ANNALES
POLONICI MATHEMATICI
LXVII.1 (1997)

Riemann problem on the double of a multiply connected circular region

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Abstract. The Riemann problem has been solved in [9] for an arbitrary closed Riemann surface in terms of the principal functionals. This paper is devoted to solution of the problem only for the double of a multiply connected region and can be treated as complementary to [9,1]. We obtain a complete solution of the Riemann problem in that particular case. The solution is given in analytic form by a Poincaré series.

1. Introduction. Consider mutually disjoint discs $D_k := \{z \in \mathbb{C} : |z-a_k|<r_k\}$ $(k = 0, 1, \dots, n)$ on the complex plane $\mathbb{C}$. Let $D := \overline{\mathbb{C}}\setminus\bigcup_{k=0}^{n} \overline{D}_k$ be a multiply connected region, and D^* be a copy of D. Let the boundary of D and the boundary of D^* be identified by the projection along $\partial D_k := \{|t - a_k| = r_k\}$ $(k = 0, 1, \dots, n)$. As a result we have a Riemann surface $\mathcal{R}$ which is called the *double* of the region D. Consider a contour Γ on $\mathcal{R}$ which consists of simple closed smooth curves. The *Riemann problem* consists in finding a function $\Phi^{\pm}(p)$ analytic in $\mathcal{R} \setminus \Gamma$ and Hölder-continuous on Γ with the boundary condition [9]

$$\Phi^{+}(p) = G(p)\Phi^{-}(p) + g(p), \quad p \in \Gamma. \tag{1.1}$$

Here $G(p)$ and $g(p)$ are known functions satisfying the Hölder condition, and $G(p) \neq 0$.

If $\Gamma = \partial D = -\bigcup_{k=0}^{n} \partial D_k$ then $\Phi(t)$ on the second sheet can be represented in the form

$$\Phi(t) = G(t)\overline{\Psi(t)} + g(t), \quad t \in \partial D. \tag{1.2}$$

Noether's theory of the last problem has been constructed by B. Bojarski [1]. If D is a simply connected region for which the conformal mapping onto the unit disc is known, then (1.2) has been solved in closed form [1]. If, moreover, $|G(t)| = 1$ and $\overline{G(t)}g(t)+\overline{g(t)} \equiv 0$ on ∂D, then $\Omega(z) = \frac{1}{2}[\Phi(z)+\Psi(z)]$, $z \in D$,

1991 *Mathematics Subject Classification*: Primary 30E25.
Key words and phrases: boundary value problems on Riemann surfaces, functional equation.

solves the *Hilbert problem*

$$\Omega(t) = G(t)\overline{\Omega(t)} + g(t), \quad t \in \partial D. \tag{1.3}$$

Conversely, if $\Omega(z)$ satisfies (1.3) then $\Phi(z) = \Psi(z) = \Omega(z)$ satisfies (1.2). Assuming that $G(t) = -1$ and $g(t)$ is a real-valued function in (1.3) we arrive at the *Schwarz problem*

$$\Omega(t) + \overline{\Omega(t)} = g(t), \quad t \in \partial D. \tag{1.4}$$

The problems (1.3) and (1.4) for a multiply connected region have been studied in [3, 4, 9]. The final solution in closed form is given in [7].

The problem (1.1) has been solved in [9] for an arbitrary closed Riemann surface in terms of the principal functionals. This paper is devoted to solution of (1.1) on the special Riemann surface $\mathcal{R}$ and can be treated as complementary to [9, 1]. We obtain a complete solution of the Riemann problem (1.1) in that particular case. In this paper the special case when $0 \le \kappa := \operatorname{ind}_\Gamma G(p) \le n$ is investigated. Solution of (1.1) is given in analytic form by a Poincaré series.

2. Reducing the Riemann problem with constant coefficients to a system of functional equations. Consider B. Bojarski's problem on $\mathbb{C}$ (see [1])

$$\phi(t) = \lambda_k\overline{\psi(t)} + g_k(t), \quad |t - a_k| = r_k, \quad k = 0, 1, \dots, n, \tag{2.1}$$

where the unknown functions $\phi(z)$ and $\psi(z)$ are analytic in D and continuously differentiable in $\overline{D}$, λ_k are given constants, $g_k(t)$ are given functions, $g_k \in C^1_\alpha$. Here C^1_α is the space of differentiable functions on $|t-a_k| = r_k$ with Hölder derivatives. The problem (2.1) is a particular case of (1.1), where $\Gamma = \bigcup_{k=0}^n \{|t - a_k| = r_k\}$ and $G(p)$ is constant over each circumference. Rewrite the boundary value problem (2.1) in the form

$$G_k\Phi(t) + \overline{G}_k\overline{\Phi(t)} = G_kH_k(t), \quad |t - a_k| = r_k, \tag{2.2}$$

where

$$\Phi(z) := \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} = (\phi(z), \psi(z))^T, \quad G_k := \begin{pmatrix} 1 & -\overline{\lambda}_k \\ i & i\overline{\lambda}_k \end{pmatrix},$$

$$H_k(t) := \begin{pmatrix} g_k(t) \\ -(1/\lambda_k)\overline{g_k(t)} \end{pmatrix}.$$

The problem (2.2) is a vector-matrix Hilbert problem for the multiply connected region D. Let us also consider the problem

$$G_k\Phi(t) + \overline{G}_k\overline{\Phi(t)} = \gamma_k, \quad |t - a_k| = r_k, \tag{2.3}$$

where $\Phi(z)$ is unknown together with the constant vectors γ_k.

