ANNALES
POLONICI MATHEMATICI
LXVII.1 (1997)

Existence and uniqueness theorems for fourth-order boundary value problems

by JOLANTA PRZYBYCIN (Kraków)

Abstract. We establish the existence and uniqueness theorems for a linear and a nonlinear fourth-order boundary value problem. The results obtained generalize the results of Usmani [4] and Yang [5]. The methods used are based, in principle, on [3], [5].

1. Let $\mathcal{L}$ be a differential operator of the form $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_0$, where $\mathcal{L}_i$ denotes the Sturm–Liouville operator defined by $\mathcal{L}_i y = -(p_i y')' + q_i y$, $i = 0, 1$. As usual we assume $p_i \in C^{3-2i}[0,1]$, $q_i \in C^{2-2i}[0,1]$ and $p_i > 0$, $q_i \geq 0$ on $[0,1]$.

Consider the nonlinear problem

$$\begin{aligned} &\mathcal{L}y = F(\cdot, y) \quad \text{in } (0,1), \\ &y(0) = y_0, \quad y(1) = y_1, \quad \mathcal{L}_0 y(0) = \widehat{y}_0, \quad \mathcal{L}_0 y(1) = \widehat{y}_1. \end{aligned} \tag{1}$$

Denote the above boundary conditions by (B.C.). By a solution of (1) we understand $u \in C^4[0,1] \cap \text{(B.C.)}$ satisfying (1).

Usmani studied a particular case of (1), namely $\mathcal{L}y = y^{(4)}$ and $F(x,y) = f(x)y + g(x)$. He proved an existence and uniqueness theorem under the condition $\sup_{x\in[0,1]} |f(x)| < \pi^4$. Yang found a better condition on f which guarantees the unique solvability of the above problem, namely $f(x) \neq j^4\pi^4$ for $j = 1, 2, \ldots$ He also showed an existence theorem for the nonlinear problem $y^{(4)} = F(\cdot, y, y'')$, (B.C.), under the assumption $|F(x,\xi,\eta)| \leq a|\xi| + b|\eta| + c$, $a/\pi^4 + b/\pi^2 < 1$, which is essential to the proof. By applying the result of Yang to $F(\cdot, y, y'') = f(\cdot, y) + qy''$, where q is a positive and continuous function on $[0,1]$ we obtain the existence of solution if $a/\pi^4 + \max_{x\in[0,1]} q(x)/\pi^2 < 1$. This sufficient condition seems to be very restrictive. To illustrate this fact consider the equation $\mathcal{L}y = y^{(4)} - k^2\pi^2 y'' = 0$ with (B.C.). It is easily verified that this problem is uniquely solvable for any $k \in \mathbb{R}$.

1991 *Mathematics Subject Classification*: Primary 34B15.
Key words and phrases: eigenvalue, Leray–Schauder degree, Fredholm alternative.

We shall now see that it is possible to find a better condition for F by proving a theorem which is more general than the result of Yang in some respects but less general in other ones.

THEOREM 1. *Let* $p_{i0} = \min_{x\in[0,1]} p_i(x)$ *and* $q_{i0} = \min_{x\in[0,1]} q_i(x)$. *Suppose that* F *is continuous on* $[0,1]$ *and satisfies the condition*

$$\exists_{a,b\geq 0,\, a<(\pi^2 p_{00}+q_{00})(\pi^2 p_{10}+q_{10})} \forall_{(x,\xi)\in[0,1]\times\mathbb{R}} \quad |F(x,\xi)| \leq a|\xi| + b. \tag{2}$$

Then for every $y_0, y_1, \widehat{y}_0, \widehat{y}_1 \in \mathbb{R}$ *problem* (1) *has a solution.*

This result may be proved in much the same way as the theorem of Yang. The main tool of the proof is the classical method of a priori bounds. Let us introduce the family of problems

$$(1_t) \qquad \begin{aligned} &\mathcal{L}y = tF(\cdot, y) \quad \text{in } (0,1),\\ &y(0) = ty_0, \quad y(1) = ty_1, \quad \mathcal{L}_0 y(0) = t\widehat{y}_0, \quad \mathcal{L}_0 y(1) = t\widehat{y}_1. \end{aligned}$$

Denote by $(\ ,\ )$ the scalar product and by $\|\ \|$ the norm in $L^2(0,1)$. The next theorem will provide a priori estimates for solutions of (1_t).

