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Successive derivatives and finite expansions involving the H-function of one and more variables

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Abstract. Certain results including the successive derivatives of the H-function of one and more variables are established. These remove the limitations of Ławrynowicz's (1969) formulas and as a result extend the results of Skibiński [13] and various other authors. As an application some finite expansion formulas are also established, which reduce to hypergeometric functions of one and more variables that are of common interest.

1. Introduction. Fox's H-function which is defined by a Mellin–Barnes type contour integral (for details see [12]) as

$$H[z]=H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right]=\frac{1}{2\pi\omega}\int_L\phi(\xi)z^{\xi}\,d\xi, \tag{1.1}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)}\cdot\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)},$$

has been extended to the multivariate H-function by Srivastava and Panda [16] in the form of the Mellin–Barnes type contour integral

$$H[z_1,\ldots,z_r]=H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\cdot\left[\begin{matrix}z_1\\\vdots\\z_r\end{matrix}\,\middle|\,\begin{matrix}((a_p;\alpha_p^{(1)};\ldots;\alpha_p^{(r)})):((c_{p_1}^{(1)},\gamma_{p_1}^{(1)}));\ldots;((c_{p_r}^{(r)},\gamma_{p_r}^{(r)}))\\ \\((b_q;\beta_q^{(1)};\ldots;\beta_q^{(r)})):((d_{q_1}^{(1)},\delta_{q_1}^{(1)}));\ldots;((d_{q_r}^{(r)},\delta_{q_r}^{(r)}))\end{matrix}\right] \tag{1.2}$$

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$$= \left(\frac{1}{2\pi\omega}\right)^r \int_{L_1} \dots \int_{L_r} \Psi(\xi_1, \dots, \xi_r) \prod_{i=1}^{r} \{\phi_i(\xi_i) z_i^{\xi_i}\}\, d\xi_1 \dots d\xi_r,$$

where

$$\Psi(\xi_1, \dots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma\Big(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\Big)}{\prod_{j=n+1}^{p} \Gamma\Big(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\Big) \prod_{j=1}^{q} \Gamma\Big(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i\Big)}$$

and for $i = 1, \dots, r$, we have

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}.$$

Moreover, $\omega^2 = -1$ and z_i ($\neq 0$) is a complex variable, $z_i = \exp\{\xi_i(\log|z_i| + \omega \arg z_i)\}$, where $\log|z_i|$ represents the natural logarithm of $|z_i|$; it is single-valued for $|\arg z_i| < \pi$.

By $((b_q; \beta_q^{(1)}, \dots, \beta_q^{(r)}))$, we mean a q-member array

$$(b_1; \beta_1^{(1)}, \dots, \beta_1^{(r)}), \dots, (b_q; \beta_q^{(1)}, \dots, \beta_q^{(r)})$$

etc. Further, the parameters a, b, c and d are complex numbers and the associated coefficients α, β, γ and δ are positive real numbers such that the poles of the integrand are separated.

The contour L_i in the ξ_i-plane runs from $-\omega\infty$ to $+\omega\infty$ along the imaginary axis, indented if necessary to ensure that the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i)$ $(j = 1, \dots, m_i)$ lie to the right of the contour and those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i)$ $(j = 1, \dots, n)$ and $\Gamma(1 - c_j^{(i)} + \delta_j^{(i)} \xi_i)$ $(i = 1, \dots, n_i)$ are to the left of the contour. The positive integers n, p, q, n_i, m_i, p_i, q_i are constrained by the inequalities:

$$0 \le n \le p, \quad 0 \le q, \quad 1 \le m_i \le q_i \quad \text{and} \quad 0 \le n_i \le p_i.$$

The multiple Mellin–Barnes contour integral (1.2) converges absolutely under the following conditions:

(i) $$\lambda_i = \sum_{j=1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} < 0,$$

(ii) $$\mu_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0 \quad \text{and}$$

(iii) $|\arg z_i| < \frac{1}{2}\mu_i\pi$.

Conditions (i) and (ii) can be relaxed to $\lambda_i \le 0$ and $\mu_i \ge 0$ (see Joshi and Joshi [6]). But then the integral exists in a shrunken region. For example, when $\lambda_i = 0$, the region of convergence reduces to the interior of the circle in the ξ_i-plane with radius D_i^{-1}, where

$$D_i = \frac{\prod_{j=1}^{p}(\alpha_j^{(i)})^{\alpha_j^{(i)}} \prod_{j=1}^{p_i}(\gamma_j^{(i)})^{\gamma_j^{(i)}}}{\prod_{j=1}^{q}(\beta_j^{(i)})^{\beta_j^{(i)}} \prod_{j=1}^{q_i}(\delta_j^{(i)})^{\delta_j^{(i)}}},$$

with the parameters satisfying

$$q_i = \frac{1}{2}(p + p_i - q - q_i) - \mathrm{Re}\Big(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + \sum_{j=1}^{p_i} c_j^{(i)} - \sum_{j=1}^{q_i} d_j^{(i)}\Big) < 0,$$

and for $\mu_i = 0$ the region of convergence shrinks to the open interval $(0, D_i^{-1})$ on the real axis in the ξ_i-plane, with the same condition on parameters as above.