LEMMA 1. *If $\Phi \in C^1(\overline{D})$ then the problem* (2.3) *has only a constant solution.*

P r o o f. Let $\Phi \neq \text{const}$. We rewrite (2.3) in the form

$$\begin{aligned} u_1 &= (\operatorname{Re}\lambda_k)u_2 - (\operatorname{Im}\lambda_k)v_2 + \operatorname{Re} g_k, \\ v_1 &= (\operatorname{Im}\lambda_k)u_2 + (\operatorname{Re}\lambda_k)v_2 + \operatorname{Im} g_k, \end{aligned} \tag{2.4}$$

where $\phi = u_1 + iv_1$ and $\psi = u_2 + iv_2$. Geometrically (2.4) is the equation of a real two-dimensional plane in $\mathbb{C}^2 \cong \mathbb{R}^4$. The vector-function $\Phi(z)$ transfers the circumferences $|t - a_k| = r_k$ to the planes (2.4). Let $\Phi(D)$ be the image of D in $\overline{\mathbb{C}^2}$.

We now show that interior points of D correspond to interior points of $\Phi(D)$. Let $z_0 \in D$, $\Phi(z_0) = w_0$, where $w_0 = (\phi(z_0), \psi(z_0))^T$ is not an interior point of $\Phi(D)$. Let $\partial\Phi(D)$ be the boundary of $\Phi(D)$. We know that $\Phi : \overline{D} \to \Phi(\overline{D})$ is continuously differentiable. If $\infty^2 \notin \Phi(D)$, then $\partial\Phi(D)$ is a three-dimensional surface. Assume that w_0 is not a flex point of $\partial\Phi(D)$. Then, by a rotation and parallel translation $A : \mathbb{R}^4 \to \mathbb{R}^4$ the set $\Phi(\overline{D})$ can be set in such a way that the plane $\widetilde{v}_1 = \widetilde{u}_1 = 0$, containing the point $\widetilde{w}_0 := Aw_0$, locally separates points belonging and not belonging to $\Phi(D)$. Moreover, the coordinates of $A \circ \Phi = (\widetilde{\phi}, \widetilde{\psi})^T$ are analytic in D. Also $\widetilde{\phi}, \widetilde{\psi} \in C^1(\overline{D})$. Every rotation in $\mathbb{R}^4$ consists of two rotations in the planes $u_1 = v_1 = 0$ and $u_2 = v_2 = 0$, and it is associated with multiplication by a complex number, i.e. it is a conformal mapping. Taking into account properties of scalar analytic functions we can find an open set U such that $z_0 \in U \subset D$ and $\widetilde{\phi}(U)$ is an open subset of the complex plane $(\widetilde{u}_1 + i\widetilde{v}_1)$. But the set $(\widetilde{\phi}(U), \widetilde{\psi}(U))^T$ is on one side of the plane $\widetilde{u}_2 = \widetilde{v}_2 = 0$, i.e. the projection of $A \circ \Phi(U)$ on the plane $\widetilde{u}_1 = \widetilde{v}_1 = 0$ cannot be $\widetilde{\phi}(U)$. We obtain a contradiction.

Hence, points of $\partial\Phi(D)$ can be flex points or can belong to the planes (2.4). But $\Phi \in C^1(\overline{D})$. This means that flex points have dimension no more than two. Hence, $\partial\Phi(D)$ has dimension no more than two. Thus, $(\infty, \infty)^T \in \Phi(\overline{D})$. The last assertion contradicts the boundedness of Φ on $\overline{D}$. Hence, $\Phi = \text{const}$.

This proves the above lemma.

Let us consider another boundary value problem

$$\Phi(t) = \Phi_k(t) - \lambda\Lambda_k\overline{\Phi_k(t)} + \gamma_k, \quad |t - a_k| = r_k, \quad k = 0, 1, \ldots, n, \tag{2.5}$$

where the unknown vector-functions $\Phi(z), \Phi_k(z)$ are analytic in D, D_k respectively and are continuously differentiable in $\overline{D}, \overline{D}_k$. Here λ is a constant,

$$\Lambda_k = G_k^{-1}\overline{G}_k = \begin{pmatrix} 0 & -\lambda_k \\ -\overline{\lambda}_k^{-1} & 0 \end{pmatrix},$$

γ_k is a constant vector. To solve the problem (2.5) we shall follow [8].

