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Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables

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Abstract. We consider a nonlinear Laplace equation $\Delta u = f(x,u)$ in two variables. Following the methods of B. Braaksma [Br] and J. Ecalle used for some nonlinear ordinary differential equations we construct first a formal power series solution and then we prove the convergence of the series in the same class as the function f in x.

0. Introduction. We consider a nonlinear Laplace equation of the form

$$\Delta u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u = f(x,u) \tag{1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. First we are going to construct a formal power series solution of (1) and then prove that every such solution is of the same class as the function f in x. Similar results for some nonlinear ordinary differential equations were proved by Braaksma [Br], following the ideas of J. Ecalle.

We denote by L the image of the positive quadrant $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$ under the unitary matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1-i & 1+i \\ 1+i & 1-i \end{smallmatrix}\right)$.

DEFINITION 1 ([Zie1]). A function F of the variable $z = (z_1, z_2) \in \mathbb{C}^2$ is said to be *Laplace holomorphic* on L if F is holomorphic on some polydisk centered at $(0,0) \in \mathbb{C}^2$, can be holomorphically continued to some sectorial neighbourhood $S = S_1 \times S_2$ of L with vertex $(0,0)$, and is of exponential growth on S, i.e. for every closed subsector $S' = S_1' \times S_2' \subset S$ there exist constants $c = (c_1, c_2)$ and C such that for $z \in S'$,

$$|F(z_1, z_2)| \le Ce^{c_1|z_1| + c_2|z_2|}. \tag{2}$$

DEFINITION 2. A function f of the variable $x = (x_1, x_2) \in \mathbb{R}^2$ is said to be a 1-*sum* of a formal power series

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$$\widehat{f}(x)=\sum_{k,l=0}^{\infty} g_{kl}\frac{1}{(x_1+ix_2)^{k+1}(ix_1+x_2)^{l+1}}$$

if there exists a Laplace holomorphic function F on L such that

$$f(x)=\int_L e^{-xz}F(z)\,dz$$

and

$$F(z)=\sum_{k,l=0}^{\infty}\frac{g_{kl}}{i2^{k+l+1}k!l!}(z_1-iz_2)^k(z_2-iz_1)^l$$

near zero. In that case we say that f is 1-*resummable*.

In this paper we assume that $f(x,u)$ on the right hand side of (1) is the 1-sum in x of a formal power series

$$\widehat{f}(x,u)=\sum_{k,l=0}^{\infty} g_{kl}(u)\frac{1}{(x_1+ix_2)^{k+1}(ix_1+x_2)^{l+1}}$$

with coefficients $g_{kl}(u)$ holomorphic for every $(k,l)\in\mathbb{N}_0^2$, on some fixed neighbourhood U of zero in $\mathbb{C}$, and $g_{kl}(0)=0$.

Therefore, if we write

$$\begin{aligned}\widehat{f}(x,u)&=\sum_{k,l=0}^{\infty}\Big(\sum_{j=1}^{\infty}c_{kl}^j u^j\Big)\frac{1}{(x_1+ix_2)^{k+1}(ix_1+x_2)^{l+1}}\\&=\sum_{j=1}^{\infty}\bigg(\sum_{k,l=0}^{\infty}c_{kl}^j\frac{1}{(x_1+ix_2)^{k+1}(ix_1+x_2)^{l+1}}\bigg)u^j\end{aligned}$$

then we can write $f(x,u)=\sum_{j=1}^{\infty}c_j(x)u^j$ where $c_j(x)$ is the 1-sum of the formal series

$$\sum_{k,l=0}^{\infty}c_{k,l}^j\frac{1}{(x_1+ix_2)^{k+1}(ix_1+x_2)^{l+1}},$$

and f is holomorphic in u on U. Hence, we have $c_j(x)=\int_L e^{-xz}T_j(z)\,dz$ for some Laplace holomorphic functions T_j. Moreover, T_j are holomorphic on the same sector S for all j, and the constants c and C in (2) are independent of j.

Theorem. *If* $T_j(0)\neq 0$, *then there exists a family of* 1-*resummable solutions of equation* (1) *of the form*

$$u(x)=\sum_{\nu=0}^{\infty}d_\nu\frac{1}{(x_1+ix_2)^{\nu_1+1}}\cdot\frac{1}{(ix_1+x_2)^{\nu_2+1}}.$$

This means that

$$u(x) = \int_L e^{-xz} T(z)\, dz \tag{3}$$

with T being a Laplace holomorphic function on L. Moreover, every formal solution $\widehat{u}$ of (1) *of the above form is* 1*-resummable.*

The proof will be divided into three parts.