2. Removal of limitations in Ławrynowicz's formulas. The successive differentiation formulas cited in Mathai and Saxena [9] (see also Srivastava *et al.* [17]) are:

$$\text{(2.1)}\quad \frac{d^r}{dx^r}\Big\{x^{-\sigma b_1/\beta_1} H_{p,q}^{m,n}\Big[x^{\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\Big]\Big\}$$
$$= \Big(\frac{-\sigma}{\beta_1}\Big)^r x^{-r-\sigma b_1/\beta_1} H_{p,q}^{m,n}\Big[x^{\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\(r+b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\end{matrix}\Big],$$

$$\text{(2.2)}\quad \frac{d^r}{dx^r}\Big\{x^{-\sigma b_q/\beta_q} H_{p,q}^{m,n}\Big[x^{\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\Big]\Big\}$$
$$= \Big(\frac{-\sigma}{\beta_q}\Big)^r x^{-r-\sigma b_q/\beta_q} H_{p,q}^{m,n}\Big[x^{\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\(b_1,\beta_1),\ldots,(b_{q-1},\beta_{q-1}),(r+b_q,\beta_q)\end{matrix}\Big],$$

$$\text{(2.3)}\quad \frac{d^r}{dx^r}\Big\{x^{-\sigma(1-a_1)/\alpha_1} H_{p,q}^{m,n}\Big[x^{-\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\Big]\Big\}$$
$$= \Big(\frac{-\sigma}{\alpha_1}\Big)^r x^{-r-\sigma(1-a_1)/\alpha_1}$$
$$\times H_{p,q}^{m,n}\Big[x^{-\sigma}\,\Big|\,\begin{matrix}(-r+a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\((b_q,\beta_q))\end{matrix}\Big],$$

$$\text{(2.4)}\quad \frac{d^r}{dx^r}\Big\{x^{-\sigma(1-a_p)/\alpha_p} H_{p,q}^{m,n}\Big[x^{-\sigma}\,\Big|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\Big]\Big\}$$

$$= \left(\frac{-\sigma}{\alpha_p}\right)^r x^{-r-\sigma(1-a_p)/\alpha_p}$$
$$\times H^{m,n}_{p,q}\left[x^{-\sigma}\,\middle|\,\begin{matrix}(a_1/\alpha_1),...,(a_{p-1},\alpha_{p-1}),(-r+a_p,\alpha_p)\\((b_q,\beta_q))\end{matrix}\right].$$

These formulas are stated to be true when $\sigma=\beta_1$, $\sigma=\beta_q$, $\sigma=\alpha_1$ and $\sigma=\alpha_p$ respectively. These restrictions are not essential and can be removed, as will be clear from what follows.

Consider for example the L.H.S. of (2.1):

$$\left(\frac{1}{2\pi\omega}\right)\int_L \phi(\xi)\frac{d^r}{dx^r}(x^{-\sigma b_1/\beta_1+\sigma\xi})\,d\xi$$
$$= \left(\frac{1}{2\pi\omega}\right)\int_L \phi(\xi)\{(-\sigma b_1/\beta_1+\sigma\xi)...(-\sigma b_1/\beta_1+\sigma\xi-r+1)x^{-\sigma b_1/\beta_1+\sigma\xi-r}\}\,d\xi.$$

Ławrynowicz writes the product

$$(-\sigma b_1/\beta_1+\sigma\xi)(-\sigma b_1/\beta_1+\sigma\xi-1)\dots(-\sigma b_1/\beta_1+\sigma\xi-r+1)$$

in the form

$$\left(-\frac{\sigma}{\beta_1}\right)^r(b_1-\beta_1\xi)(b_1-\beta_1\xi+\beta_1/\sigma)\dots(b_1-\beta_1\xi+(r-1)\beta_1/\sigma)$$

and then interprets it as a product resulting by imposing the restriction $\sigma=\beta_1$. But this is not necessary, since obviously it can be put in the form

$$(-1)^r(\sigma b_1/\beta_1-\sigma\xi)(\sigma b_1/\beta_1-\sigma\xi+1)\dots(\sigma b_1/\beta_1-\sigma\xi+r-1)$$
$$= (-1)^r(b_1/\beta_1-\sigma\xi)_r = (-1)^r\frac{\Gamma(\sigma b_1/\beta_1-\sigma\xi+r)}{\Gamma(\sigma b_1/\beta_1-\sigma\xi)}.$$

Similarly in the case of (2.2), (2.3) and (2.4), the restriction can be removed.

Thus after removal of limitations Ławrynowicz's formulas take the following forms:

$$(2.5)\quad \frac{d^r}{dx^r}\left\{x^{-\sigma b_1/\beta_1}H^{m,n}_{p,q}\left[x^{\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right]\right\}$$
$$= (-1)^r x^{-r-\sigma b_1/\beta_1}H^{m+1,n}_{p+1,q+1}\left[x^{\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p)),(\sigma b_1/\beta_1,\sigma)\\(r+\sigma b_1/\beta_1,\sigma),((b_q,\beta_q))\end{matrix}\right],$$

$$(2.6)\quad \frac{d^r}{dx^r}\left\{x^{-\sigma b_q/\beta_q}H^{m,n}_{p,q}\left[x^{\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right]\right\}$$
$$= (-1)^r x^{-r-\sigma b_q/\beta_q}H^{m,n+1}_{p+1,q+1}\left[x^{\sigma}\,\middle|\,\begin{matrix}(\sigma b_q/\beta_q,\sigma),((a_p,\alpha_p))\\((b_q,\beta_q)),(r+\sigma b_q/\beta_q,\sigma)\end{matrix}\right],$$

$$(2.7)\quad \frac{d^r}{dx^r}\left\{x^{-\sigma(1-a_1)/\alpha_1}H^{m,n}_{p,q}\left[x^{-\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right]\right\}$$

$$= (-1)^r x^{-r-\sigma(1-a_1)/\alpha_1}$$
$$\times H^{m,n+1}_{p+1,q+1}\left[x^{-\sigma}\,\middle|\,\begin{matrix}(1-r-\sigma(1-a_1)/\alpha_1,\sigma),((a_p,\alpha_p))\\((b_q,\beta_q)),(1-\sigma(1-a_1)/\alpha_1,\sigma)\end{matrix}\right],$$

and

$$(2.8)\qquad \frac{d^r}{dx^r}\left\{x^{-\sigma(1-a_p)/\alpha_p}H^{m,n}_{p,q}\left[x^{-\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right]\right\}$$
$$= (-1)^r x^{-r-\sigma(1-a_p)/\alpha_p}H^{m+1,n}_{p+1,q+1}\left[x^{\sigma}\,\middle|\,\begin{matrix}((a_p,\alpha_p)),(1-r-\sigma(1-a_p)/\alpha_p,\sigma)\\(1-\sigma(1-a_p)/\alpha_p,\sigma),((b_q,\beta_q))\end{matrix}\right].$$

It follows from the foregoing discussion that the result of Srivastava *et al.* [17], which is stated without any restriction, is wrong and that formula (2.5) above gives its correct version.