Lemma 2. *If* $|\lambda| < 1$ *then the problem* (2.5) *has only constant solutions.*

Proof. We shall use the idea of B. Bojarski [2]. Let us put

$$U(z) = \begin{cases} \Phi(z), & z \in D, \\ \Phi_k(z) - \lambda \Lambda_k \overline{\Phi_k(z)} + \gamma_k, & z \in D_k. \end{cases}$$

Then the vector-function $U(z)$ is a solution of the partial differential equation

$$U_{\overline{z}} + Q\overline{U}_z = 0, \quad z \in \overline{\mathbb{C}} \setminus \bigcup_{k=0}^{n} \partial D_k, \tag{2.6}$$

where

$$Q = \begin{cases} 0, & z \in D, \\ \lambda \Lambda_k, & z \in D_k. \end{cases}$$

The system (2.6) is elliptic because $U_{\overline{z}} = 0$ in D, and we can rewrite (2.6) in D_k in the form of two scalar elliptic equations

$$(G_k U)_{\overline{z}} + \lambda \overline{(G_k U)_z} = 0.$$

The condition $U^+ = U^-$ holds on ∂D_k. The boundary values $U^{\pm}$ are in $L_2(\partial D_k)$. Hence by [2], (2.6) is valid in $\overline{\mathbb{C}}$. By the general Liouville theorem we get the equality $U = \text{const}$. Therefore, the problem (2.5) for $|\lambda| < 1$ has only constant solutions.

The lemma is proved.

The problem (2.2) is equivalent to the following $\mathbb{R}$-linear boundary value problem:

$$\Phi(t) = \Phi_k(t) - \Lambda_k \overline{\Phi_k(t)} + H_k^+(t), \quad |t - a_k| = r_k, \tag{2.7}$$

where

$$H_k(z) := \begin{pmatrix} g_k^+(z) \\ (1/\lambda_k)\overline{g_k^-(z_k^*)} \end{pmatrix},$$

$g_k(t) = g_k^+(t) - g_k^-(t)$ is the representation of $g_k(t)$ in the form of a difference of analytic functions by Sokhotski's formulas, and $z_k^* := r_k^2/(\overline{z - a_k}) + a_k$ is the inversion of z with respect to the circumference ∂D_k. We take an orientation of ∂D_k such that D_k is located to the left of ∂D_k. The unknown vector-function $\Phi_k(z)$ is analytic in D_k and is continuously differentiable in $\overline{D}_k$. If $\Phi(z)$ is a solution of (2.7), then $\Phi(z)$ is a solution of (2.2). If $\Phi(z)$ is a solution of (2.2), then $\Phi_k(z)$ can be found from the Schwarz problem

$$2 \operatorname{Im} G_k \Phi_k(t) = \operatorname{Im} G_k(\Phi(t) - H_k^+(t)), \quad |t - a_k| = r_k.$$

Actually, these are two scalar problems. Their solution depends additively on the vector $G_k^{-1}\gamma_k$, where γ_k is an arbitrary real constant vector [4]. The function $\Phi_k(z)$ and the initial function $\Phi(z)$ satisfy (2.7).

Let us consider the vector-function

$$\Omega(z) := \begin{cases} \Phi_k(z) + \sum\limits_{\substack{m=0 \\ m\neq k}}^{n} \Lambda_m \overline{\Phi_m(z_m^*)} + H_k^+(z), & |z-a_k| \le r_k, \\ & k = 0,1,\dots,n, \\ \Phi(z) + \sum\limits_{m=0}^{n} \Lambda_m \overline{\Phi_m(z_m^*)} + H_k^+(z), & z \in D. \end{cases}$$

Let us show that $\Omega(z)$ is analytic in $\overline{\mathbb{C}}$. From (2.7) we obtain

$$\Omega^+(t) - \Omega^-(t) = \Phi_k(t) + H_k^+(t) - \Phi(t) - \Lambda_k \overline{\Phi_k(t)} = 0, \qquad |t-a_k| = r_k.$$

By the principle of analytic continuation and the Liouville theorem we get

$$\Omega(z) = q + \sum_{m=0}^{n} \Lambda_m \overline{\Phi_m(z_m^*)} = \text{const},$$

where w is a fixed point belonging to $\overline{D} \setminus \{\infty\}$, and

$$\Phi(w) = \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix} =: \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = q.$$

From the definition of $\Omega(z)$ in D_k we obtain the following relations:

$$\Phi_k(z) = -\sum_{\substack{m=0 \\ m\neq k}}^{n} \Lambda_m (\overline{\Phi_m(z_m^*)} - \overline{\Phi_m(w_m^*)}) + \Lambda_k \overline{\Phi_k(w_k^*)} - H_k^+(z) + q, \tag{2.8}$$
$$|z-a_k| \le r_k,\ k = 0,1,\dots,n,$$

These relations constitute a system of n linear functional equations for n unknown functions $\Phi_k(z)$ $(k = 0,1,\dots,n)$ which are analytic in D_k and are continuously differentiable in $\overline{D}_k$.