THEOREM 2. *Let* y_t *denote a solution of* (1_t). *Then*

$$\exists M > 0\ \forall t \in [0,1] \quad \|y_t\| + \|\mathcal{L}_0 y_t\| \leq M. \tag{3}$$

Proof. Choose a smooth function $w : [0,1] \to \mathbb{R}$ satisfying the boundary conditions $w(0) = y_0$, $w(1) = y_1$, $\mathcal{L}_0 w(0) = \widehat{y}_0$, $\mathcal{L}_0 w(1) = \widehat{y}_1$. Let $z_t = y_t - tw$. Setting $G(x, z(x)) = tF(x, z(x) + tw(x)) - t\mathcal{L}w(x)$, we see that z_t satisfies the equation

$$\begin{aligned} &\mathcal{L}z = G(\cdot, z) \quad \text{in } (0,1),\\ &z(0) = z(1) = \mathcal{L}_0 z(0) = \mathcal{L}_0 z(1) = 0. \end{aligned}$$

From (2) we have $|G(x,\xi)| \leq a|\xi| + b_1$, where b_1 depends on b and w. Setting $u = \mathcal{L}_0 z$ we can study the following coupled problem:

$$\begin{aligned} \mathcal{L}_0 z &= u, & z(0) &= z(1) = 0,\\ \mathcal{L}_1 u &= G(\cdot, z), & u(0) &= u(1) = 0. \end{aligned}$$

By applying the Schwarz inequality combined with the Poincaré inequality we have the estimate

$$\begin{aligned} (p_{00}\pi^2 + q_{00})\|z\|^2 \leq p_{00}\|z'\|^2 + q_{00}\|z\|^2 &\leq \int_0^1 (p_0(x)[z'(x)]^2 + q_0(x)[z(x)]^2)\, dx\\ &= (\mathcal{L}_0 z, z) = (u, z) \leq \|u\| \cdot \|z\|. \end{aligned}$$

Hence

$$\|z\| \leq \frac{1}{p_{00}\pi^2 + q_{00}} \|u\|.$$

Proceeding analogously we obtain for arbitrary $\varepsilon > 0$,

$$\begin{aligned}(p_{10}\pi^2 + q_{10})\|u\|^2 &\le (u, \mathcal{L}_1 u) = (u, G(\cdot, z))\\ &\le \int_0^1 (a|u(x)| \cdot |z(x)| + b_1|u(x)|)\,dx\\ &\le a\|u\| \cdot \|z\| + \frac{1}{2}\varepsilon\|u\|^2 + \frac{b_1^2}{2\varepsilon}\\ &\le \left(\frac{a}{p_{00}\pi^2 + q_{00}} + \frac{1}{2}\varepsilon\right)\|u\|^2 + \frac{b_1^2}{2\varepsilon}.\end{aligned}$$

Since a satisfies (2) we can choose ε sufficiently small such that

$$1 - \frac{a}{(p_{00}\pi^2 + q_{00})(p_{10}\pi^2 + q_{10})} - \frac{\varepsilon}{2(\pi^2 p_{10} + q_{10})} = k > 0.$$

Hence

$$\|u\| \le \frac{b_1}{[2\varepsilon k(p_{10}\pi^2 + q_{10})]^{1/2}} = b_2$$

and consequently

$$\|z\| \le \frac{b_2}{p_{00}\pi^2 + q_{00}}.$$

Thus the proof is complete.

Proof of Theorem 1. Problem (1_t) can be written in the form

$$(1'_t) \qquad \begin{aligned}&\mathcal{L}_0 y = u, && y(0) = ty_0,\ y(1) = ty_1,\\ &\mathcal{L}_1 u = tF(\cdot, y), && u(0) = t\widehat{y}_0,\ u(1) = t\widehat{y}_1.\end{aligned}$$

