissipative term $-\nu u_{xx}$ dominates dispersive effects, then for sufficiently large t the equation (1.1) generates a family of nonlinear contractions on L^1. We make this precise in the next proposition inspired by [16].

PROPOSITION 3.2. *Suppose* $\nu > |b|$, $u_0, w_0 \in L^1 \cap H^2$, *and* u, w *are the solutions of* (1.1) *corresponding to the initial data* u_0, w_0. *Then there exists* $T > 0$ *such that for all* $T \le s \le t$,

$$|u(\cdot,t) - w(\cdot,t)|_1 \le |u(\cdot,s) - w(\cdot,s)|_1. \tag{3.1}$$

Proof. We multiply (1.1) by $K(x-y)$, where $K(z) = e^{-|z|}/2$, and we integrate with respect to y over $\mathbb{R}$ to obtain

$$u_t(x,t) = -\nu u(x,t) + \nu K * u(x,t) + K' * [-bu + f(u)](x,t). \tag{3.2}$$

Now multiplying by $e^{\nu t}$ and integrating over $[s,t]$ we get

$$e^{\nu t}u(x,t) = e^{\nu s}u(x,s) + \int_s^t e^{\nu\tau}(\nu K*u(x,\tau) + K'*[-bu+f(u)](x,\tau))\,d\tau. \tag{3.3}$$

Considering a similar formula for w and subtracting it from (3.3) we obtain

$$e^{\nu t}(u-w)(x,t) = e^{\nu s}(u-w)(x,s) + \int_s^t e^{\nu\tau} A(x,\tau)\,d\tau, \tag{3.4}$$

where

$$\begin{aligned} A(x,\tau) &= \nu K * (u-w)(x,\tau) + K' * [-b(u-w) + f(u) - f(w)](x,\tau) \\ &= \frac{1}{2}\int_x^\infty e^{x-y}\{\nu(u-w)(y,\tau) + [-b(u-w) + f(u) - f(w)](y,\tau)\}\,dy \\ &\quad + \frac{1}{2}\int_{-\infty}^x e^{y-x}\{\nu(u-w)(y,\tau) - [-b(u-w) + f(u) - f(v)](y,\tau)\}\,dy. \end{aligned} \tag{3.5}$$

Since $(f(u)-f(w))(x,t)/(u-w)(x,t) \to f'(0) = 0$ as $t \to \infty$ uniformly with respect to x (from Proposition 3.1 and Remark 1.1), by the assumption on ν there exists $T > 0$ such that $|(f(u)-f(w))(x,t)/(u-w)(x,t)| < \nu - |b|$ for every $t > T$ and $x \in \mathbb{R}$, which implies

$$\left|\nu - b + \frac{(f(u)-f(w))(x,t)}{(u-w)(x,t)}\right| + \left|\nu + b + \frac{(f(u)-f(w))(x,t)}{(u-w)(x,t)}\right| = 2\nu.$$

Keeping this in mind, using (3.5) and changing the order of integration we get

$$(3.6)\quad |A(\cdot,\tau)|_1 \le \frac{1}{2}\int\int_x^{\infty} e^{x-y}|(\nu+b)(u-w)(y,\tau)+(f(u)-f(w))(y,\tau)|\,dy\,dx$$
$$+\frac{1}{2}\int\int_{-\infty}^{x} e^{y-x}|(\nu-b)(u-w)(y,\tau)-(f(u)-f(w))(y,\tau)|\,dy\,dx$$
$$=\frac{1}{2}\int\{|(\nu+b)(u-w)(y,\tau)+(f(u)-f(w))(y,\tau)|$$
$$+|(\nu-b)(u-w)(y,\tau)-(f(u)-f(w))(y,\tau)|\}\,dy$$
$$\le \nu|u(\cdot,\tau)-w(\cdot,t)|_1$$

for $\tau > T$. Hence if we assume $T \le s \le t$, and if we take the L^1-norm of (3.4) with respect to x, using (3.6), we obtain

$$e^{\nu t}|u(\cdot,t)-w(\cdot,t)|_1 \le e^{\nu s}|u(\cdot,s)-w(\cdot,s)|_1 + \nu\int_s^t e^{\nu\tau}|u(\cdot,\tau)-w(\cdot,\tau)|_1\,d\tau.$$

Now the Gronwall lemma applied to the function $e^{\nu t}|u(\cdot,t)-w(\cdot,t)|_1$ concludes the proof.

If $t > T$, then the contraction property (3.1) allows us to obtain a sort of maximum principle for solutions to (1.1). This is an immediate consequence of a modification of a lemma proved by M. G. Crandall and L. Tartar [9]. We present here a full proof to show that we do not need their additional assumptions about the set where our mappings are defined.

LEMMA 3.1 (Crandall and Tartar). *Fix $M \subset L^1$ and a mapping $\mathcal{T} : M \to L^1$ preserving the integral, i.e. $\int \mathcal{T}f(x)\,dx = \int f(x)\,dx$, and satisfying $|\mathcal{T}f-\mathcal{T}g|_1 \le |f-g|_1$ for every $f,g \in M$. Then $f \le g$ a.e. implies $\mathcal{T}f \le \mathcal{T}g$ a.e.*

Proof. Let $f, g \in M$. We write $s^+ = (|s|+s)/2$. The inequality $f \le g$ a.e. gives

$$2\int(\mathcal{T}f-\mathcal{T}g)^+ = \int|\mathcal{T}f-\mathcal{T}g| + \int(\mathcal{T}f-\mathcal{T}g)$$
$$\le \int|f-g| + \int(f-g) = 2\int(f-g)^+ = 0.$$

Thus $\mathcal{T}f - \mathcal{T}g \le 0$ a.e.