Consider the Banach space C consisting of all functions continuous on $\bigcup_{k=0}^{n} \partial D_k$ with the norm

$$\|\Psi\| := \max_{0\le k\le n} \max_{\partial D_k} (|\Psi^1(t)|^2 + |\Psi^2(t)|^2)^{1/2}, \qquad \text{where } \Psi = (\Psi^1, \Psi^2)^T.$$

We introduce the subspace $C^+ \subset C$, which consists of all vector-functions analytic in each D_k. We differentiate the system (2.8):

$$\Psi_k(z) = -\sum_{\substack{m=0 \\ m\neq k}}^{n} \Lambda_m \, (\,\overline{z_m^*}\,)' \, \overline{\Psi_m(z_m^*)} - H_k^{+\prime}(z), \qquad |z-a_k| \le r_k. \tag{2.9}$$

Let us rewrite the last system in the form of the equation

$$\Psi(z) = A\Psi(z) - H'(z) \tag{2.10}$$

in the space C^+, where the operator A is defined by the right hand side of the system (2.9), $\Psi(z) := \Psi_k(z)$, $H(z) := H_k^{+\prime}(z)$ when $|z-a_k| \le r_k$; $\Psi, H' \in C^+$.

Lemma 3. *The homogeneous equation* (2.10) ($H'(z) \equiv 0$) *has the zero solution only.*

Proof. By integrating the homogeneous system (2.9) we obtain

$$\Phi_k(z) = -\sum_{\substack{m=0\\ m\neq k}}^{n} \Lambda_m \overline{\Phi_m(z_m^*)} + \gamma_k, \qquad |z-a_k| \le r_k, \tag{2.11}$$

where γ_k is a constant. Let us introduce the vector-function

$$\Phi(z) := -\sum_{m=0}^{n} \Lambda_m \overline{\Phi_m(z_m^*)},$$

which is analytic in $\overline{D}$. From (2.11) we obtain

$$\Phi(t) = \Phi_k(t) - \Lambda_k \overline{\Phi_k(t)} + \gamma_k, \qquad |t-a_k| = r_k.$$

It follows from Lemma 1 that $\Phi(z) \equiv \text{const}$. Hence $\Phi_k = \text{const}$ and $\Psi_k(z) = \Phi_k'(z) \equiv 0$. This proves the lemma.

Lemma 4. *The equation* (2.10) *has a unique solution in* C^+. *This solution can be found by the method of successive approximations in* C^+.

Proof. Let us rewrite the system (2.10) on ∂D_k in the form of a system of integral equations:

$$\Psi_k(t) = -\sum_{\substack{m=0\\ m\neq k}}^{n} (\overline{t_m^*})' \Lambda_m \overline{\frac{1}{2\pi i} \int\limits_{\partial D_m} \frac{\Psi_m(\tau_m^*)}{\tau - t_m^*}\, d\tau} - H_k^{+\prime}(t), \qquad |t-a_k| = r_k.$$

It can be written as an equation in C^+:

$$\Psi(t) = A\Psi(t) - H'(t). \tag{2.12}$$

Since integral operators are compact in C and multiplication by the matrix $(\overline{t_m^*})' \Lambda_m$ and complex conjugation are bounded in C, it follows that A is a compact operator in C. If Ψ is a solution of (2.12) in C, then $\Psi \in C^+$. This follows from the properties of the Cauchy integral. Therefore, the equation (2.12) in C and the equation (2.10) in C^+ are equivalent when $H' \in C^+$. It follows from Lemma 3 that the homogeneous equation $\Psi = A\Psi$ has the zero solution only. Then the Fredholm theorem implies that the system (2.12) or the system (2.10) has a unique solution.

Let us demonstrate the convergence of the successive approximations. It is sufficient to prove the inequality $\varrho(A) < 1$, where $\varrho(A)$ is the spectral radius of A. The spectrum of the compact operator A consists of eigenvalues only [6]. The inequality $\varrho(A) < 1$ is satisfied iff there exists a complex number λ such that $|\lambda| \le 1$ and the equation

$$\Psi(t) = \lambda A\Psi(t)$$

has the zero solution only. This equation can be written in the form

$$\Psi_k(z) = -\lambda \sum_{\substack{m=0 \\ m\neq k}}^{n} \Lambda_m \, (\,\overline{z_m^*}\,)' \, \overline{\Psi_m(z_m^*)}, \qquad |z-a_k| \le r_k. \tag{2.13}$$

Let $|\lambda| < 1$. Then integrating (2.13) we obtain

$$\Phi_k(z) = -\lambda \sum_{\substack{m=0 \\ m\neq k}}^{n} \Lambda_m \overline{\Phi_m(z_m^*)} + \gamma_k, \qquad |z-a_k| \le r_k,$$

where $\Phi_k'(z) = \Psi_k(z)$ and γ_k are arbitrary constant vectors. Introduce the vector-function

$$\Phi(z) := -\lambda \sum_{m=0}^{n} \Lambda_m \overline{\Phi_m(z_m^*)},$$

which is analytic in $\overline{D}$. Then $\Phi(z)$ and $\Phi_k(z)$ satisfy the $\mathbb{R}$-linear boundary value problem

$$\Phi(t) = \Phi_k(t) - \lambda \Lambda_k \overline{\Phi_k(t)} - \gamma_k, \qquad |t-a_k| = r_k.$$

It follows from Lemma 2 that this problem has constant solutions only. Thus $\Phi_k'(z) = \Psi_k(z) = 0$.