3. The successive derivative formulas. The operator $T_{k,q}$ was defined by Joshi and Prajapat [7] as

$$T_{k,q} = x^q\left(k + x\frac{d}{dx}\right),$$

k and q being integer constants. We now define two new operators as

$$(3.1)\qquad T(s,k:cx+d) = (cx+d)^s\left\{k+(cx+d)\frac{d}{dx}\right\},$$

$$(3.2)\qquad T_1(s,k:cx+d) = (cx+d)^s\left\{k+(cx+d)\frac{\partial}{\partial x}\right\},$$

where c, d, s are real numbers and k may be a real or complex number. From the definition it is clear that

$$(3.3)\qquad T(s,k:cx+d)(cx+d)^\alpha = (k+c\alpha)(cx+d)^{\alpha+s},$$

$$(3.4)\quad T^N(s,k:cx+d)(cx+d)^\alpha = (cs)^N\frac{\Gamma\left(\dfrac{k+c\alpha}{cs}+N\right)}{\Gamma\left(\dfrac{k+cx}{cs}\right)}(cx+d)^{\alpha+Ns}$$

and

$$(3.5)\qquad T^N(0,k:cx+d)(cx+d)^\alpha = (k+c\alpha)^N(cx+d)^\alpha.$$

Now assuming u to be a constant, we have

$$T^N(s,k:cx+d)\{(cx+d)^\alpha H[u(cx+d)^\sigma]\}$$
$$= \frac{1}{2\pi\omega}\int_L u^\xi\phi(\xi)\{T^N(s,k:cx+d)(cx+d)^{\alpha+\sigma\xi}\}\,d\xi$$

$$= \begin{cases} \dfrac{1}{2\pi\omega} \displaystyle\int_L u^{\xi} \phi(\xi)(k+c\alpha+c\sigma\xi)^N (cx+d)^{\alpha+\sigma\xi}\, d\xi & \text{for } s=0, \\ \dfrac{1}{2\pi\omega} \displaystyle\int_L u^{\xi} \phi(\xi)(cs)^N \\ \quad \times \dfrac{\Gamma\left(\dfrac{k+c\alpha+c\sigma\xi}{cs}+N\right)}{\Gamma\left(\dfrac{k+c\alpha+c\sigma\xi}{cs}\right)} (cx+d)^{\alpha+\sigma\xi+Ns}\, d\xi & \text{for } s\neq 0. \end{cases}$$

Thus interpreting the R.H.S., we have

$$\begin{aligned} (3.6)\quad & T^N(s,k:cx+d)\{(cx+d)^{\alpha} H[u(cx+d)^{\sigma}]\} \\ & = (cs)^N (cx+d)^{\alpha+Ns} \\ & \quad \times H^{m,n+1}_{p+1,q+1}\left[u(cx+d)^{\sigma} \left| \begin{matrix} \left(1-\dfrac{k+c\alpha}{cs}-N, \dfrac{\sigma}{s}\right); \ldots \\ \ldots; \left(1-\dfrac{k+c\alpha}{cs}, \dfrac{\sigma}{s}\right) \end{matrix} \right. \right], \qquad s\neq 0, \end{aligned}$$

$$\begin{aligned} (3.7)\quad & T^N(s,k:cx+d)\{(cx+d)^{\alpha} H[u(cx+d)^{\sigma}]\} \\ & = (cx+d)^{\alpha} \\ & \quad \times H^{m,n+N}_{p+N,q+N}\left[u(cx+d)^{\sigma} \left| \begin{matrix} (-k-c\alpha, c\sigma), \ldots, (-k-c\alpha, c\sigma), \ldots \\ \ldots, (1-k-c\alpha, c\sigma), \ldots, (1-k-c\alpha, c\sigma) \end{matrix} \right. \right], \\ & \qquad s=0. \end{aligned}$$

Proceeding in a similar manner, the two and r-variable analogues of this result are

$$\begin{aligned} (3.8)\quad & T_1^N(s_1,k_1:c_1x_1+d_1) T_1^N(s_2,k_2:c_2x_2+d_2)\{(c_1x_1+d_1)^{\alpha_1}(c_2x_2+d_2)^{\alpha_2} \\ & \times H[u_1(c_1x_1+d_1)^{\sigma_1}, u_2(c_2x_2+d_2)^{\sigma_2}]\} \\ & = (c_1s_1)^N (c_2s_2)^N (c_1x_1+d_1)^{\alpha_1+Ns_1} (c_2x_2+d_2)^{\alpha_2+Ns_2} \\ & \quad \times H^{0,n:m_1,n_1+1;\ldots;m_r,n_r+1}_{p,q:p_1+1,q_1+1;\ldots;p_r+1,q_r+1} \left[\begin{matrix} u_1(c_1x_1+d_1)^{\sigma_1} \\ u_2(c_2x_2+d_2)^{\sigma_2} \end{matrix} \right. \\ & \qquad \left| \begin{matrix} \ldots : \left(1-\dfrac{k_1+c_1\alpha_1}{c_1s_1}-N, \dfrac{\sigma_1}{s_1}\right); \left(1-\dfrac{k_2+c_2\alpha_2}{c_2s_2}-N, \dfrac{\sigma_2}{s_2}\right); \ldots \\ \ldots : \ldots; \left(1-\dfrac{k_1+c_1\alpha_1}{c_1s_1}, \dfrac{\sigma_1}{s_1}\right); \ldots; \left(1-\dfrac{k_2+c_2\alpha_2}{c_2s_2}, \dfrac{\sigma_2}{s_2}\right) \end{matrix} \right] \end{aligned}$$