COROLLARY 3.1. *Let ν, u, w satisfy the assumptions of Proposition 3.2. Suppose that $u(x,s) \le w(x,s)$ for every $x \in \mathbb{R}$ and some $s > T$. Then $u(x,t) \le w(x,t)$ for all $t \ge s$. In particular, if, for some $s > T$, $u(x,s) \ge 0$ for every $x \in \mathbb{R}$, then $u(x,t) \ge 0$ for every $x \in \mathbb{R}$ and $t \ge s$.*

P r o o f. We apply directly Lemma 3.1 with $M = L^1 \cap H^2$, $f(x) = u(x,s)$, and $g(x) = v(x,s)$. We define $\mathcal{T}u(x,s) = u(x,t)$ for $s \le t$. The equality $\int u(x,t)\,dx = \int u_0(x)\,dx$, which holds for every $t \ge 0$ and $u_0 \in M$, shows that $\mathcal{T}$ preserves the integral. Proposition 3.2 implies that $\mathcal{T}$ is a contraction for $t > T$.

Properties of solutions to (1.1) established in Proposition 3.2 and Corollary 3.1 were observed by B. L. Lucier [16], but only for f globally Lipschitz continuous. In that case the inequality (3.1) holds for every $t \ge s \ge 0$. We also refer to [4, Theorem 5.4], where similar considerations are presented for $f(u) = -u^2/2$, and to the paper of E. DiBenedetto and M. Pierre [10] investigating linear pseudoparabolic equations on a bounded open set $\Omega \subset \mathbb{R}^n$.

R e m a r k 3.1. The application of the L^1-norm to (3.3) with $s = 0$ and the Gronwall lemma give

$$|u(\cdot,t)|_1 \le |u_0|_1 e^{ct} \tag{3.7}$$

for a positive constant c independent of t. To get it we can use the estimate $|f(u)| \le C|u|$, which is valid for all $f \in C^1(\mathbb{R})$, $f(0) = 0$, and for u belonging to any bounded set. We stress the fact that the inequality (3.7) holds for every $u_0 \in L^1 \cap H^1$ and $\nu > 0$. On the other hand, Theorem 2.2 says that some additional assumptions give the bound $|u(\cdot,t)|_1 \le C$ for some $C > 0$ depending on u_0 but independent of t.

To end this section, we formulate a basic technical tool useful in estimating integrals with an exponential oscillating factor.

LEMMA 3.2 (van der Corput). *Suppose h, g are sufficiently smooth functions defined on $[a,b]$ and h is real-valued. Then*

$$\Big|\int_a^b \exp(ih(\xi))g(\xi)\,d\xi\Big| \le \Theta(h)\Big(|g(b)| + \int_a^b |g'(\xi)|\,d\xi\Big),$$

where

$$\Theta(h) = \begin{cases} 8(\min_{[a,b]}|h''|)^{-1/2} & \text{if } h'' \ne 0 \text{ on } [a,b], \\ 18(\min_{[a,b]}|h^{(3)}|)^{-1/3} & \text{if } h^{(3)} \ne 0 \text{ on } [a,b]. \end{cases}$$

This is a well-known fact concerning one-dimensional oscillatory integrals and the proof can be found e.g. in the book of Stein [19, Ch. VIII, §1, Corollary of Proposition 2].

Here we also recall two results frequently used in the nonlinear scattering theory for small solutions.

LEMMA 3.3. *Suppose that a positive continuous function q defined on $[0,\infty)$ satisfies the inequality $q(t) \le c_1 + c_2 q(t)^\kappa$ with some $c_1, c_2 \ge 0$, and $\kappa > 1$. Then there exists a constant $\delta > 0$ such that if either $c_1 \le \delta$ or $c_2 \le \delta$, then q is bounded from above.*

Proof. The maximum of the function $q \mapsto q - c_1 - c_2 q^\kappa$ on $[0,\infty)$ is positive and finite if either c_1 or c_2 is small enough.

LEMMA 3.4. *Assume $\alpha \in (-1,0]$ and $\beta \le 0$. There exists C independent of t such that*

$$\int_0^t (1+(t-\tau))^\alpha (1+\tau)^\beta \, d\tau \le \begin{cases} C(1+t)^{\alpha+\beta+1} & \text{if } -1<\beta, \\ C(1+t)^\alpha & \text{if } \beta<-1. \end{cases}$$

Proof. After splitting the integral into $\int_0^{t/2} + \int_{t/2}^t$ the above inequality is obtained by estimating each term by the supremum of one of the integrated factors.

4. Decay of solutions to (1.1) in the presence of dissipative effects ($\nu > 0$).

The proof of Proposition 2.1 is preceded by a lemma, where we compute the decay rate of the oscillatory integral $S(t)\varphi$ and its derivatives for a cut-off function φ.

LEMMA 4.1. *Let $1 \le p \le \infty$. Fix φ such that $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi(\xi) = 1$ on $[-1/2,1/2]$, and $\varphi(\xi) = 0$ for $|\xi| \ge 1$. Define $\mathcal{I}_k(x,t) = \int \xi^k \varphi(\xi) \exp(t\Phi(\xi) + ix\xi)\, d\xi$. Then for every nonnegative integer k there exists a constant C independent of t such that*

$$|\mathcal{I}_k(\cdot,t)|_p \le C(1+t)^{-(k+1-1/p)/2}.$$

Proof. First we show that

$$|\mathcal{I}_k(\cdot,t)|_2 \le C(1+t)^{-k/2-1/4}, \tag{4.1}$$

which is a consequence of the Plancherel formula:

$$\begin{aligned} |\mathcal{I}_k(\cdot,t)|_2^2 &= 2\pi|\widehat{\mathcal{I}}_k(\cdot,t)|_2^2 = 2\pi \int |\xi|^{2k} \varphi(\xi)^2 \exp\left(\frac{-2\nu t\xi^2}{1+\xi^2}\right) d\xi \\ &\le C \int_{|\xi|\le 1} |\xi|^{2k} \exp\left(\frac{-2\nu t\xi^2}{1+\xi^2}\right) d\xi \le \int_{|\xi|\le 1} |\xi|^{2k} \exp(-\nu t\xi^2)\, d\xi \\ &\le Ct^{-1/2-k} \int |w|^{2k} \exp(-\nu w^2)\, dw \le Ct^{-1/2-k}. \end{aligned}$$

In the estimations above we use the inequality $2\xi^2/(1+\xi^2) \ge \xi^2$ valid for $|\xi| \le 1$ and the change of variables $t^{1/2}\xi = w$. Since the L^2-norm is also bounded independently of t, (4.1) is proved.