1. Convolution equation. Applying Δ to u in the form (3) we arrive at the complex symbol of Δ as the complex polynomial

$$P(z_1, z_2) = z_1^2 + z_2^2 = \left(\frac{1+i}{\sqrt{2}} z_1 + \frac{1-i}{\sqrt{2}} z_2\right)\left(\frac{1-i}{\sqrt{2}} z_1 + \frac{1+i}{\sqrt{2}} z_2\right).$$

In the new variables

$$\zeta_1 = \frac{1+i}{\sqrt{2}} z_1 + \frac{1-i}{\sqrt{2}} z_2, \qquad \zeta_2 = \frac{1-i}{\sqrt{2}} z_1 + \frac{1+i}{\sqrt{2}} z_2,$$

P becomes the polynomial $\widetilde{P}(\zeta_1, \zeta_2) = \zeta_1 \cdot \zeta_2$, and after changing variables on the left hand side of (1) we get

$$\begin{aligned}\Delta u(x_1, x_2) &= (P(z_1, z_2)T)[e^{-x_1 z_1 - x_2 z_2}] \\ &= (\widetilde{P}(\zeta_1, \zeta_2)\widetilde{T})[e^{-x_1(\frac{1-i}{2\sqrt{2}}\zeta_1 + \frac{1+i}{2\sqrt{2}}\zeta_2) - x_2(\frac{1+i}{2\sqrt{2}}\zeta_1 + \frac{1-i}{2\sqrt{2}}\zeta_2)}] \\ &= (\widetilde{P}(\zeta_1, \zeta_2)\widetilde{T})[e^{-(\frac{1-i}{2\sqrt{2}}x_1 + \frac{1+i}{2\sqrt{2}}x_2)\zeta_1 - (\frac{1+i}{2\sqrt{2}}x_1 + \frac{1-i}{2\sqrt{2}}x_2)\zeta_2}].\end{aligned}$$

So we are looking for a solution

$$\widetilde{u}(y_1, y_2) = \widetilde{T}[e^{-y_1\zeta_1 - y_2\zeta_2}] = u\left(\frac{1-i}{2\sqrt{2}} x_1 + \frac{1+i}{2\sqrt{2}} x_2, \frac{1+i}{2\sqrt{2}} x_1 + \frac{1-i}{2\sqrt{2}} x_2\right)$$

of the convolution equation

$$\zeta_1 \zeta_2 \widetilde{T} = f^* \widetilde{T} \tag{4}$$

where $f^*\widetilde{T} = \sum_{j=1}^{\infty} \widetilde{T}_j * \widetilde{T}^{*j}$ with $\widetilde{T}^{*j}$ denoting the jth convolution power of $\widetilde{T}$, i.e. $T^{*j} = T * \ldots * T$ (j times). From now on we write T instead of $\widetilde{T}$.

We can assume that $T_1(0) = 1$, for otherwise we modify slightly the change of variables after dividing equation (1) by $T_1(0)$.

Since our existence proof for the solution of (4) essentially follows that of Braaksma [Br], we shall consider T having the formal expansion

$$T = \sum_{k,l=0}^{\infty} d_{kl} \widetilde{\zeta}_1^k \widetilde{\zeta}_2^l \tag{5}$$

with $\widetilde{\zeta}^p = \zeta^p / \Gamma(p+1)$. Then due to the convolution formula