and

$$(3.9)\quad T_1^N(s_1,k_1:c_1x_1+d_1)...T_1^N(s_r,k_r:c_rx_r+d_r)\prod_{j=1}^{r}\{(c_jx_j+d_j)^{\alpha_j}\times H[u_1(c_1x_1+d_1)^{\sigma_1},...,u_r(c_rx_r+d_r)^{\sigma_r}]\}$$

$$=\prod_{i=1}^{r}(c_is_i)^N(c_ix_i+d_i)^{\alpha_i+Ns_i}$$

$$\times H^{0,n:m_1,n_1+1;...;m_r,n_r+1}_{p,q:p_1+1,q_1+1;...;p_r+1,q_r+1}\left[\begin{matrix}u_1(c_1x_1+d_1)^{\sigma_1}\\ \vdots\\ u_r(c_rx_r+d_r)^{\sigma_r}\end{matrix}\right.$$

$$\left|\begin{matrix}...:\left(1-\dfrac{k_1+c_1\alpha_1}{c_1s_1}-N,\dfrac{\sigma_1}{s_1}\right),...;...;\left(1-\dfrac{k_r+c_r\alpha_r}{c_rs_r}-N,\dfrac{\sigma_r}{s_r}\right);...\\ ...:...;\left(1-\dfrac{k_1+c_1\alpha_1}{c_1s_1},\dfrac{\sigma_1}{s_1}\right);...;\left(1-\dfrac{k_r+c_r\alpha_r}{c_rs_r},\dfrac{\sigma_r}{s_r}\right);...\end{matrix}\right].$$

For $s_i=0$, we can give corresponding formulas similar to (3.6) above.

Again by using (3.3) and interpreting the result, we get the following formulas:

$$(3.10)\quad T^N(s,k:at+b)\{(at+b)^{\alpha}H[z(at+b)^{\sigma}]\}$$

$$=(as)^N(at+b)^{\alpha+Ns}H^{m,n+1}_{p+1,q+1}\left[z(at+b)^{\sigma}\,\left|\,\begin{matrix}\left(1-\dfrac{k+a\alpha}{as}-N,\dfrac{\sigma}{s}\right),...\\ ...,\left(1-\dfrac{k+a\alpha}{as},\dfrac{\sigma}{s}\right)\end{matrix}\right.\right],$$

$$(3.11)\quad T^N(s,k:at+b)\{(at+b)^{\alpha}H[z_1(at+b)^{\sigma_1};z_2(at+b)^{\sigma_2}]\}$$

$$=(as)^N(at+b)^{\alpha+Ns}$$

$$\times H^{0,n+1:...}_{p+1,q+1:...}\left[\begin{matrix}z_1(at+b)^{\sigma_1}\\ z_2(at+b)^{\sigma_2}\end{matrix}\,\left|\,\begin{matrix}\left(1-\dfrac{k+a\alpha}{as}-N,\dfrac{\sigma_1}{s},\dfrac{\sigma_2}{s}\right),...\\ ...,\left(1-\dfrac{k+a\alpha}{as},\dfrac{\sigma_1}{s},\dfrac{\sigma_2}{s}\right)\end{matrix}\right.\right]$$

and

$$(3.12)\quad T^N(s,k:at+b)\{(at+b)^{\alpha}H[z_1(at+b)^{\sigma_1};...;z_r(at+b)^{\sigma_r}]\}$$

$$=(as)^N(at+b)^{\alpha+Ns}$$

$$\times H^{0,n+1:...}_{p+1,q+1:...}\left[\begin{matrix}z_1(at+b)^{\sigma_1}\\ \vdots\\ z_r(at+b)^{\sigma_r}\end{matrix}\,\left|\,\begin{matrix}\left(1-\dfrac{k+a\alpha}{as}-N;\dfrac{\sigma_1}{s_1},...,\dfrac{\sigma_r}{s_r}\right),...\\ ...,\left(1-\dfrac{k+a\alpha}{as};\dfrac{\sigma_1}{s_1},...,\dfrac{\sigma_r}{s_r}\right):...\end{matrix}\right.\right].$$

4. Particular cases. In (3.5), let us set $c = 1$, $d = 0$, $k = 0$. Then on replacing one by one α by $-\sigma(b_1/\beta_1)$; α by $-\sigma(b_q/\beta_q)$; α by $-\sigma(1-a_1/\alpha_1)$, σ by $-\sigma$, α by $-\sigma(1-a_p/\alpha_p)$, σ by $-\sigma$, we shall respectively get formulas (2.5) to (2.8). These correspond to Ławrynowicz's formulas for $\sigma = \beta_1$, $\sigma = \beta_q$, $\sigma = \alpha_1$ and $\sigma = \alpha_p$. Again for the same set of values of c, d, k, s, this formula will yield the result of Mathai and Saxena [9] for $\alpha = 0$, that of Skibiński [13] for $\alpha = 0$, $\sigma = -\delta$, and that of Anandani [1] for $\alpha = \lambda$.

The particular cases of interest of (3.8) when $s = 0$ and $s = -1$ are the two results due to Gupta and Jain [4], when $c = 1$, $d = 0$, $k = 0$, and a result of Oliver and Kalla [10] when $s = -1$, $k = 0$, $\alpha = 0$.

Formula (3.9) on the other hand corresponds to two results of Goyal [3], for $s = 0$ and $s = -1$, and that of Rakesh [12] for $s = -2$, when the other parameters involved are given the values $a = 1$, $b = 0$, $k = 0$. In the case $k = 0$, $s = -1$, we get the result of Raina [11].