Let $|\lambda| = 1$. Then, changing the variable $z = \sqrt{\lambda} Z$, the system (2.13) is reduced to the same system with $\lambda = 1$, the constants $a_k = \sqrt{\lambda} A_k$ and the functions $\Omega_k(Z) := \Psi_k(z)$. It follows from Lemma 3 that $\Omega_k(Z) = \Psi_k(z) = 0$. Hence, $\varrho(A) < 1$. This inequality proves the lemma.

Let us introduce the mappings

$$z^*_{k_m k_{m-1} \dots k_1} := (z^*_{k_{m-1} \dots k_1})^*_{k_m}.$$

In the sequence $k_1, \dots, k_m$ no two neighboring numbers are equal. When m is even, these are Möbius transformations in z. If m is odd, we have transformations in $\overline{z}$. The number m is called the *level* of the mapping. The mapping can be written in the form

$$\gamma_j(z) = (\widehat{a}_j z + b_j)/(c_j z + d_j), \quad m \text{ is even},$$
$$\gamma_j(\overline{z}) = (\widehat{a}_j \overline{z} + b_j)/(c_j \overline{z} + d_j), \quad m \text{ is odd},$$

where $\widehat{a}_j d_j - c_j b_j = 1$. Here $\gamma_0(z) := z$, $\gamma_1(\overline{z}) := z_0^*$, $\gamma_2(\overline{z}) := z_1^*, \dots$ $\dots, \gamma_{n+1}(\overline{z}) := z_n^*$, $\gamma_{n+2}(z) := z_{01}^*$, $\gamma_{n+3}(z) := z_{02}^*$, and so on. The indices j of γ_j are fixed in such a way that the level is increasing. The functions γ_j generate a Kleinian group [7].

Let us investigate the vector systems (2.8) and (2.9) as scalar systems. Let $\Phi_k(z) = (\phi_k(z), \psi_k(z))^T$. Then from (2.8) and (2.9) we have, for

$|z-a_k|\le r_k,\ k=0,1,\dots,n,$

$$
\begin{aligned}
\phi_k(z) =& -\sum_{\substack{m=0\\ m\ne k}}^{n}\lambda_m(\overline{\psi_m(z_m^*)}-\overline{\psi_m(w_m^*)})\\
&-\lambda_k\overline{\psi_k(w_k^*)}-g_k^+(z)+q_1,\\
\psi_k(z) =& -\sum_{\substack{m=0\\ m\ne k}}^{n}\frac{1}{\overline{\lambda}_m}(\overline{\phi_m(z_m^*)}-\overline{\phi_m(w_m^*)})\\
&-\frac{1}{\overline{\lambda}_k}\overline{\phi_k(w_k^*)}-\frac{1}{\overline{\lambda}_k}\overline{g_k^-(z_k^*)}+q_2,\\
\phi_k'(z) =& -\sum_{\substack{m=0\\ m\ne k}}^{n}\lambda_m(\overline{\psi_m(z_m^*)})'-(g_k^+(z))',\\
\psi_k'(z) =& -\sum_{\substack{m=0\\ m\ne k}}^{n}\frac{1}{\overline{\lambda}_m}(\overline{\phi_m(z_m^*)})'-\left(\frac{1}{\overline{\lambda}_k}\overline{g_k^-(z_k^*)}\right)'.
\end{aligned}
\tag{2.14}
$$

It follows from Lemma 4 that we can apply the method of successive approximations. Thus, for $|z-a_k|\le r_k$,

$$
\begin{aligned}
\phi_k'(z) =& -(g_k^+(z))'+\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}(\overline{g_{k_1}^-(z_{k_1}^*)})'\\
&+\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}\sum_{\substack{k_2=0\\ k_2\ne k}}^{n}\frac{\lambda_{k_1}}{\lambda_{k_2}}(g_{k_2}^+(z_{k_2k_1}^*))'\\
&-\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}\sum_{\substack{k_2=0\\ k_2\ne k}}^{n}\sum_{\substack{k_3=0\\ k_3\ne k}}^{n}\frac{\lambda_{k_1}}{\lambda_{k_2}}(g_{k_3}^-(z_{k_2k_1}^*))'-\dots,\\
\psi_k'(z) =& \frac{1}{\overline{\lambda}_k}\Bigg(-(\overline{g_k^-(z_k^*)})'+\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}\frac{\overline{\lambda}_k}{\overline{\lambda}_{k_1}}(\overline{g_{k_1}^+(z_{k_1}^*)})'\\
&+\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}\sum_{\substack{k_2=0\\ k_2\ne k}}^{n}\frac{\overline{\lambda}_k}{\overline{\lambda}_{k_1}}(\overline{g_{k_2}^-(z_{k_2k_1}^*)})'\\
&-\sum_{\substack{k_1=0\\ k_1\ne k}}^{n}\sum_{\substack{k_2=0\\ k_2\ne k}}^{n}\sum_{\substack{k_3=0\\ k_3\ne k}}^{n}\frac{\overline{\lambda}_{k_2}}{\overline{\lambda}_{k_1}}\cdot\frac{\overline{\lambda}_k}{\overline{\lambda}_{k_3}}(\overline{g_{k_3}^+(z_{k_3k_2k_1}^*)})'-\dots\Bigg).
\end{aligned}
\tag{2.15}
$$