$$\widetilde{\zeta}^l * \widetilde{\zeta}^k = \widetilde{\zeta}^{l+k+1}$$

we find

$$\begin{aligned}
f^*T &= \sum_{j=1}^{\infty}\sum_{m_1,m_2=0}^{\infty} c^j_{m_1m_2}\widetilde{\zeta}_1^{m_1}\widetilde{\zeta}_2^{m_2} * \Big(\sum_{k,l=0}^{\infty} d_{kl}\widetilde{\zeta}_1^{k}\widetilde{\zeta}_2^{l}\Big)^{*j}\\
&= \sum_{j=1}^{\infty}\sum_{m_1,m_2=0}^{\infty} c^j_{m_1m_2}\widetilde{\zeta}_1^{m_1}\widetilde{\zeta}_2^{m_2} * \sum_{\nu_1+\ldots+\nu_j=\mathbf{0}}^{\infty} d_{\nu_1}\ldots d_{\nu_j}\widetilde{\zeta}^{\nu_1+\ldots+\nu_j+\mathbf{j}-\mathbf{1}}\\
&= \sum_{j=1}^{\infty}\sum_{m+\nu_1+\ldots+\nu_j=0}^{\infty} c^j_m d_{\nu_1}\ldots d_{\nu_j}\widetilde{\zeta}^{m+\nu_1+\ldots+\nu_j+\mathbf{j}}\\
&= \sum_{k=\mathbf{0}}^{\infty}\Big(\sum_{j=1}^{\overline{k}+1}\sum_{m+\nu_1+\ldots+\nu_j=k+\mathbf{1}-\mathbf{j}} c^j_m d_{\nu_1}\ldots d_{\nu_j}\Big)\widetilde{\zeta}^{k+\mathbf{1}}
\end{aligned}$$

for $k,m,\nu_j\in\mathbb{N}_0^2$, $\mathbf{j}=(j,j)$, $\mathbf{1}=(1,1)$, $\overline{k}=\min\{k_1,k_2\}$.

Inserting this in (4) we find

$$d_k(k+\mathbf{1}) = \sum_{j=1}^{\overline{k}+1}\sum_{m+\nu_1+\ldots+\nu_j=k+\mathbf{1}-\mathbf{j}} c^j_m d_{\nu_1}\ldots d_{\nu_j}, \tag{6}$$

since

$$\zeta\cdot\widetilde{\zeta}^p = (p+1)\widetilde{\zeta}^{p+1}.$$

In particular, we can take d_{00} arbitrarily (since $c^1_{00}=1$), $d_{10}=c^1_{01}d_{00}, d_{01}$ $=c^1_{01}d_{00}$,

$$\begin{aligned}
&2d_{20} = c^1_{20}d_{00}+c^1_{10}d_{10}, \qquad 2d_{02} = c^1_{02}d_{00}+c^1_{01}d_{01},\\
&3d_{11} = c^1_{11}d_{00}+c^1_{10}d_{01}+c^1_{01}d_{10}+c^2_{00}d_{00},\ \ldots
\end{aligned}$$

We are going to prove that T defined formally by (5) with coefficients d_ν satisfying the recurrence (6) is a holomorphic function of exponential growth in some sector S.

Before starting the resummation proof for the expansion (5), we consider the resummation problem with respect to one variable. Therefore, let us write (5) in the form

$$T = \sum_{k=0}^{\infty} T_{1k}(\zeta_1)\widetilde{\zeta}_2^k$$

where $T_{1k}(\zeta_1)=\sum_{l=0}^{\infty} d_{lk}\widetilde{\zeta}_1^l$.

In a way similar to that of deriving (6), we find that T_{1k} satisfy the convolution equation

$$\zeta_1(k+1)T_{1k} = \sum_{j=1}^{k+1} \sum_{m+\nu_1+\ldots+\nu_j=k+1-j} T_m^j * T_{1\nu_1} * \ldots * T_{1\nu_j}$$

for $k, m, \nu_j \in \mathbb{N}_0$, where $T_m^j = \sum_{p=0}^{\infty} c_{pm}^j \widetilde{\zeta}_1^p$. For $k = 0$ this gives

$$\zeta_1 T_{10} = T_0^1 * T_{10}, \tag{7}$$

which is equivalent to the equation

$$\frac{d}{dt} u_0 = c_0^1(t) u_0, \tag{7'}$$

in the variable $t = \frac{1-i}{2\sqrt{2}} x_1 + \frac{1+i}{2\sqrt{2}} x_2$.

For $k = 1$ we get

$$2\zeta_1 T_{11} = T_0^1 * T_{11} + T_1^1 * T_{10} + T_0^2 * T_{10}^{*2}$$

or equivalently

$$2\frac{d}{dt} u_1 = c_0^1(t) u_1 + c_1^1(t) u_0 + c_0^2(t)(u_0)^2.$$

We easily see that the jth equation is linear in u_j with $u_0, \ldots, u_{j-1}$ regarded as coefficients. Since the solutions of linear equations with resummable coefficients are resummable themselves (cf. [Br], [Zie1]), we see that all T_{1k} are Laplace holomorphic functions. The same is also true for