5. Applications. A particular case of (3.6) for $c = 1$, $d = 0$ gives

$$(5.1)\qquad T^N(s,k:x)\{x^\alpha H[ux^\sigma]\} = s^N x^{\alpha+Ns} H^{m,n+1}_{p+1,q+1}\left[ux^\sigma \,\middle|\, \begin{matrix} (1-(k+\alpha)/s-N,\sigma/s),\ldots \\ \ldots,(1-(k+\alpha)/s,\sigma/s) \end{matrix}\right].$$

By using (3.4), we have

$$(5.2)\qquad T^N(s,k:x)\{x^{\beta_1+\beta_2} H[ux^\sigma]\} = s^N x^{\beta_1+\beta_2+Ns} \sum_{M=0}^{N} \binom{N}{M} \frac{\Gamma\left(\frac{k+\beta_1}{s}+N-M\right)}{\Gamma\left(\frac{k+\beta_1}{s}\right)} \times H^{m,n+1}_{p+1,q+1}\left[ux^\sigma \,\middle|\, \begin{matrix} \left(1-\frac{\beta_2}{s}-M,\frac{\sigma}{s}\right),\ldots \\ \ldots,\left(1-\frac{\beta_2}{s},\frac{\sigma}{s}\right) \end{matrix}\right].$$

Thus, if $\alpha = \beta_1 + \beta_2$, from (5.1) and (5.2), we get

$$(5.3)\qquad H^{m,n+1}_{p+1,q+1}\left[ux^\sigma \,\middle|\, \begin{matrix} \left(1-\frac{k+\alpha}{s}-N,\frac{\sigma}{s}\right),\ldots \\ \ldots,\left(1-\frac{k+\alpha}{s},\frac{\sigma}{s}\right) \end{matrix}\right] = \sum_{M=0}^{N} \binom{N}{M} \frac{\Gamma\left(\frac{k+\beta_1}{s}+N-M\right)}{\Gamma\left(\frac{k+\beta_1}{s}\right)} H^{m,n+1}_{p+1,q+1}\left[ux^\sigma \,\middle|\, \begin{matrix} \left(1-\frac{\beta_2}{s}-M,\frac{\sigma}{s}\right),\ldots \\ \ldots,\left(1-\frac{\beta_2}{s},\frac{\sigma}{s}\right) \end{matrix}\right].$$

Various such summation results are available in the literature. It is stated in [16] that all summation formulas that give finite and infinite series expansions of the H-function in terms of the H-function itself can be expressed as equation (2.11.1) of [15] (see also R. N. Jain [5]), derived by using Gauss's, Kummer's, Saalshutz's, Dixon's, Watson's theorem and a known theorem (Slater [14], p. 243, III eq. 10). It appears that our formula (5.3) and its multivariable analogue do not conform to this general form, and therefore may be considered as a new type of finite series representation for the H-function.

Similarly, comparing (3.8)–(3.11) for $c = 0$, $d = 0$ with their alternative forms obtained using Leibniz's theorem, we get the finite expansion formulas:

$$(5.4)\quad H^{0,n:m_1,n_1+1;\ldots;m_r,n_r+1}_{p,q:p_1+1,q_1+1;\ldots;p_r+1,q_r+1} \cdot \left[\begin{matrix} u_1x_1^{\sigma_1} \\ \vdots \\ u_rx_r^{\sigma_r}\end{matrix}\left|\begin{matrix} \ldots:\left(1-\frac{k_1+\alpha_1}{s_1}-N,\frac{\sigma_1}{s_1}\right),\ldots;\left(1-\frac{k_r+\alpha_r}{s_r},\frac{\sigma_r}{s_r}\right),\ldots \\ \ldots:\ldots,\left(1-\frac{k_1+\alpha_1}{s_1},\frac{\sigma_1}{s_1}\right);\ldots;\ldots,\left(1-\frac{k_r+\alpha_r}{s_r},\frac{\sigma_r}{s_r}\right)\end{matrix}\right.\right]$$

$$= \sum_{M_1,\ldots,M_r=0}^{N}\prod_{i=1}^{r}\left\{\binom{N}{M_i}\frac{\Gamma((k_i+\beta_i^{(i)})/s_i+N-M_i)}{\Gamma((k_i+\beta_i^{(i)})/s_i)}\right\}$$

$$\times H^{0,n:m_1,n_1+1;\ldots;m_r,n_r+1}_{p,q:p_1+1,q_1+1;\ldots;p_r+1,q_r+1} \cdot \left[\begin{matrix} u_1x_1^{\sigma_1} \\ \vdots \\ u_rx_r^{\sigma_r}\end{matrix}\left|\begin{matrix} \ldots:\left(1-\frac{\beta_2^{(1)}}{s_2}-M_1,\frac{\sigma_1}{s_1}\right),\ldots,\left(1-\frac{\beta_2^{(r)}}{s_r}-M_r,\frac{\sigma_r}{s_r}\right),\ldots \\ \ldots:\ldots,\left(1-\frac{\beta_2^{(1)}}{s_1},\frac{\sigma_1}{s_1}\right);\ldots;\ldots,\left(1-\frac{\beta_2^{(r)}}{s_r},\frac{\sigma_r}{s_r}\right)\end{matrix}\right.\right]$$

provided $\alpha_i = \beta_1^{(i)} + \beta_2^{(i)},\ i = 1, \ldots, r;$

$$(5.5)\quad H^{0,n+1:\ldots}_{p+1,q+1:\ldots}\left[\begin{matrix} z_1t_1^{\sigma_1} \\ z_2t_2^{\sigma_2}\end{matrix}\left|\begin{matrix} \left(1-\frac{k+\alpha}{s}-N;\frac{\sigma_1}{s_1},\frac{\sigma_2}{s_2}\right),\ldots:\ldots;\ldots \\ \ldots,\left(1-\frac{k+\alpha}{s};\frac{\sigma_1}{s_1},\frac{\sigma_2}{s_2}\right):\ldots;\ldots\end{matrix}\right.\right]$$