From the definition of $\Omega(z)$ we obtain, for $z \in \overline{D}$,

$$\phi(z) = q_1 + \sum_{k=0}^{n} \lambda_k (\overline{\psi_k(z_k^*)} - \overline{\psi_k(w_k^*)}) = q_1 + \sum_{k=0}^{n} \int_w^z \lambda_k (\overline{\psi_k(z_k^*)})' \, dz,$$
$$\psi(z) = q_2 + \sum_{k=0}^{n} \frac{1}{\overline{\lambda}_k} (\overline{\phi_k(z_k^*)} - \overline{\phi_k(w_k^*)}) = q_2 + \sum_{k=0}^{n} \int_w^z \frac{1}{\overline{\lambda}_k} (\overline{\phi_k(z_k^*)})' \, dz.$$

The series (2.15) converge in C^+, i.e. uniformly. Then, calculating the last integrals term, by term we get

$$\begin{aligned}
\phi(z) = q_1 &- \sum_{k=0}^{n} [g_k^-(z) - g_k^-(w)] \\
&+ \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \frac{\lambda_k}{\lambda_{k_1}} [g_{k_1}^+(z_{k_1 k}^*) - g_{k_1}^+(w_{k_1 k}^*)] \\
&+ \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \frac{\lambda_k}{\lambda_{k_1}} [g_{k_2}^-(z_{k_1 k}^*) - g_{k_2}^-(w_{k_1 k}^*)] \\
&- \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \sum_{\substack{k_3=0 \\ k_3 \neq k_2}}^{n} \frac{\lambda_{k_2}}{\lambda_{k_1}} \cdot \frac{\lambda_k}{\lambda_{k_3}} [g_{k_3}^+(z_{k_3 k_2 k_1 k}^*) \\
&- g_{k_3}^+(w_{k_3 k_2 k_1 k}^*)] - \dots \\
\psi(z) = q_2 &- \sum_{k=0}^{n} \frac{1}{\overline{\lambda}_k} [\overline{g_k^+(z_k^*)} - \overline{g_k^+(w_k^*)}] \\
&+ \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \frac{1}{\overline{\lambda}_k} [\overline{g_{k_1}^-(z_{k_1}^*)} - \overline{g_{k_1}^-(w_{k_1}^*)}] \\
&+ \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \frac{\overline{\lambda}_{k_1}}{\overline{\lambda}_k \overline{\lambda}_{k_2}} [\overline{g_{k_2}^+(z_{k_2 k_1 k}^*)} - \overline{g_{k_2}^+(w_{k_2 k_1 k}^*)}] \\
&- \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \sum_{\substack{k_3=0 \\ k_3 \neq k_2}}^{n} \frac{\overline{\lambda}_{k_1}}{\overline{\lambda}_k \overline{\lambda}_{k_2}} [\overline{g_{k_3}^-(z_{k_2 k_1 k}^*)} \\
&- \overline{g_{k_3}^-(w_{k_2 k_1 k}^*)}] - \dots
\end{aligned} \tag{2.16}$$

The functions (2.16) are solutions of the problem (2.1) if and only if the system of functional equations (2.14) is solvable. We get solvability conditions

if we set $z = w_k^*$ in (2.14):

$$\begin{aligned} \phi_k(z_k^*) &= \sum_{\substack{m=0\\ m\neq k}}^{n} \lambda_m P_{km} - \lambda_k \overline{\psi_k(w_k^*)} - g_k^+(w_k^*) + q_1, \\ \psi_k(z_k^*) &= \sum_{\substack{m=0\\ m\neq k}}^{n} \frac{1}{\overline{\lambda}_m} R_{km} - \frac{1}{\overline{\lambda}_k} \overline{\phi_k(w_k^*)} - \frac{1}{\overline{\lambda}_k} g_k^-(w_k^*) + q_2, \end{aligned} \tag{2.17}$$

for $k = 0, 1, \dots, n$, where $P_{km} = \overline{\psi_m(w_{mk}^*)} - \overline{\psi_m(w_m^*)}$ and $R_{km} = \overline{\phi_m(w_{mk}^*)} - \overline{\phi_m(w_m^*)}$. Here, for example, $R_{km} := \int_w^{w_k^*} (\overline{z_m^*})' \overline{\phi_m'(z_m^*)}\, dz$, and the functions $\phi_m'(z)$ have the form (2.15). The relations (2.17) hold if and only if, for $k = 0, 1, \dots, n$,

$$q_1 - \lambda_k \overline{q}_2 + \sum_{\substack{m=0\\ m\neq k}}^{n} \lambda_m P_{km} - g_k^+(w_k^*) = \sum_{\substack{m=0\\ m\neq k}}^{n} \frac{\lambda_k}{\lambda_m} \overline{R}_{km} - g_k^-(w). \tag{2.18}$$

Definition (see [4]). If $\lambda_k = \lambda$ for every $k = 0, 1, \dots, n$, then we shall say that *the conditions of single-valuedness hold*. If there exist k and m such that $\lambda_k \neq \lambda_m$, then we say that *the conditions of single-valuedness do not hold*.