$$T_{l2}(\zeta_2) = \sum_{j=0}^{\infty} d_{lj} \widetilde{\zeta}_2^j.$$

Now we pass to the proof of the convergence of the formal series (5) with d_ν satisfying (6). Since the series (5) satisfies (4), for a fixed $N \in \mathbb{N}_0$ the series

$$T_N = \sum_{l,j=N+1}^{\infty} d_{lj} \widetilde{\zeta}_1^l \widetilde{\zeta}_2^j = T - S_N$$

satisfies the equation

$$\begin{aligned}\zeta_1\zeta_2 T_N = G^N(\zeta, T_N) &= \sum_{j=1}^{\infty} \sum_{k=0}^{j} \binom{j}{k} T_j * T_N^{*k} * S_N^{*(j-k)} - \zeta_1\zeta_2 S_N \\ &= \sum_{k=0}^{\infty} \Bigg(\sum_{\substack{j=k \\ j\geq 1}}^{\infty} \binom{j}{k} T_j * S_N^{*(j-k)} \Bigg) * T_N^{*k} - \zeta_1\zeta_2 S_N.\end{aligned}$$

We write

$$G^N(\zeta, \psi) = \sum_{k=0}^{\infty} g_k(\zeta) * \psi^{*k} \tag{8}$$

where $g_0 = \sum_{j=1}^{\infty} T_j * S_N^{*j} - \zeta_1\zeta_2 S_N$, and $g_k = \sum_{j=k}^{\infty} \binom{j}{k} T_j * S_N^{*(j-k)}$ for $k > 0$. The series g_k are convergent near $(0,0)$ due to the remarks about the resummation problem with respect to one variable and the fact that the series $\sum_{j=k}^{\infty} \binom{j}{k} T_j(\zeta) u^{j-k}$ is convergent near 0. Moreover, we can see that for every subsector $S' \subset S$ there exist K and $c = (c_1, c_2)$ such that for $\zeta \in S'$,

$$\begin{cases} |g_0(\zeta)| \le K|\zeta_1|^{N+1}|\zeta_2|^{N+1} e^{c_1|\zeta_1|+c_2|\zeta_2|}, \\ |g_k(\zeta)| \le K e^{c_1|\zeta_1|+c_2|\zeta_2|} \quad \text{for } k \ge 1. \end{cases} \tag{9}$$

For $p = (p_1, p_2)$, $p_i > 0$, $s = (s_1, s_2) \in \mathbb{R}^2$, we denote by $W_s(p)$ the space of functions ψ holomorphic in the polydisc $\{|\zeta_1| \le p_1, |\zeta_2| \le p_2\}$ and such that

$$\|\psi\|_{s,p} = \sup_{|\zeta_i| \le p_i} |\zeta^{-s}\psi(\zeta)| < \infty.$$

Observe that for $\zeta \in \{|\zeta_i| \le p_i\}$ and $s_1 > -1, s_2 > -1$,

$$|\psi^{*m}(\zeta)| \le \|\psi\|_{s,p}^m \frac{\Gamma(s_1+1)^m \Gamma(s_2+1)^m}{\Gamma(m(s_1+1))\Gamma(m(s_2+1))} \times |\zeta_1|^{m(s_1+1)-1}|\zeta_2|^{m(s_2+1)-1}. \tag{10}$$

Therefore by the properties of the Γ-function the function (8) makes sense for $\psi \in W_s(p)$ if s is large enough.

Consider the operator

$$R\psi(\zeta) = \frac{1}{\zeta} g_0(\zeta) + \frac{1}{\zeta}(g_1 * \psi)(\zeta) + \sum_{m=2}^{\infty} \frac{1}{\zeta}(g_m * \psi^{*m})(\zeta). \tag{11}$$

Denoting the summands by $R_0\psi$, $R_{\text{lin}}\psi$ and $Q\psi$ respectively, for $\psi \in W_{N-1}(p)$ and $\zeta \in \{|\zeta_i| \le p_i,\ i = 1, 2\}$ we get the estimates

$$\begin{cases} |R_0\psi(\zeta)| \le K|\zeta_1|^N|\zeta_2|^N, \\ |R_{\text{lin}}\psi(\zeta)| \le K\|\psi\|_{N-1,p}|\zeta_1|^{N-1}|\zeta_2|^{N-1}\left(\dfrac{\Gamma(N)}{\Gamma(N+1)}\right)^2 \\ \qquad\qquad = \dfrac{K}{N^2}\|\psi\|_{N-1,p}|\zeta_1\zeta_2|^{N-1}, \\ |Q\psi(\zeta)| \le K\left(\displaystyle\sum_{m=2}^{\infty} \|\psi\|_{N-1,p}^m \frac{\Gamma(N)^{2m}}{\Gamma(mN)^2}|\zeta_1\zeta_2|^{(m-1)N-1}\right)|\zeta_1\zeta_2|^N. \end{cases} \tag{12}$$