Let G_i for $i = 0, 1$ be the Green function of the equation $\mathcal{L}_i v = h$ in $(0,1)$, with $v(0) = v(1) = 0$. Then $v(x) = \int_0^1 G_i(x,s)h(s)\,ds$. Using G_i we can transform $(1'_t)$ into the equivalent system of integral equations

$$(*) \qquad y(x) = ty_0 + xt(y_1 - y_0) + \int_0^1 G_0(x,s)u(s)\,ds,$$

$$(**) \qquad u(x) = t\widehat{y}_0 + xt(\widehat{y}_1 - \widehat{y}_0) + \int_0^1 tG_1(x,s)F(s, y(s))\,ds.$$

Let $E = L^2(0,1) \times L^2(0,1)$. It is a Banach space equipped with the norm $\|(y,u)\| = \|y\| + \|u\|$. Define a map $T_t : E \to E$ by $T_t = (T_t^0, T_t^1)$ where $T_t^0(y,u), T_t^1(y,u)$ are the right-hand sides of $(*)$ and $(**)$ respectively. To prove that problem (1) has a C^4-solution it is enough to search for solutions of $(I - T_1)(y,u) = 0$ in E. It is easily seen that T_t is a compact operator for every $t \in [0,1]$. Thus we see that the Leray–Schauder degree theory applies to $I - T_t$ and t is an allowable homotopy parameter. Consider $\overline{B}_{M+1} = \{(y,z) \in E : \|(y,u)\| \le M + 1\}$. The estimate (3) guarantees that

$\deg(I - T_t, B_{M+1}, 0)$ is well defined for each $t \in [0,1]$ and, by using the homotopy invariance of the degree we have

$$\deg(I - T_1, B_{M+1}, 0) = \deg(I - T_0, B_{M+1}, 0) = \deg(I, B_{M+1}, 0) = 1.$$

Consequently, $(I - T_1)(y, u) = 0$ has a solution in B_{M+1}, which completes the proof.

Remark 3. Let μ_1^i denote the first eigenvalue of the problem $\mathcal{L}_i y = \mu y$ subject to $y(0) = y(1) = 0$. From the above proof it is clear that using the variational definition of μ_1^i we can replace the assumption (2) by

$$\exists a, b \geq 0,\ a < \mu_1^0 \mu_1^1\ \forall x, \xi \quad |F(x,\xi)| \leq a|\xi| + b.$$

Remark 4. The equation $\mathcal{L}y = y^{(4)} - 3\pi^2 y'' = 4\pi^4 y$ has no solutions when $y_0 + y_1 + (1/(4\pi^2))(\widehat{y}_0 + \widehat{y}_1) \neq 0$, which means that assumption (2) is sharp.

2. Let us return to problem (1) in a linear version similar to that which was investigated by Usmani. The function F has the form $F(x,y) = f(x)y + g(x)$, where f and g are continuous on $[0,1]$. So, we consider the problem

$$\mathcal{L}y = fy + g \quad \text{in } (0,1) \tag{4}$$

together with the boundary conditions (B.C.). If we assume additionally that the operator $\mathcal{L}$ is symmetric and positive definite (this is satisfied in particular when $\mathcal{L}_0 = \mathcal{L}_1$) then the linear problem

$$\mathcal{L}v = \mu v$$

together with the boundary conditions

$$v(0) = v(1) = \mathcal{L}_0 v(0) = \mathcal{L}_0 v(1) = 0$$

has an increasing sequence of positive eigenvalues $0 < \mu_1 < \mu_2 < \ldots$

Our main result for (4) is:

THEOREM 5. *If $f(x) \neq \mu_j$, $j = 1, 2, \ldots$, then for any chosen $y_0, y_1, \widehat{y}_0, \widehat{y}_1$ and an arbitrary function g problem* (4) *has a unique solution.*

This result may be obtained by applying a mapping theorem for nonlinear operators of the form $L - N$ in a Hilbert space, with L linear and N nonlinear, proved by Mawhin in [3]. Nevertheless, for clarity and simplicity we give the direct proof of Theorem 5 which is based in great part on Mawhin's idea.