$$= \sum_{M=0}^{N}\binom{N}{M} H^{0,n:m_1,n_1+1;m_2,n_2+1}_{p,q:p_1+1,q_1+1;p_2+1,q_2+1} \cdot \left[\begin{matrix} z_1t_1^{\sigma_1} \\ z_2t_2^{\sigma_2}\end{matrix}\left|\begin{matrix} \ldots:\left(1-\frac{k+\beta_1}{s}+m-N,\frac{\sigma_1}{s_1}\right),\ldots,\left(1-\frac{\beta_2}{s}-M,\frac{\sigma_2}{s}\right),\ldots \\ \ldots:\ldots,\left(1-\frac{k+\alpha}{s}-N,\frac{\sigma_1}{s_1}\right),\ldots;\ldots,\left(1-\frac{\beta_2}{s_2},\frac{\sigma_2}{s_2}\right)\end{matrix}\right.\right]$$

and

$$(5.6)\qquad H^{0,n+1:...}_{p+1,q+1:...}\left[\begin{matrix} z_1t_1^{\sigma_1}\\ \vdots \\ z_rt_r^{\sigma_r}\end{matrix}\left|\begin{matrix}\left(1-\dfrac{k+\alpha}{s}-N;\dfrac{\sigma_1}{s_1},...,\dfrac{\sigma_r}{s_r}\right),...:...;...\\ ...,\left(1-\dfrac{k+\alpha}{s};\dfrac{\sigma_1}{s_1},...,\dfrac{\sigma_r}{s_r}\right):...;...\end{matrix}\right.\right]$$

$$=\sum_{M=0}^{N}\binom{N}{M}H^{0,n+2:...}_{p+2,q+2:...}$$

$$\cdot\left[\begin{matrix} z_1t_1^{\sigma_1}\\ \vdots \\ z_rt_r^{\sigma_r}\end{matrix}\left|\begin{matrix}...:\left(1-\frac{k+\beta_1}{s}+M-N;\frac{\sigma_1}{s},...,\frac{\sigma_{r-1}}{s},0\right),...;\left(1-\frac{\beta_2}{s}-M,\frac{\sigma_r}{s}\right),...\\ ...:...,\left(1-\frac{k+\beta_1}{s};\frac{\sigma_1}{s},...,\frac{\sigma_{r-1}}{s},0\right);...;...;\left(1-\frac{\beta_2}{s},\frac{\sigma_r}{s}\right)\end{matrix}\right.\right].$$

Iteration then leads to

$$(5.7)\qquad H^{0,n+A:...}_{p+A,q+A:...}\left[\begin{matrix} z_1\prod\limits_{i=1}^{A}t_i^{\sigma_{i1}}\\ \vdots \\ z_r\prod\limits_{i=1}^{A}t_i^{\sigma_{ir}}\end{matrix}\left|\begin{matrix}\left(\left(1-\frac{k_A+\alpha_A}{s_A}-N;\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{rA}}{s_A}\right)\right),...:...\\ ...,\left(\left(1-\frac{k_A+\alpha_A}{s_A};\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{rA}}{s_A}\right)\right):...\end{matrix}\right.\right]$$

$$=\sum_{M_1,...,M_A=0}^{N}\prod_{i=1}^{A}\binom{N}{M_i}H^{0,n+A:m_1,n_1;...;m_r,n_r+A}_{p+A,q+A:p_1,q_1;...;p_r+A,q_r+A}\left[\begin{matrix} z_1\prod\limits_{i=1}^{A}t_i^{\sigma_{i1}}\\ \vdots \\ z_r\prod\limits_{i=1}^{A}t_i^{\sigma_{ir}}\end{matrix}\right.$$

$$\left|\begin{matrix}...:\left(\left(1-\frac{k_A+\beta_1^{(A)}}{s_A}+M_A-N;\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{(r-1)A}}{s_A},0\right)\right);...;\left(\left(1-\frac{\beta_2^{(A)}}{s_A}-M_A;\frac{\sigma_{rA}}{s_A}\right)\right),....\\ ....:...,\left(\left(1-\frac{k_A+\beta_1^{(A)}}{s_A};\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{(r-1)A}}{s_A},0\right)\right);...;...;\left(\left(1-\frac{\beta_2^{(A)}}{s_A},\frac{\sigma_{rA}}{s_A}\right)\right)\end{matrix}\right.\right]$$

and

$$(5.8)\qquad H^{0,n+A:...}_{p+A,q+A:...}\left[\begin{matrix} z_1\prod\limits_{i=1}^{A}t_i^{\sigma_{i1}}\\ \vdots \\ z_r\prod\limits_{i=1}^{A}t_i^{\sigma_{ir}}\end{matrix}\left|\begin{matrix}\left(\left(1-\frac{k_A+\alpha_A}{s_A}-N;\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{rA}}{s_A}\right)\right),...:...\\ ...,\left(\left(1-\frac{k_A+\alpha_A}{s_A};\frac{\sigma_{1A}}{s_A},...,\frac{\sigma_{rA}}{s_A}\right)\right):...\end{matrix}\right.\right]$$

$$= \sum_{M_1,\dots,M_A=0}^{N} \prod_{i=1}^{A} \binom{N}{M_i} H^{0,n+2A:\dots}_{p+2A,q+2A:\dots} \left[\begin{matrix} z_1 \prod_{i=1}^{A} t_i^{\sigma_{i1}} \\ \vdots \\ z_r \prod_{i=1}^{A} t_i^{\sigma_{ir}} \end{matrix} \right.$$