Set $M = \left\|\frac{1}{\zeta} g_0(\zeta)\right\|_{N-1,p}$. If $\|\psi\|_{N-1,p} \le 2M$ then by choosing p small and N large we may have (by (12))

$$\|Q\psi\|_{N-1,p} \le \tfrac{1}{3}M \quad \text{and} \quad \|R_{\text{lin}}\psi\| \le \tfrac{1}{3}M.$$

Therefore the operator R acts in the space

$$B_{N-1,p} = \{\psi \in W_{N-1}(p) : \|\psi\|_{N-1,p} \le 2M\}.$$

Observe that for $\psi, \psi + \chi \in B_{N-1,p}$ we have

$$\begin{aligned}|(\psi+\chi)^{*m}(\zeta)-\psi^{*m}(\zeta)| &= \left|\sum_{l=1}^{m}\binom{m}{l}(\psi^{*(m-l)}*\chi^{*l})(\zeta)\right| \\ &\le \sum_{l=1}^{m}\binom{m}{l}\|\psi\|_{N-1,p}^{m-l}\|\chi\|_{N-1,p}^{l}\frac{\Gamma(N)^{2m}}{\Gamma(mN)^2}|\zeta_1\zeta_2|^{mN-1} \\ &\le \frac{\Gamma(N)^{2m}}{\Gamma(mN)^2}|\zeta_1\zeta_2|^{mN-1}\sum_{l=1}^{m}\binom{m}{l}\|\psi\|_{N-1,p}^{m-l}\|\chi\|_{N-1,p}^{l}.\end{aligned}$$

We have

$$\begin{aligned}\sum_{l=1}^{m}\binom{m}{l}\|\psi\|_{N-1,p}^{m-l}\|\chi\|_{N-1,p}^{l} &\le \|\chi\|_{N-1,p}\sum_{l=1}^{m}\binom{m}{l}(2M)^{m-l}(4M)^{l-1} \\ &\le \frac{\|\chi\|_{N-1,p}}{4M}(2M+4M)^m = \frac{(6M)^m}{4M}\|\chi\|_{N-1,p}\end{aligned}$$

since $\|\chi\|\le\|\psi+\chi\|+\|\psi\|\le 4M$. Hence

$$|(\psi+\chi)^{*m}(\zeta)-\psi^{*m}(\zeta)| \le \frac{(6M\Gamma(N)^2)^m}{4M\Gamma(mN)^2}\|\chi\|_{N-1,p}(|\zeta_1\zeta_2|)^{mN-1},$$

and

$$\begin{aligned}&\left|\frac{1}{\zeta}(g_m*((\psi+\chi)^{*m}-\psi^{*m}))(\zeta)\right| \\ &\qquad\le \frac{K}{4M}\left(\frac{(6M\Gamma(N)^2)^m}{\Gamma(mN)^2}|\zeta_1\zeta_2|^{(m-1)N-1}\right)|\zeta_1\zeta_2|^N\|\chi\|_{N-1,p}.\end{aligned}$$

From this and from (12), we derive that

$$\begin{aligned}\|R(\psi+\chi)-R\psi\|_{N-1,p} &\le \|R_{\rm lin}\psi\|_{N-1,p}+\|Q(\psi+\chi)-Q\psi\| \\ &\le \tfrac{1}{3}\|\psi\|_{N-1,p}+K'\|\chi\|_{N-1,p}\le\tfrac{2}{3}\|\chi\|_{N-1,p}\end{aligned}$$

provided p is small enough. Therefore, for p small and N large, the operator R is a contraction on $B_{N-1,p}$. Hence we get a unique function ψ_N solving the nonlinear convolution equation

$$\zeta_1\zeta_2\psi_N = G^N(\zeta,\psi_N), \qquad \psi_N\in B_{N-1,p}. \tag{13}$$

From the construction of G^N it follows that for every N (sufficiently large) the function ψ_N+S_N satisfies the equation (4), hence the kth Taylor coefficient of ψ_N (at 0) must satisfy (6) (for $k_i\ge N+1$), so T defined formally by (5) and (6) converges on $\{|\zeta_i|\le p_i\}$.