If the conditions of single-valuedness hold, then the constant $p := q_1 - \lambda \overline{q}_2$ is defined from (2.18) for $k = 0$. The other n relations give necessary and sufficient solvability conditions for (2.1). The solution of (2.1) has the form (2.16), where the arbitrary constants q_1 and q_2 are related by $p = q_1 - \lambda \overline{q}_2$. If the conditions of single-valuedness do not hold, then two equalities of (2.18) define q_1 and q_2 and the other $n - 1$ equalities give necessary and sufficient solvability conditions for (2.1). The solution of (2.1) has the form (2.16).

3. Solution of the Riemann problem for circumferences. Consider the following boundary value problem on $\overline{\mathbb{C}}$:

$$\phi(t) = \lambda_k(t) \overline{\psi(t)} + g_k(t), \qquad |t - a_k| = r_k,\ k = 0, 1, \dots, n, \tag{3.1}$$

where $\lambda_k(t)$ are Hölder-continuous functions, $\lambda_k(t) \neq 0$. There exists a complete Noether theory for (3.1) (see [1]). But we construct the solution of (3.1) in analytic form. Set

$$\kappa_k := \operatorname*{ind}_{\partial D_k} \lambda_k(t), \quad R(z) := \prod_{m=0}^{n} (z - a_m)^{-\kappa_m}, \quad \kappa := \sum_{m=0}^{n} \kappa_m.$$

Let $\kappa \geq 0$. Let us introduce the function

$$\omega(z) := \frac{\psi(z)}{\overline{R(z)}} - \sum_{s=1}^{\kappa} \delta_s z^s \tag{3.2}$$

($\sum_{s=1}^{0} := 0$), where $\sum_{s=1}^{\kappa} \delta_s z^s$ is the principle part of $\psi(z)/R(z)$ at infinity. Then (3.1) transforms to

$$\phi(t) = \lambda_k(t)\overline{R(t)}\,\overline{\psi(t)} + \lambda_k(t) \sum_{s=1}^{\kappa} \overline{\delta_s t^s R(t)} + g_k(t), \qquad |t - a_k| = r_k. \tag{3.3}$$

Let us apply the factorization method [4] to the problem (3.3). We consider the auxiliary problem

$$X_1(t) - \overline{X_2(t)} = \ln \lambda_k(t)\overline{R(t)} + c_k, \qquad |t - a_k| = r_k,$$
$$X_1(w) = X_2(w) = 0,$$

where c_k are unknown complex constants. Since $\operatorname{ind}_{\partial D_k} \lambda_k(t)\overline{R(t)} = 0$, the logarithms are correctly defined. If c_k are fixed, then the problem (3.3) is a particular case of the problem (2.1) for $\lambda_k = 1$. From the necessary and sufficient solvability conditions for (2.18) we obtain

$$c_k = \sum_{\substack{m=0 \\ m \neq k}}^{n} (P_{km} - \overline{R}_{km}), \qquad k = 0, 1, \ldots, n. \tag{3.4}$$

The solution has the form (2.16), where $q_1 = q_2 = 0$, $\lambda_k = 1$, $\phi(z) = X_1(z)$, $\psi(z) = X_2(z)$ and $\ln \lambda_k(t)\overline{R(t)} = g_k^+(t) - g_k^-(t)$.

Let us introduce the auxiliary unknown functions

$$\alpha(z) := \phi(z) \exp(-X_1(z)), \qquad \beta(z) := \omega(z) \exp(-X_2(z)),$$

satisfying the following boundary value problem:

$$\alpha(t) = \mu_k \overline{\beta(t)} + f_k(t), \qquad |t - a_k| = r_k, \tag{3.5}$$

where $\mu_k := \exp c_k$, c_k has the form (3.4),

$$f_{k0}(t) := g_k(t) \exp(-X_1(t)), \qquad f_{ks}(t) := \lambda_k(t)\overline{t^s R(t)} \exp(-X_1(t)),$$
$$f_k(t) := f_{k0}(t) + \sum_{s=1}^{\kappa} f_{ks}(t)\delta_s.$$