Proof of Theorem 5. Using the Green functions introduced in Section 1 we can convert problem (4) into an equivalent integral equation over $C[0,1]$:

$$y - Ty = h, \tag{5}$$

where

$$Ty(x) = \int_0^1 G_0(x,s)\Big[\int_0^1 G_1(s,t)f(t)y(t)\,dt\Big]\,ds,$$

$$h(x) = y_0 + x(y_1 - y_0) + \int_0^1 G_0(x,s)\Big[\widehat{y}_0 + s(\widehat{y}_1 - \widehat{y}_0) + \int_0^1 G_1(s,t)(f(t)y(t) + g(t))\,dt\Big]\,ds.$$

It is clearly enough to show that (5) is uniquely solvable for arbitrary $h \in C[0,1]$. Since T is a compact operator we can apply the Fredholm alternative. So, it is sufficient to prove that the boundary value problem

$$\text{(6)} \qquad \begin{aligned} &\mathcal{L}y = fy \quad \text{in } (0,1),\\ &y(0) = y(1) = \mathcal{L}_0 y(0) = \mathcal{L}_0 y(1) = 0, \end{aligned}$$

has only the trivial solution. The differential operator $\mathcal{L}$ together with the boundary conditions $y(0) = y(1) = \mathcal{L}_0 y(0) = \mathcal{L}_0 y(1) = 0$ defines an unbounded selfadjoint operator L in $L^2(0,1)$, so that problem (6) can be rewritten as

$$\text{(7)} \qquad (L - kI)y = \widehat{F}(y),$$

where $k \in \mathbb{R}$ and $\widehat{F}$ denotes the operator of multiplication by $f - k$, namely $\widehat{F}(y)(x) = (f(x) - k)y(x)$.

We denote by $\sigma(L)$ the spectrum of L. For $k \neq \mu_j$, $L - kI$ is invertible, so that (7) is equivalent to

$$y = (L - kI)^{-1}\widehat{F}(y).$$

Since $\|(L - kI)^{-1}\|^{-1} = \operatorname{dist}(k, \sigma(L))$ ([2]), we obtain

$$\begin{aligned} \|(L - kI)^{-1}\widehat{F}\| &\le \|(L - kI)^{-1}\| \cdot \|\widehat{F}\| \\ &= \frac{\|\widehat{F}\|}{\operatorname{dist}(k, \sigma(L))} \le \frac{\max_{x\in[0,1]} |f(x) - k|}{\operatorname{dist}(k, \sigma(L))}. \end{aligned}$$

There are two possibilities: either $\max_{x\in[0,1]} f(x) < \mu_1$, or there exists $j \in \mathbb{N}$ such that $\mu_j < \min_{x\in[0,1]} f(x) \le \max_{x\in[0,1]} f(x) < \mu_{j+1}$.

Note that

$$\operatorname{dist}(k, \sigma(L)) = \begin{cases} \mu_1 - k & \text{for } k < \mu_1, \\ \inf\{k - \mu_j, \mu_{j+1} - k\} & \text{for } k \in (\mu_j, \mu_{j+1}). \end{cases}$$

It is clear that we can choose k depending on f such that $\|(L - kI)^{-1}\widehat{F}\| < 1$. So (7) has only the trivial solution. This completes the proof.

Consider the particular case of problem (4), namely

$$\text{(8)} \qquad y^{(4)} = f(x)y + q_1 y'' + g(x)$$

with the boundary conditions (B.C.). The next result is an immediate consequence of Theorem 5.

THEOREM 6. *If* $f(x) \neq j^4\pi^4, j = 1, 2, \ldots,$ *then for any chosen* $y_0, y_1, \widehat{y}_0, \widehat{y}_1$ *and arbitrary functions* g *and* q_1 *problem* (8) *has a unique solution.*

Notice that $y^{(4)} = -\pi^2 y''$ has no solutions when $\widehat{y}_0 + \widehat{y}_1 \neq 0$, which shows that the condition $q_i \geq 0$ is sharp.

References

[1] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.
[2] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1966.
[3] J. Mawhin, *Contractive mappings and periodically perturbed conservative systems*, Arch. Math. (Brno) 12 (1976), 67–74.
[4] R. A. Usmani, *A uniqueness theorem for a boundary value problem*, Proc. Amer. Math. Soc. 77 (1979), 329–335.
[5] Y. Yang, *Fourth-order two-point boundary value problems*, ibid. 104 (1988), 175–180.

Institute of Mathematics
University of Mining and Metallurgy
Mickiewicza 30
30-059 Kraków, Poland

Reçu par la Rédaction le 7.2.1996
Révisé le 20.8.1996