2. Analytic continuation of solutions. Define $S(r)=\{\zeta\in\mathbb{C}:|\zeta|\le r\}\cap S_1$ (see Introduction) and let p be such that the solution ψ_N of (13) is holomorphic in the interior of $S^2(p)=S(p)\times S(p)$. We shall extend this solution to a unique solution on some complex neighbourhood of $\mathbb{R}_+^2$.

Choose $\delta, p_1 \in \mathbb{R}_+,\ \delta < p_1 < p$. Define

$$S_0 = S(p_1) \times S(p),$$
$$S_+ = \{\zeta \in \mathbb{C}^2 : (\zeta_1 - p_1, \zeta_2) \in S(\delta) \times S(p) \text{ or } \zeta_1 = p_1\},$$
$$S^1 = S_0 \cup S_+.$$

Then $S_0 \cap S_+ = \{p_1\} \times S(p)$.

Let W_0 denote the space of functions on S^1 which are continuous on $S^1 \setminus (S_0 \cap S_+)$ and analytic in its interior. Next define $\widetilde{\psi} \in W_0$ by setting $\widetilde{\psi} = \psi_N$ on S_0 and $\widetilde{\psi} \equiv 0$ on S_+. Introduce the space

$$V_{N-1}(\delta) = \{\phi \in C^0(S_+) \cap \mathcal{O}(\mathrm{int} S_+) : \sup_{\zeta \in S_+} |\zeta_2^{-N+1}\phi(\zeta)| < \infty\}.$$

For $\phi \in V_{N-1}(\delta)$ define $\phi_0 \in W_0$ by extending ϕ by zero on S_0. Then

$$(\phi_0 * \phi_0)(\zeta) = \int\limits_{C(\zeta)} \phi_0(\zeta - \gamma)\phi_0(\gamma)\, d\gamma \equiv 0$$

where $C(\zeta) = C(\zeta_1) \times C(\zeta_2), C(\zeta_i)$ is a path from 0 to ζ_i. Hence also $\phi_0^{*m} \equiv 0$ for $m \ge 2$. Clearly, $\widetilde{\psi}^{*m} = \psi_N^{*m}$ on S_0 for all m. Therefore $(\widetilde{\psi} + \phi)^{*m} = \widehat{\psi}^{*m} + m\widehat{\psi}^{*(m-1)} * \phi_0$.

Consequently, for $G(\zeta, \psi) = G^N(\zeta, \psi)$ given by (8) we have

$$G(\zeta, \widetilde{\psi} + \phi_0) = G(\zeta, \widetilde{\psi}) + (B * \phi_0)(\zeta) \quad \text{where}$$
$$B(\zeta) = g_1(\zeta) + \sum_{m=2}^{\infty} m(g_m * \widetilde{\psi}^{*(m-1)})(\zeta).$$

Thus the equation

$$\zeta(\widetilde{\psi} + \phi_0) = G(\zeta, \widetilde{\psi} + \phi_0)$$

gives rise to a linear convolution equation

$$\phi_0 = \chi + \frac{1}{\zeta}(B * \phi_0)(\zeta) \tag{14}$$

for $\phi_0 \in V_{N-1}(\delta)$, with $\chi(\zeta) = \frac{1}{\zeta}G(\zeta, \widetilde{\psi}) - \widetilde{\psi}$.

For $\zeta \in S_+$ and $\phi \in V_{N-1}(\delta)$ we have

$$\begin{aligned}\left|\frac{1}{\zeta}(B * \phi_0)(\zeta)\right| &= \left|\frac{1}{\zeta}\int\limits_{p_1}^{\zeta_1}\int\limits_0^{\zeta_2} B(\zeta_1 - \eta_1, \zeta_2 - \eta_2)\phi(\eta_1, \eta_2)\, d\eta_1\, d\eta_2\right| \\ &= \left|\frac{1}{\zeta}\int\limits_0^{\zeta_2}\Big[\int\limits_0^{\zeta_1 - p_1} B(\zeta_1 - p_1 - \gamma_1, \zeta_2 - \eta_2)\phi(\gamma_1 + p_1, \eta_2)\, d\gamma_1\Big]\, d\eta_2\right| \\ &\le \frac{1}{|\zeta|}\|\phi\|_{N-1}\left|\int\limits_0^{\zeta_2} \eta_2^{N-1}\Big[\int\limits_0^{\zeta_1 - p_1} B(\tau, \zeta_2 - \eta_2)\, d\tau\Big]\, d\eta_2\right|\end{aligned}$$