Now the inequality $|w|_\infty \le C|w|_2^{1/2}|w'|_2^{1/2}$ and (4.1) give the decay of the L^∞-norm:

$$\begin{aligned} |\mathcal{I}_k(\cdot,t)|_\infty &\le C|\mathcal{I}_k(\cdot,t)|_2^{1/2}|\mathcal{I}_{k+1}(\cdot,t)|_2^{1/2} \\ &\le C(1+t)^{-(1/2+k)/4}(1+t)^{-(1/2+k+1)/4} \\ &= C(1+t)^{-(k+1)/2}. \end{aligned}$$

Before estimating the L^1-norm we observe that for all (smooth, rapidly decreasing) functions $w = w(x)$ defined on $\mathbb{R}$,

$$|w|_1 \le C|w|_2^{1/2}|(\widehat{w})'|_2^{1/2} \tag{4.2}$$

with a constant C independent of w. Indeed, taking $R = |(\widehat{w})'|_2/|w|_2$, we obtain

$$\begin{aligned}
|w|_1 &= \int_{|\xi|\le R} |w(\xi)|\,d\xi + \int_{|\xi|>R} |w(\xi)|\,d\xi \\
&\le \Big(\int_{|\xi|\le R} d\xi\Big)^{1/2}\Big(\int_{|\xi|\le R} |w(\xi)|^2\,d\xi\Big)^{1/2} \\
&\quad + \Big(\int_{|\xi|>R} |\xi|^{-2}\,d\xi\Big)^{1/2}\Big(\int_{|\xi|>R} |w(\xi)|^2|\xi|^2\,d\xi\Big)^{1/2} \\
&\le C(R^{1/2}|w|_2 + R^{-1/2}|(\widehat{w})'|_2) \le C|w|_2^{1/2}|(\widehat{w})'|_2^{1/2}.
\end{aligned}$$

Now the assumptions about φ, (4.2), and the argument used in the proof of (4.1) allow us to compute

$$\begin{aligned}
|\mathcal{I}_k(\cdot,t)|_1 &\le C\bigg(\int |\xi|^{2k}|\varphi(\xi)|^2 \exp\Big(\frac{-2\nu t\xi^2}{1+\xi^2}\Big)\,d\xi\bigg)^{1/4} \\
&\quad\times\bigg(\int\bigg|\frac{d}{d\xi}\Big(\xi^k\varphi(\xi)\exp\Big(\frac{-\nu\xi^2+ib\xi^3}{1+\xi^2}t\Big)\Big)\bigg|^2\,d\xi\bigg)^{1/4} \\
&\le C((1+t)^{-(1/2+k)})^{1/4}\{(1+t)^{-(1/2+k-1)} + t^2((1+t)^{-(1/2+k+1)} \\
&\quad + (1+t)^{-(1/2+k+2)}) + (1+t)^{-(1/2+k)}\}^{1/4} \le C(1+t)^{-k/2}.
\end{aligned}$$

An application of the interpolation inequality

$$|w|_p \le |w|_\infty^{1-1/p}|w|_1^{1/p}, \tag{4.3}$$

which holds for all $w \in L^1 \cap L^\infty$, completes the proof.

P r o o f o f P r o p o s i t i o n 2.1. First observe that $L^2_1 \subset L^1$, i.e. $|u_0|_1 \le C|u_0|_{2,1}$, which can be proved by the Schwarz inequality. We use the cut-off function φ from Lemma 4.1 to decompose $S(t)u_0$ into two integrals

$$\begin{aligned}
S(t)u_0(x) &= \int \varphi(\xi)\exp(t\Phi(\xi) + ix\xi)\widehat{u}_0(\xi)\,d\xi \\
&\quad + \int (1-\varphi(\xi))\exp(t\Phi(\xi) + ix\xi)\widehat{u}_0(\xi)\,d\xi \\
&\equiv I_1(x,t) + I_2(x,t).
\end{aligned}$$

The integral I_1 can be interpreted as the convolution of the function $\mathcal{I}_0(x,t)$ and the initial data u_0. Consequently, applying the Young inequality and

Lemma 4.1 we conclude

$$|I_1(\cdot,t)|_p \le C(1+t)^{(1/p-1)/2}|u_0|_1.$$

Next we prove the exponential decay of the second integral. We have

$$|I_2(\cdot,t)|_\infty \le e^{-\nu t/5}|\widehat{u}_0|_1 \le Ce^{-\nu t/5}\|u_0\|_1,$$

because $|\exp(t\Phi(\xi))| \le e^{-\nu t/5}$ for every $\xi \in \mathbb{R}\setminus[-1/2,1/2]$.