$$\left| \begin{matrix} \dots : \left(\left(1-\frac{k_A+\beta_1^{(A)}}{s_A}+M_A-N; \frac{\sigma_{1A}}{s_A},\dots,\frac{\sigma_{r_1A}}{s_A},0,\dots,0\right)\right), \\ \left(\left(1-\frac{\beta_2^{(A)}}{s_A}-M_A; 0,\dots,0,\frac{\sigma_{(r_1+1)A}}{s_A},\dots,\frac{\sigma_{rA}}{s_A}\right)\right),\dots:\dots \\ \dots,\left(\left(1-\frac{k_A+\beta_1^{(A)}}{s_A}; \frac{\sigma_{1A}}{s_A},\dots,\frac{\sigma_{r_1A}}{s_A},0,\dots,0\right)\right), \left(\left(1-\frac{\beta_2^{(A)}}{s_A},0,\dots,0,\frac{\sigma_{(r_1+1)A}}{s_A},\dots,\frac{\sigma_{rA}}{s_A}\right)\right) \end{matrix} \right]$$

provided $\alpha_i = \beta_1^{(i)} + \beta_2^{(i)}$, $\sigma^i/s^i > 0$ $(i = 1,\dots,A)$.

Alternatively these expansion formulas can also be obtained directly by using the identity $T^N(s,k:x)(x^{\alpha+\beta}) = T^N(s,k:x)(x^\alpha \cdot x^\beta)$, i.e.

$$s^N x^{\alpha+\beta+Ns} \frac{\Gamma\left(\frac{k+\alpha+\beta}{s}+N\right)}{\Gamma\left(\frac{k+\alpha+\beta}{s}\right)}$$

$$= s^N x^{\alpha+\beta+Ns} \sum_{M=0}^{N} \binom{N}{M} \frac{\Gamma\left(\frac{k+\alpha}{s}+N-M\right)}{\Gamma\left(\frac{k+\alpha}{s}\right)} \cdot \frac{\Gamma\left(\frac{\beta}{s}+M\right)}{\Gamma\left(\frac{\beta}{s}\right)}$$

(see also Joshi and Prajapat [7]).

6. Expansion of hypergeometric functions. The results established in the preceding section enable us to derive expansion formulas for Appell functions in terms of products of Gauss hypergeometric functions of the type considered earlier by Burchnall and Chaundy [2] and their multiple variable analogues. Indeed, if we assume $n = p = q = 0$ and specialize the parameters, for example in (5.6), as

(i) $k = 0$, $s = 1$, $\sigma_1 = 1$, $\sigma_2 = 1$,

(ii) $m_1 = 1$, $n_1 = r_1$, $p_1 = r_1$, $q_1 = s_1$, $(c_j^{(1)}, \delta_j^{(1)}) = (1 - b_j, 1)$, $j = 1,\dots,r_1$, $(d_1^{(1)}, \delta_1^{(1)}) = (0,1)$ and $(d_j^{(1)}, \delta_j^{(1)}) = (1 - c_j, 1)$, $j = 2,\dots,s_1$,

(iii) $m_2 = 1$, $n_2 = r_2$, $p_2 = r_2$, $q_2 = s_2 + 1$, $(c_j^{(2)}, \delta_j^{(2)}) = (1 - b'_j, 1)$, $j = 1,\dots,r_2$, $(d_1^{(2)}, \delta_1^{(2)}) = (0,1)$ and $(d_j^{(2)}, \delta_j^{(2)}) = (1 - c'_j, 1), j = 2,\dots,s_2$, and replace z_1 by $-z_1$, z_2 by $-z_2$ and t by 1, we get

$$\frac{\Gamma(\alpha+N)}{\Gamma(\alpha)} F^{1:r_1;r_2}_{1:s_1;s_2}\left(\begin{matrix} a:(b_{r_1});(b_{r_2}); \\ c:(c_{s_1});(c_{s_2}); \end{matrix}\, z_1, z_2\right)$$
$$= \sum_{M=0}^{N} \binom{N}{M} \frac{\Gamma(\beta_1+N-M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2+M)}{\Gamma(\beta_2)}$$
$$\times {}_{r_1+1}F_{s_1+1}\left(\begin{matrix} \beta_1+N-M, (b_{r_1}); \\ \beta_1, (c_{s_1}); \end{matrix}\, z_1\right)$$
$$\times {}_{r_2+1}F_{s_2+1}\left(\begin{matrix} \beta_2+M, (b_{r_2}); \\ \beta_2, (c_{s_2}); \end{matrix}\, z_2\right).$$

If we put $\alpha+N=a$ and $\alpha=a-c$, we have

$$\frac{\Gamma(a)}{\Gamma(c)} F^{1:r_1;r_2}_{1:s_1;s_2}\left(\begin{matrix} a+N:(b_{r_1});(b_{r_2}); \\ c:(c_{s_1});(c_{s_2}); \end{matrix}\, z_1, z_2\right) \tag{6.1}$$
$$= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1+a-c-M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2+M)}{\Gamma(\beta_2)}$$
$$\times {}_{r_1+1}F_{s_1+1}\left(\begin{matrix} \beta_1+a-c-M, (b_{r_1}); \\ \beta_1, (c_{s_1}); \end{matrix}\, z_1\right)$$
$$\times {}_{r_2+1}F_{s_2+1}\left(\begin{matrix} \beta_2+M, (b_{r_2}); \\ \beta_2, (c_{s_2}); \end{matrix}\, z_2\right),$$

provided

(i) $a-c>0$ is a positive integer, $c=\beta_1+\beta_2$,

(ii) $|z_1|<1$, $|z_2|<1$,

(iii) $\theta_1 \equiv \frac{1}{2}(r_1-s_1) - \mathrm{Re}\left(\sum_{j=1}^{r_1} b_j - \sum_{j=1}^{s_1} c_j\right) < 0,$

$$\theta_2 \equiv \frac{1}{2}(r_2-s_2) - \mathrm{Re}\left(\sum_{j=1}^{r_2} b_j - \sum_{j=1}^{s_2} c_j\right) < 0,$$

which is of a different form than that of eq. (242) of Srivastava and Karlsson [15].