with $\|\phi\|_{N-1} = \sup_{\zeta\in S_+} |\zeta_2^{-N+1}\phi(\zeta)|$. Now, from the definition of B, we see that for $\tau \in S(\delta)\times S(p)$,

$$|B(\tau)| \le C\biggl(1+\sum_{m=2}^{\infty} m\|\psi_N\|_{N-1,p}^{m-1}\frac{\Gamma(N)^{2(m-1)}}{\Gamma((m-1)N)^2}(|\tau_1|\cdot|\tau_2|)^{(m-1)N}\biggr) < M.$$

Thus for $\zeta\in S_+$ (and consequently for $|\zeta_1|\ge p_1$) we have

$$\left|\frac{1}{\zeta}(B*\phi_0)(\zeta)\right| \le K\|\phi\|_{N-1}|\zeta_2^{N-1}| \quad \text{with } K=\frac{M\delta}{Np_1}.$$

Hence if we take $\delta < Np_1/M$, then the operator $\phi\to\frac{1}{\zeta}B*\phi_0$ is a contraction in the space $V_{N-1}(\delta)$. Thus there exists a unique solution $\phi\in V_{N-1}(\delta)$ satisfying (14). Hence $\phi=\psi_N$ on the interior of $S^2(p)\cap S_+$ and it is clear that ϕ extends ψ_N to S_+.

A repeated application of this procedure yields an extension of ψ_N to some region $U\times S(p)$, where U is a sectorial neighbourhood of $\mathbb{R}_+$ in $\mathbb{C}$. By interchanging variables and proceeding by the same method we get an extension of ψ_N to some region $S(p)\times V$ with V being a sectorial neighbourhood of $\mathbb{R}_+$ in $\mathbb{C}$. Finally, in the same way we obtain an extension of ψ_N to some sector $U\times V$.

3. Exponential estimation. It follows from the results on analytic continuation of the solution of (13) that there exists a function ψ, holomorphic in some sector S containing $\mathbb{R}_+^2$, satisfying (13) and such that $\zeta^{-N+1}\psi$ is locally bounded. We shall prove a global exponential estimate: for every closed subsector $S'\subset S$,

$$|\psi(\zeta_1,\zeta_2)| \le K|\zeta^{N-1}|e^{c_1|\zeta_1|+c_2|\zeta_2|}$$

for $\zeta\in S'$, with appropriate constants K and c_1, c_2. The proof is again a two-dimensional variant of the reasoning given in [Br].

For $p>0$ define

$$M(p)=\sup\{|\zeta_1^{-N+1}\psi(\zeta_1,\zeta_2)| : 0<|\zeta_1|<1,\ |\zeta_2|=p,\ \zeta\in S'\}.$$

It follows from the local estimates for ψ that $M(p)$ makes sense for each fixed $p>0$. Then for $0<|\zeta_1|<1,\ |\zeta_2|=p,\ \zeta\in S'$,

$$|\psi(\zeta_1,\zeta_2)| \le M(p)|\zeta_1^{N-1}|,$$

and as in (10),

$$|\psi^{*m}(\zeta_1,\zeta_2)| \le M^{*m}(p)\biggl(\frac{\Gamma(N)^m}{\Gamma(mN)}|\zeta_1^{(m-1)N-1}|\biggr)|\zeta_1|^N.$$

Then, by (9), we find that for any $\widehat{c}_2 > c_2$,

$$\left|\frac{1}{\zeta}(g_m*\psi^{*m})(\zeta)\right| \le Ke^{\hat{c}_2 p}*q^mM^{*m}(p)|\zeta_1|^N$$

where q is a sufficiently small constant such that

$$\left(\frac{\Gamma(N)^m}{\Gamma(mN)}|\zeta_1^{(m-1)N-1}|\right)^{1/m} \le q \quad \text{for } m \in \mathbb{N}.$$