We deduce the decay of L^1-norm from the inequality (4.2):

$$\begin{aligned} |I_2(\cdot,t)|_1 &\le C\Big(\int|(1-\varphi(\xi))\exp(t\Phi(\xi))\widehat{u}_0(\xi)|^2\,d\xi\Big)^{1/4} \\ &\quad\times\bigg(\int\bigg|\frac{d}{d\xi}\{(1-\varphi(\xi))\exp(t\Phi(\xi))\widehat{u}_0(\xi)\}\bigg|^2\,d\xi\bigg)^{1/4} \\ &\le Ce^{-\alpha t}|u_0|_2^{1/2}(e^{-\alpha t}|u_0|_2^{1/2}+t^{1/2}e^{-\alpha t}|u_0|_{2,1}^{1/2}+e^{-\alpha t}|(\widehat{u}_0)'|_2^{1/2}) \\ &\le Ce^{-\alpha t}|u_0|_{2,1}. \end{aligned}$$

The interpolation inequality (4.3) yields

$$|I_2(\cdot,t)|_p \le A(p,\|u_0\|_1,|u_0|_{L^2_1})e^{-\alpha t}.$$

Finally, we stress that here α represents positive numbers which depend on p, but are independent of t and u_0.

Before proving Theorem 2.1 we establish the decay of $S(t-\tau)K'$. We point out that the estimations are somewhat subtle, because $\widehat{K'} \notin L^1$ (see the proof of Lemma 4.2).

LEMMA 4.2. *For every $1\le p\le\infty$ there exists a constant $C_p>0$ such that*

$$|S(t)K'|_p \le C_p(1+t)^{-(1-1/(2p))}.$$

Proof. First we observe that $\widehat{K}'(\xi)=i\xi(1+\xi^2)^{-1}$, which can be checked by a direct computation. Using the cut-off function φ from Lemma 4.1 we decompose $S(t)K'$ into two integrals: over a neighborhood of 0 and over its complement:

$$\begin{aligned} S(t)K'(x) &= \int\exp(t\Phi(\xi)+ix\xi)i\xi(1+\xi^2)^{-1}\,d\xi \\ &= \int\varphi(\xi)\exp(t\Phi(\xi)+ix\xi)i\xi(1+\xi^2)^{-1}\,d\xi \\ &\quad+\int(1-\varphi(\xi))\exp(t\Phi(\xi)+ix\xi)i\xi(1+\xi^2)^{-1}\,d\xi \\ &\equiv A(x,t)+B(x,t). \end{aligned}$$

Lemma 4.1 with $k=1$ shows immediately that

$$|A(\cdot,t)|_p \le C(1+t)^{-(1-1/(2p))}.$$

For the remainder term $B(x,t)$, we get the exponential decay of $|B(\cdot,t)|_1$ by the inequality (4.2):

$$|B(\cdot,t)|_1 \le C\Big(\int \psi(\xi)\exp\left(\frac{-2\nu t\xi^2}{1+\xi^2}\right)\frac{\xi^2}{(1+\xi^2)^2}\,d\xi\Big)^{1/4}$$
$$\times\left(\int\left|\frac{d}{d\xi}\left(\psi(\xi)\exp\left(\frac{-\nu\xi^2+ib\xi^3}{1+\xi^2}t\right)\frac{i\xi}{1+\xi^2}\right)\right|^2 d\xi\right)^{1/4}$$
$$\le C\exp(-\nu t/4)(t^{1/2}\exp(-\nu t/4)+\exp(-\nu t/4)).$$

Here $\psi = 1-\varphi$ is a smooth function supported on $\mathbb{R}\setminus[-1/2,1/2]$.

To estimate the L^∞-norm of $B(x,t)$ we fix $\zeta \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\zeta \subset \{\xi : 2 > |\xi| > 1/2\}$ and $\varphi(\xi)+\sum_{k=0}^\infty \zeta(2^{-k}\xi) = 1$ for $\xi\in\mathbb{R}$, and we assume in addition that ζ is an even function, i.e. $\zeta(-\xi)=\zeta(\xi)$. A construction of such a dyadic symmetric partition of identity can be found e.g. in [19]. Now we decompose $B(x,t)$ into a series of integrals:

$$B(x,t) = \int \psi(\xi)\varphi(\xi)i\xi(1+\xi^2)^{-1}\exp(t\Phi(\xi)+ix\xi)\,d\xi$$
$$+\sum_{k=0}^\infty\int\psi(\xi)\zeta(2^{-k}\xi)i\xi(1+\xi^2)^{-1}\exp(t\Phi(\xi)+ix\xi)\,d\xi$$
$$\equiv \mathcal{J}(x)+\sum_{k=0}^\infty I_k(x).$$

$|\mathcal{J}|_\infty$ and $|I_0|_\infty$ decay exponentially, because $t\Phi(\xi)\le -\alpha t$ for some positive α on the supports of $\psi\varphi$ and $\psi\zeta$. Noting that $\psi\equiv 1$ on the support of $\zeta(2^{-k}\xi)$ for $k\ge 1$, and changing the variables $\omega = 2^{-k}\xi$ we compute

$$\begin{aligned}(4.4)\qquad I_k(x) &= 2^{2k}\int\zeta(\omega)i\omega(1+2^{2k}\omega^2)^{-1}\exp(t\Phi(2^k\omega)+ix2^k\omega)\,d\omega\\ &= \int\zeta(\omega)i\omega(2^{-2k}+\omega^2)^{-1}\\ &\quad\times\exp\left(t\frac{-2^{2k}\nu\omega^2+2^{3k}ib\omega^3}{1+2^{2k}\omega^2}+ix2^k\omega\right)d\omega.\end{aligned}$$

First suppose that $b\ne 0$. Lemma 3.2 with $h(\omega) = t(2^{3k}b\omega^3)(1+2^{2k}\omega^2)^{-1} + x2^k\omega$ and $g(\omega) = \zeta(\omega)i\omega(2^{-2k}+\omega^2)^{-1}\exp(t(-2^{2k}\nu\omega^2)(1+2^{2k}\omega^2)^{-1})$ gives the estimate for I_k:

$$|I_k|_\infty \le C(t2^k)^{-1/2}\exp(-t\alpha),$$

where the positive constants C and α are independent of t, x and k. Hence $|B(\cdot,t)|_\infty \le |\mathcal{J}|_\infty + \sum_{k=0}^\infty |I_k|_\infty \le Ct^{-1/2}\exp(-t\alpha)$.