For particular values $r_1=1$, $s_1=0$, $r_2=1$, $s_2=0$, $b_1=b$, $b_1'=b'$ this takes the form

$$\frac{\Gamma(a)}{\Gamma(c)} F_1(a,b,b';c;z_1,z_2) \tag{6.2}$$

$$= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1+a-c-M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2+M)}{\Gamma(\beta_2)}$$

$$\times {}_2F_1\left(\begin{matrix}\beta_1+a-c-M, b; \\ \beta_1;\end{matrix} z_1\right) {}_2F_1\left(\begin{matrix}\beta_2+M, b'; \\ \beta_2;\end{matrix} z_2\right)$$

and the condition of convergence reduces to

(i) $a-c$ is a positive integer, $c = \beta_1+\beta_2$,
(ii) $|z_1|<1$, $|z_2|<1$ and
(iii) $1/2 - \mathrm{Re}(b-c) < 0$ and $1/2 - \mathrm{Re}(b'-c') < 0$.

Proceeding in a similar manner from (5.7) to (5.11), we have respectively:

$$\text{(6.3)} \quad \frac{\Gamma(a)}{\Gamma(c)} F_D^{(r)}(a, b_1, \ldots, b_r; c; z_1, \ldots, z_r)$$

$$= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1+a-c)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2+m)}{\Gamma(\beta_2)}$$

$$\times F_D^{(r-1)}(\beta_1+a-c-m, b_1, \ldots, b_r; c, z_1, \ldots, z_{r-1})$$

$$\times {}_2F_1(\beta_2+m, b_r; \beta_2; z_r),$$

provided $a-c$ is a positive integer and $c = \beta_1+\beta_2$;

$$\text{(6.4)} \quad \frac{\Gamma(a)}{\Gamma(c)} F_D^{(r)}(a, b_1, \ldots, b_r; c; z_1, \ldots, z_r)$$

$$= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1+a-c)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2+m)}{\Gamma(\beta_2)}$$

$$\times F_D^{(r_1)}(\beta_1+a-c-m, b_1, \ldots, b_{r_1}; \beta_1; z_1, \ldots, z_{r_1})$$

$$\times F_D^{(r-r_1)}(\beta_2+m, b_{r_1+1}, \ldots, b_r; \beta_2; z_{r_1+1}, \ldots, z_r);$$

$$\text{(6.5)} \quad \prod_{i=1}^{n} \frac{\Gamma(a_i)}{\Gamma(c_i)} F_{n:q_1;q_2}^{n:p_1;p_2}\left(\begin{matrix}(a_n) : (b_p^{(1)}); (b_p^{(2)}); \\ (c_n) : (c_q^{(1)}); (c_q^{(2)});\end{matrix} z_1, z_2\right)$$

$$= \sum_{m_1=0}^{a_1-c_1} \cdots \sum_{m_n=0}^{a_n-c_n} \prod_{i=1}^{n} \binom{a_i-c_i}{m_i} \frac{\Gamma(\beta_1^{(i)}+a_i-c_i-m_i)}{\Gamma(\beta_1^{(i)})} \cdot \frac{\Gamma(\beta_2^{(i)}+m_i)}{\Gamma(\beta_2^{(i)})}$$

$$\times {}_{p_1+n}F_{q_1+n}\left(\begin{matrix}(\beta_1^{(n)}+a_n-c_n-m_n), (b_{p_1}); \\ (\beta_1^{(n)}), (c_{q_1});\end{matrix} z_1\right)$$

$$\times {}_{p_2+n}F_{q_2+n}\left(\begin{matrix}(\beta_2^{(n)}+m_n), (b_{p_2}); \\ (\beta_2^{(n)}), (c_{q_2});\end{matrix} z_2\right)$$

provided $a_i - c_i$ is a positive integer and $c_i = \beta_1^{(i)} + \beta_2^{(i)},\ i = 1, \ldots, n$;

$$(6.6)\quad \prod_{i=1}^{n} \frac{\Gamma(a_i)}{\Gamma(c_i)} F^{n:p_1;\ldots;p_r}_{n:q_1;\ldots;q_r}\left(\begin{matrix}(a_n) : (b^{(1)}_{p_1}); \ldots; (b^{(r)}_{p_r});\\ (c_n) : (c^{(1)}_{q_1}); \ldots; (c^{(r)}_{q_r});\end{matrix}\ z_1, \ldots, z_r\right)$$

$$= \sum_{m_1=0}^{a_1-c_1} \cdots \sum_{m_n=0}^{a_n-c_n} \prod_{i=1}^{n} \binom{a_i - c_i}{m_i} \frac{\Gamma(\beta_1^{(i)} + a_i - c_i - m_i)}{\Gamma(\beta_1^{(i)})} \cdot \frac{\Gamma(\beta_2^{(i)} + m_i)}{\Gamma(\beta_2^{(i)})}$$

$$\times\ {}_{p_r+n}F_{q_r+n}\left(\begin{matrix}(\beta_2^{(n)} + m_n), (b_{p_r});\\ (\beta_2^{(n)}), (c_{q_r});\end{matrix}\ z_r\right)$$

$$\times F^{n:p_1;\ldots;p_{r-1}}_{n:q_1;\ldots;q_{r-1}}\left(\begin{matrix}(\beta_1^{(n)} + a_n - c_n - m_n), (b_{p_1}); \ldots; (b^{r-1}_{p_{r-1}});\\ (\beta_1^{(n)}), (c_{q_1}); \ldots; (c_{q_{r-1}});\end{matrix}\ z_1, \ldots, z_r\right)$$

provided $c_i = \beta_1^{(i)} + \beta_2^{(i)},\ i = 1, \ldots, n$, and $a_i - c_i$ are positive integers.

The above results can be combined to yield interesting identities but for reasons of brevity we omit the details.

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