Therefore we have for $0 < |\zeta_1| < 1,\ |\zeta_2| = p,\ \zeta \in S'$,

$$|\psi(\zeta)| = |R\psi(\zeta)| \le K|\zeta_1|^N e^{\hat{c}_2 p} + K|\zeta_1|^N \Big(e^{\hat{c}_2 p} * \sum_{m=1}^{\infty} q^m M^{*m}(p)\Big)$$

and for all $p > 0$,

$$|\zeta_1^{-N+1}\psi(\zeta)| \le \widetilde{K} e^{\hat{c}_2 p} + \widetilde{K}\Big(e^{\hat{c}_2 p} * \sum_{m=1}^{\infty} q^m M^{*m}(p)\Big). \tag{15}$$

Denoting the right hand side of (15) by SM we get $M(p) \le SM(p)$ for $p > 0$.

Consider the equation

$$N(p) = SN(p). \tag{16}$$

Under the Laplace transformation

$$v(s) = \mathcal{L}N(s) = \int_0^\infty e^{-ps} N(p)\, dp$$

equation (16) becomes

$$v(s) = \frac{\widetilde{K}}{s - \widehat{c}_2} + \frac{\widetilde{K}}{s - \widehat{c}_2} \cdot \sum_{m=1}^{\infty} (qv(s))^m = \frac{\widetilde{K}}{s - \widehat{c}_2} \cdot \frac{1}{1 - qv(s)}$$

or equivalently

$$qv^2 - v + \frac{\widetilde{K}}{s - \widehat{c}_2} = 0.$$

This equation has a unique solution analytic in $1/s$ at infinity, of the form

$$v(s) = \frac{\widetilde{K}}{s} + \sum_{l=1}^{\infty} \frac{b_l}{s^{l+1}} \quad \text{for } s \text{ large enough}$$

with coefficients $b_l \in \mathbb{R}$. Hence

$$N(p) = \widetilde{K} + \sum_{l=1}^{\infty} \frac{b_l}{l!} p^l$$

is a solution of (16) real-valued for $p > 0$ and of exponential growth: $N(p) \le \widetilde{\widetilde{K}} e^{\hat{c}_2 p}$ with some $\widetilde{\widetilde{K}} < \infty$.

Since $M(0) = 0$ and $N(0) = \widetilde{K} > 0$, and therefore $M(p) \le N(p)$, it follows from the definition of M that for $\zeta \in S' \cap \{|\zeta| \le 1,\ |\zeta_2| \ge 1\}$,

$$|\psi(\zeta)| \le \widehat{\widetilde{K}}(|\zeta_1| \cdot |\zeta_2|)^{N-1} e^{\hat{c}_2|\zeta_2|}. \tag{17}$$

By the same method we get for $\zeta \in S' \cap \{|\zeta_1| \ge 1, |\zeta_2| \le 1\}$,

$$|\psi(\zeta)| \le \widehat{\widetilde{K}}(|\zeta_1| \cdot |\zeta_2|)^{N-1} e^{\bar{c}_1|\zeta_1|}. \tag{17'}$$

Now we pass to the global estimate on S'. By (9) we get for $\bar{c}_i > c_i$ $(i = 1, 2)$,

$$|R\psi(\zeta)| \le K\bigg(e^{\bar{c}_1|\zeta_1|+\bar{c}_2|\zeta_2|} + \frac{1}{|\zeta_1| \cdot |\zeta_2|}\Big(e^{\bar{c}_1|\zeta_1|+\bar{c}_2|\zeta_2|} * \sum_{m=1}^{\infty} |\psi|^{*m}(\zeta)\Big)\bigg).$$

Using this for $|\zeta_1| \ge 1,\ |\zeta_2| \ge 1$, since $\psi = R\psi$, we get

$$|\psi(\zeta_1, \zeta_2)| \le \widetilde{K}\Big(e^{\langle \bar{c}, |\zeta| \rangle} + e^{\langle \bar{c}, |\zeta| \rangle} * \sum_{m=1}^{\infty} |\psi|^{*m}(\zeta)\Big)$$

As above, under the two-dimensional Laplace transformation we are led to considering the equation

$$v(s_1, s_2) = \frac{\widetilde{K}}{(s_1 - \bar{c_1})(s_2 - \bar{c}_2)} \cdot \frac{1}{1 - v(s_1, s_2)}$$

with $v = \mathcal{L}\psi$. Again we prove that it has a solution v analytic in $(1/s_1, 1/s_2)$ at infinity, so ψ satisfies the exponential growth condition.

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