We also need to prove that $|B(\cdot,t)|_\infty\le C$ for $t\in[0,1]$, and this bound is a standard consequence of the inequalities (see e.g. [19, the proof of Proposition 1, Ch. VI, §4])

$$|I_k| \le \begin{cases} C(2^k|x+bt|)^{-1} & \text{if } 2^k|x+bt| \ge 1, \\ C(2^k|x+bt|+2^{-k}) & \text{if } 2^k|x+bt| < 1, \end{cases} \tag{4.5}$$

with C independent of k, x, t. To show (4.5), we rewrite (4.4) in a more convenient form

$$I_k(x) = \int \zeta(\omega) i\omega (2^{-2k} + \omega^2)^{-1} \exp\left(t \frac{-2^{2k}\nu\omega^2 - 2^k ib\omega}{1+2^{2k}\omega^2} \right) e^{i2^k(bt+x)\omega}\, d\omega$$

and observe that $I_k(x) = \widehat{h}_{k,t}(-2^k(bt+x))$, where

$$h_{k,t}(\omega) = \zeta(\omega) i\omega (2^{-2k} + \omega^2)^{-1} \exp\left(t \frac{-2^{2k}\nu\omega^2 - 2^k ib\omega}{1+2^{2k}\omega^2} \right)$$

is a smooth function with compact support, uniformly bounded with respect to k and $t \in [0,1]$.

We obtain the first estimate in (4.5) using integration by parts. To get the second one, we estimate using the properties of ζ and assumptions on t:

$$\begin{aligned} |\widehat{h}_{k,t}(0)| &= \left| \int_0^\infty 2\zeta(\omega) i\omega (2^{-2k}+\omega^2)^{-1} \exp\left(t\frac{-2^{2k}\nu\omega^2}{1+2^{2k}\omega^2}\right) \sin\left(\frac{2^k bt\omega}{1+2^{2k}\omega^2}\right) d\omega \right| \\ &\le 2\int_0^\infty |\zeta(\omega)\omega| (2^{-2k}+\omega^2)^{-1} \exp\left(t\frac{-2^{2k}\nu\omega^2}{1+2^{2k}\omega^2}\right) \frac{2^k t|\omega b|}{1+2^{2k}\omega^2}\, d\omega \\ &\le C2^{-k}. \end{aligned}$$

Since the first derivatives of all $\widehat{h}_{k,t}(y)$ have a uniform bound for every k, bounded t, and $|y| < 1$, the inequality $|\widehat{h}_{k,t}(y) - \widehat{h}_{k,t}(0)| \le C|y|$ ends the proof of (4.5).

If $b = 0$, then the proof of (4.5) is simpler and follows similar arguments as before, hence we skip it.

Since both the norms $|B(\cdot,t)|_1$ and $|B(\cdot,t)|_\infty$ decay exponentially, the same holds for other L^p-norms of $B(\cdot,t)$ from the interpolation inequality (4.3). The proof of Lemma 4.2 is complete.

Now we proceed with the proof of the extrapolation theorem.

Proof of Theorem 2.1. Multiplying equation (2.1) by v, and integrating over $\mathbb{R}$ and over $[0,t]$ we get

$$|v(\cdot,t)|_2^2 + |v_x(\cdot,t)|_2^2 + 2\nu \int_0^t\int v_x^2(x,s)\,dx\,ds = |u_0|_2^2 + |u_{0x}|_2^2. \tag{4.6}$$

Suppose $p_0 \in [1,2)$. In this case we improve the argument from [4, Theorem 5.1]. By (4.6) and the Plancherel formula we deduce

$$\begin{aligned}(4.7)\quad \frac{d}{dt}\Big((1+t)\int(v^2+v_x^2)\,dx\Big) &= \int(v^2+v_x^2)\,dx - 2\nu(1+t)\int v_x^2\,dx \\ &= \int|\widehat{v}(\xi,t)|^2\,d\xi - 2\nu(1+t)\int\xi^2|\widehat{v}(\xi,t)|^2\,d\xi + \int v_x^2\,dx \\ &\le \int\limits_{|\xi|<(2\nu(1+t))^{-1/2}} |\widehat{v}(\xi,t)|^2\,d\xi + \int v_x^2\,dx.\end{aligned}$$

We estimate the first term of the right-hand side of (4.7). We fix $p,q\in[1,\infty]$ such that $1/p+1/q=1$ and $1/p_0+1/(2q)=1$, and next we apply the Hölder and Hausdorff–Young inequalities:

$$\begin{aligned}(4.8)\quad \int\limits_{|\xi|<(2\nu(1+t))^{-1/2}} |\widehat{v}(\xi,t)|^2\,d\xi \\ &\le (2\nu(1+t))^{-1/(2p)}|\widehat{v}(\cdot,t)|_{2q}^2 \le C(1+t)^{-1/(2p)}|v(\cdot,t)|_{p_0}^2 \\ &\le C(1+t)^{-1/(2p)}(1+t)^{1/p_0-1} = C(1+t)^{-1/2}.\end{aligned}$$

Now (4.8) and the integration of (4.7) over $[0,t]$ give $|v(\cdot,t)|_2 \le C(1+t)^{-1/4}$ since $u_x\in L^2(\mathbb{R}\times\mathbb{R}^+)$ by Proposition 3.1. This shows that we can consider $p_0\ge 2$ only.

To estimate the L^p-norm $(1\le p\le\infty)$ of the solution $v(x,t)$ we use the Duhamel formula (2.4), where the first term is bounded by Proposition 2.1, and we apply to the second term the Young inequality for the convolution of $S(t-\tau)K'$ and $f(u)$ with respect to x and Lemma 4.2. Note that by Remark 1.1 and the inequalities $|u(\cdot,t)|_\infty^2 \le 2|u(\cdot,t)|_2|u_x(\cdot,t)|_2 \le \|u_0\|_1^2$, we can assume $|f(u)|\le C|u|^2$.

Supposing that for some $p_0\in[2,\infty)$ the estimate (2.5) holds, we compute

$$\begin{aligned}(4.9)\quad |v(\cdot,t)|_p &\le |S(t)u_0|_p + \int_0^t |S(t-\tau)K'|_{(1+1/p-2/p_0)^{-1}}|f(v(\cdot,\tau))|_{p_0/2}\,d\tau \\ &\le C(u_0)(1+t)^{(1/p-1)/2} \\ &\quad + C\int_0^t (1+(t-\tau))^{(1/p-1)/2-1/p_0}|v(\cdot,\tau)|_{p_0}^2\,d\tau \\ &\le C(u_0)(1+t)^{(1/p-1)/2} \\ &\quad + C\int_0^t (1+(t-\tau))^{(1/p-1)/2-1/p_0}(1+\tau)^{1/p_0-1}\,d\tau.\end{aligned}$$

Since $1/p_0-1\in(-1,0)$ and $(1/p-1)/2-1/p_0\in(-1,0)$ for every $p\in[1,\infty]$

and $p_0 > 2$, by Lemma 3.4 the last integral is bounded by $C(1+t)^{(1/p-1)/2}$, which finishes the proof of Theorem 2.1 in this case. We obtain the same estimates if $p_0 = 2$ and $1 \le p < \infty$. Now, if we repeat our arguments for some $p \in (2,\infty)$ (say for $p = 4$), we get the decay for $p = \infty$. This completes the proof of Theorem 2.1.

Remark 4.1. We can extend this extrapolation principle to the L^∞-case. If we assume that $|v(\cdot,t)|_\infty \le C(1+t)^{-1/2}$ for some constant $C > 0$, then the idea from the second part of the proof of Theorem 2.1 gives the following estimate of the L^p-norm for $p \in [1,\infty)$:

$$|v(\cdot,t)|_p \le C(1+t)^{(1/p-1)/2}(1+\log(1+t)).$$

For the proof we observe that

$$\int_0^t (1+(t-\tau))^\alpha (1+\tau)^{-1}\,d\tau \le C(1+t)^\alpha(1+\log(1+t))$$

for every $\alpha > -1$. This fact improves Lemma 3.4. Similar estimates give the optimal decay (2.1), if we suppose additionally that $|f(u)| \le C|u|^{2+\varepsilon}$ for some $\varepsilon > 0$.

Proof of Corollary 2.1. By Remark 1.1, $f(0) = f'(0) = 0$. If $f''(0) = 0$, then by Proposition 3.1(iii) for every $\varepsilon > 0$ there exists $T > 0$ such that $|f(v(x,t))| \le \varepsilon |v(x,t)|^2$ for every $t > T$ and $x \in \mathbb{R}$. Using this fact we fix $T > 0$ and we decompose the range of integration in (4.8) into two pieces: $[0,T]$ and $[T,t]$. Now, by direct computations similar to (4.8), we get

$$\begin{aligned}\Big|\int_0^T S(t-\tau)K' * f(v(\cdot,\tau))\,d\tau\Big|_p &\\ &\le \int_0^T (1+(t-\tau))^{(1/p-1)/2-1/p_0}(1+\tau)^{1/p_0-1}\,d\tau\\ &\le C(T)(1+t-T)^{(1/p-1)/2-1/p_0} = o((1+t)^{(1/p-1)/2}),\end{aligned}$$

because $p_0 \in [1,\infty)$. Following (4.9) we can also estimate the second integral:

$$\begin{aligned}\Big|\int_T^t S(t-\tau)K' * f(v(\cdot,\tau))\,d\tau\Big|_p &\\ &\le \varepsilon C\int_T^t (1+(t-\tau))^{1/p-1)/2-1/p_0}(1+\tau)^{1/p_0-1}\,d\tau\\ &\le \varepsilon C(1+t)^{(1/p-1)/2},\end{aligned}$$

for C independent of T and an arbitrary $\varepsilon > 0$. This ends the proof of Corollary 2.1.

P r o o f o f T h e o r e m 2.2. We know from Proposition 3.2 that if $\nu > |b|$, then $|v(\cdot,t)|_1 \le |u_0|_1$ for every $t \ge T \ge 0$. Since $v \in C([0,\infty); L^1)$, there exists C depending only on u_0 such that $|v(\cdot,t)|_1 \le C$ for every $t \ge 0$. Therefore Theorem 2.1 gives the proof under the condition (i).

Now we need an estimate of $|S(t)v|_2$ better than that in Proposition 2.1. The Plancherel formula combined with the method from this proposition give

$$\begin{aligned} |S(t)w|_2 &\le C(1+t)^{-1/4}|w|_1 + e^{-\alpha t}|\widehat{w}|_1 \\ &\le C(1+t)^{-1/4}|w|_1 + Ce^{-\alpha t}\|w\|_1^{1/2}|w_x|_2^{1/2}. \end{aligned} \tag{4.10}$$

For fixed $T \ge 0$ we have the following integral representation of solutions to the equation (2.1):

$$v(t) = S(t-T)v(T) + \int_T^t S(t-\tau)K' * f(v(\tau))\,d\tau, \tag{4.11}$$

where $t \ge T$. We use it in order to estimate the quantity

$$q_T(t) = \sup_{\tau\in[T,t]} \{(1+\tau-T)^{1/4}|v(\cdot,\tau)|_2\}. \tag{4.12}$$

Applying the L^2-norm to (4.11), by (4.10) and Lemma 4.2, we get

$$\begin{aligned} |v(\cdot,t)|_2 \le{}& C(1+t-T)^{-1/4}|v(T)|_1 + Ce^{-\alpha t}(t-T)\|u_0\|_1^{1/2}|v_x(T)|_2^{1/2} \\ &+ C\int_T^t (1+(t-\tau))^{-3/4}|f(v(\cdot,\tau))|_1\,d\tau. \end{aligned} \tag{4.13}$$

Since by Remark 1.1, we can assume $|f(u)| \le C|u|^2$, after multiplication by $(1+t-T)^{1/4}$, and then taking supremum over $[T,t]$, we can rewrite (4.13) as

$$q_T(t) \le C(|v(T)|_1 + \|u_0\|_1^{1/2}|v_x(T)|_2^{1/2} + q_T^2(t)).$$

But $|v_x(T)|_2 \to 0$ as $T \to \infty$, and therefore by Lemma 3.3, $q_T(t)$ is a bounded function, provided T is sufficiently large and $|v(T)|_1$ is sufficiently small. However, the inequality (3.7) shows that for fixed T, $|v(T)|_1$ is small if $|u_0|_1$ is small. Again, an application of Theorem 2.1 finishes the proof in the case (ii).

If we suppose (iii), then

$$|F(v)_x|_1 = |F'(v)v_x|_1 \le C|v^2|_2|v_x|_2 \le C|v|_4^2|v_x|_2 \le C|v|_1|v_x|_2^2. \tag{4.14}$$

In the above estimates, we use the Gagliardo–Nirenberg inequality $|v|_4^2 \le C|v|_1|v_x|_2$. Hence computing the L^1-norm of (2.4), by Lemma 4.2, and (4.14), we obtain

$$|u(t)|_1 \le C|u_0|_1 + C\int_0^t |F(u(\tau))_x|_1\,d\tau \le c|u_0|_1 + C\int_0^t |u(\tau)|_1|u_x(\tau)|_2^2\,d\tau.$$

Now we use the Gronwall lemma for $|u(t)|_1$ and (4.6) in order to get

$$|u(t)|_1 \le C|u_0|_1 \exp\Big(C\int_0^t |u_x(\tau)|_2^2\,d\tau\Big) \le C|u_0|_1 \exp(C(2\nu)^{-1}\|u_0\|_1^2).$$

Theorem 2.1 again completes the proof.

5. Decay in the pure dispersive case ($\nu = 0$)**.** Our analysis of the nonlinear equation (2.6) is based on the L^∞-estimate of the operator $T(t)K$.

Lemma 5.1. *If* $\widehat{K}(\xi) = (1+\xi^2)^{-1}$, *then there exists a constant* $C \ge 0$ *such that*

$$|T(t)K|_\infty \le C(1+t)^{-1/3}.$$

Proof. We first observe that $|T(t)K(x)| \le \int (1+\xi^2)^{-1}\,d\xi < \infty$, which means that we can assume $t \ge 1$. The main idea of the proof consists in the dyadic decomposition of $T(t)K$ into a series of integrals and estimating all the terms using the van der Corput lemma.

We take functions $\varphi, \zeta \in C_c^\infty(\mathbb{R})$ such that $\varphi(\xi) + \sum_{k=0}^\infty \zeta(2^{-k}\xi) = 1$ for every $\xi \in \mathbb{R}$, $\varphi(\xi) = 1$ for $\xi \in [-2,2]$ and $\operatorname{supp}\varphi \subseteq [-3,3]$, and we write

$$\begin{aligned} T(t)K(x) &= \int \varphi(\xi)\exp(it\Psi(\xi) + ix\xi)(1+\xi^2)^{-1}\,d\xi \\ &\quad + \sum_{k=0}^\infty \int \zeta(2^{-k}\xi)\exp(it\Psi(\xi)+ix\xi)(1+\xi^2)^{-1}\,d\xi \\ &\equiv \mathcal{J} + \sum_{k=0}^\infty \mathcal{J}_k. \end{aligned}$$

An easy computation shows that the second and third derivatives of the function $h(\xi) = t\Psi(\xi) + x\xi$ are independent of x, and $h''(\xi) = 0$ only for $\xi \in \{0, -\sqrt{3}, \sqrt{3}\}$, but $h^{(3)}(\xi) \ne 0$ at those points. Therefore using Lemma 3.2 and the remark from the beginning of the proof we obtain $|\mathcal{J}| \le C(1+t)^{-1/3}$.

In the case of $\mathcal{J}_k$ we first change variables $2^{-k}\xi = \omega$, and next use Lemma 3.2 with $h(\omega) = t\Psi(2^k\omega) + x2^k\omega$ and $g(\omega) = \zeta(\omega)(2^{-2k}+\omega^2)^{-1}$. We have $|h''(\omega)| \ge Ct2^{-k}$ for $\omega \in \operatorname{supp}\zeta(\omega)$, where C is independent of k and t, and this implies

$$|\mathcal{J}_k| = 2^{-k}\Big|\int \zeta(\omega)\exp(it\Psi(2^k\omega) + ix2^k\omega)(2^{-2k}+\omega^2)^{-1}\,d\omega\Big| \le Ct^{-1/2}2^{-k/2}.$$

Thus we get $|T(t)K(x)| \le |\mathcal{J}| + \sum_{k=0}^\infty |\mathcal{J}_k| \le Ct^{-1/3}$ for $t \ge 1$.

Proof of Theorem 2.3. We use the Duhamel formula (2.4) with $S(t)$ replaced by $T(t)$ in order to show that the quantity

$$q(t) = \sup_{0\le\tau\le t}((1+\tau)^{1/3}|u(\cdot,\tau)|_\infty)$$

satisfies the inequality

$$q(t) \le C(\|u_0\|_{2,1} + \|u_0\|_1 q(t)^{p-1}). \tag{5.1}$$

First observe that by Lemma 5.1 we get

$$\begin{aligned}|T(t)u_0|_\infty &= |T(t)K * (1-\partial_x^2)u_0|_\infty \le |T(t)K|_\infty |(1-\partial_x^2)u_0|_1 \\ &\le C(1+t)^{-1/3}\|u_0\|_{2,1}.\end{aligned}$$

Now we deduce from the formula (2.4) the inequality

$$\begin{aligned}|u(\cdot,t)|_\infty &\le C(1+t)^{-1/3}\|u_0\|_{2,1} \\ &\quad + \int_0^t (1+(t-\tau))^{-1/3} |\{f'(u(\cdot,\tau))\}u_x(\cdot,\tau)|_1 \, d\tau \\ &\le C(1+t)^{-1/3}\|u_0\|_{2,1} + \|u_0\|_1 \int_0^t (1+(t-\tau))^{-1/3} |u(\cdot,\tau)|_\infty^{p-1} \, d\tau,\end{aligned}$$

giving (5.1), because by Lemma 3.5,

$$\int_0^t (1+(t-\tau))^{-1/3}(1+\tau)^{-(p-1)/3} \, d\tau \le C(1+\tau)^{-1/3}$$

for $p>4$. Now Lemma 3.4 proves that $q(t)$ remains bounded if either $\|u_0\|_{2,1}$ or $\|u_0\|_1$ are sufficiently small.

The proof of existence of $u^{\mp}$ follows in the same manner as in [12].

6. Decay of solutions to the equation (1.2). In this section we briefly review analogous properties of solutions to the equation (1.2). We assume here that $\nu > 0$. An analysis of the case $\nu = 0$ can be found e.g. in [8], [15], and in the references given there.

The first step in our considerations is to investigate the decay of solutions to the linearized problem

$$v_t + v_{xxx} - \nu v_{xx} + b v_x = 0, \qquad v(x,0) = v_0(x). \tag{6.1}$$

The L^p-estimates of solutions to (6.1) for $p \in [2,\infty]$ are well known (see [11]) and using the inequality (4.5) it is possible to extend these results to $p \in [1,2)$. Since the solution of (6.1) for $v_0 \in L^2 \cap L^1$ can be represented in the form $v(x,t) = S(t) * v_0(x)$, where

$$S(t)(x) = \int e^{(-\nu\xi^2 + i\xi^3 - ib\xi)t + ix\xi} \, d\xi,$$

following the idea of the proof of Lemma 4.1 one can show

Proposition 6.1. *Suppose* $v_0 \in W^{k,1}$. *For every* $p \in [1,\infty]$ *and any nonnegative integer* k *there exists* $C > 0$ *depending on* v_0 *but independent of* t *such that*

$$|(\partial^k/\partial x^k)v(\cdot,t)|_p \le Ct^{-(k+1-1/p)/2}.$$

It is important in the analysis of the nonlinear equation that replacing b by $b - f'(0)$ and $f(u)$ by $f(u) - f'(0)u$ allows us to postulate that $f(0) = f'(0) = 0$ in (1.2). Now, in order to adapt the proofs from Section 4 we need an *a priori* bound

$$\sup_{t\ge 0} |u(\cdot,t)|_\infty < \infty, \tag{6.2}$$

which gives the estimate $|f(u)| \le C|u|^2$ for u in a bounded set. It is known that $|u(\cdot,t)|_2 \le |u_0|_2$. If we showed that $\sup_{t\ge0} |u_x(\cdot,t)|_2 < \infty$, then we would obtain (6.2) by the Sobolev imbedding theorem. It is known (see [2], [6], [11], [14]) that some additional assumptions on u_0 and $f(u)$ give the boundedness in time of $|u_x(\cdot,t)|_2$. We formulate the extrapolation theorem for (1.2) under the assumption (6.2).

Theorem 6.1. *Let* $u(x,t)$ *be the solution of* (1.2) *with* $f \in C^2(\mathbb{R})$ *and* $\nu > 0$ *corresponding to the initial data* $u_0 \in L^1 \cap H^1$. *Suppose that* $\sup_{t\ge0} |u(\cdot,t)|_\infty < \infty$. *Assume that for some* $p_0 \in [1,\infty)$ *and a constant* $C > 0$,

$$|u(\cdot,t)|_{p_0} \le Ct^{(1/p_0-1)/2} \quad \text{for all} \quad t \ge 0. \tag{6.3}$$

Then for every $p \in [1,\infty]$ *there exists a constant* $C = C(p, |u_0|_1, \|u_0\|_1)$ *such that the inequality* (6.3) *holds for* $p_0 = p$.

The proof is similar to the proofs of the results from Section 2 and consequently we skip it.

Remark 6.1. If we know that $|u(\cdot,t)|_\infty \to 0$ as $t \to \infty$ and $f''(0) = 0$, then under the assumptions of Theorem 6.1 one can show

$$|u(\cdot,t) - S(t)u_0(\cdot)|_p = o(t^{(1/p-1)/2}) \quad \text{as } t \to \infty$$

for $p \ge 2$, following the proof of Corollary 2.1.

Here we do not consider the decay of solutions to (1.2) for small initial conditions. For a deeper discussion of this case we refer the reader to [11].

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