The necessary and sufficient solvability conditions for (3.5),

$$q_1 - \mu_k \overline{q}_2 + \sum_{\substack{m=0 \\ m \neq k}}^{n} \left(\mu_m P_{km} - \frac{\lambda_k}{\lambda_m} \overline{R}_{km} \right) \tag{3.6}$$
$$= f_{k0}^+(w_k^*) - \overline{f_{k0}^-(w)} + \sum_{s=1}^{\kappa} f_{ks}^+(w_k^*)\delta_s - \sum_{s=1}^{\kappa} \overline{f_{ks}^-(w)}\,\overline{\delta}_s = 0, \qquad k = 0, 1, \ldots, n,$$

form a system of $2(n+1)$ $\mathbb{R}$-linear algebraic equations for the $2(\kappa+2)$ real quantities $\operatorname{Re}\delta_s,\ \operatorname{Im}\delta_s,\ \operatorname{Re}q_1,\ \operatorname{Im}q_2$ $(s=1,\dots,\kappa)$. Here $f_{ks}(t)=f_{ks}^{+}(t)-f_{ks}^{-}(t),\ |t-a_k|=r_k$.

THEOREM 1 (index $\kappa\geq 0$). *The problem* (3.1) *is solvable if and only if the system* (3.6) *is solvable. If* (3.6) *is solvable then the solution*

$$\phi(z):=\alpha(z)\exp(X_1(z)),$$
$$\psi(z):=\Big[\beta(z)\exp\Big(X_2(z)+\sum_{s=1}^{\kappa}\delta_s z^s\Big)\Big]R(z),\quad z\in\overline{D},$$

depends on the constants $\operatorname{Re}\delta_s,\ \operatorname{Im}\delta_s,\ \operatorname{Re}q_1,\ \operatorname{Im}q_2$, *which are arbitrary solutions of the system* (3.6). *The functions* $X_1(z), X_2(z)$ *have the form* (2.16). *The functions* $\alpha(z)$ *and* $\beta(z)$ *have the same form as* $\phi(z)$ *and* $\psi(z)$, *respectively, with* $\lambda_k=\mu_k,\ f_k(t)=g_k^{+}(t)-g_k^{-}(t),\ |t-a_k|=r_k$.

Remark. It follows from the general theory [9] that if $\kappa\geq n$, then the system (3.6) is solvable. The number of linearly independent solutions is $l=2(\kappa-n+1)$.

Let $\kappa<0$. Then we put $\omega(z):=\psi(z)[R(z)]^{-1}$. Analogously, we obtain the boundary value problem

$$\phi(t)=\lambda_k(t)\overline{R(t)}\,\overline{\omega(t)}+g_k(t),\qquad |t-a_k|=r_k.$$

That problem is solved like (3.1). We have

$$\psi(z)=R(z)\omega(z),$$

where $R(z)$ has a pole at $z=\infty$. The order of the pole is $-\kappa$. Since $\psi(z)$ has to be analytic at $z=\infty$, the function $\omega(z)$ has a zero. The order of the zero is no less than $-\kappa$:

$$\operatorname*{res}_{z=\infty} z^s\omega(z)=0,\qquad s=0,1,\dots,-\kappa-1. \tag{3.7}$$

THEOREM 2 (index $\kappa<0$). *Suppose the conditions of single-valuedness hold* $(\mu_k=\mu,\ k=0,1,\dots,n)$. *Then the problem* (20) *is solvable if and only if* n *of the conditions* (3.6) *are satisfied. The remaining condition of* (3.6) *defines the constant* $p:=q_1-\mu\overline{q}_2$. *Necessary and sufficient solvability conditions for* (3.1) *have the form* (3.6) *and* (3.7). *When* (3.6) *and* (3.7) *hold the solution of* (3.1) *has the form*

$$\phi(z):=\alpha(z)\exp(X_1(z)),\qquad \psi(z):=\beta(z)\exp(X_2(z)), \tag{3.8}$$

depending on the arbitrary complex constant q_1. *The functions* $\alpha(z)$, $\beta(z)$, $X_1(z)$, $X_2(z)$ *have the form* (2.16).

If the conditions of single-valuedness do not hold then the problem (3.1) *is solvable when* $n-1$ *conditions* (3.6) *are satisfied. The other two conditions of*

(3.6) *define the constants* q_1 *and* q_2. *When* (3.6) *and* (3.7) *hold the problem* (3.1) *has the unique solution* (3.8).

4. Solution of the Riemann problem (1.1). According to the classical scheme [9], we shall transfer the boundary conditions from Γ to ∂D_k. Let some components of Γ lie in D and D^* simultaneously. Let us continue these components in D and D^* in the disc $|z-a_k|<r_k$ up to a_k. Denote the resulting contour by Γ_0 and Γ_0^* on D and D^* respectively. Consider the following Riemann problem:

$$F_1^+(t)=G(t)F_1^-(t),\qquad t\in\Gamma_0, \tag{4.1}$$

on the complex plane $\mathbb{C}$. Here $G(t)=1$ when $t\notin\Gamma$. We assume that $F_1(z)$ can have singularities at $z=a_k$ of order such that the problem (4.1) has a solution. Let $F_1(z)$ be a solution of (4.1). One can find identities for the function $F_1(z)$ in [4]. Let us introduce an auxiliary unknown function on the first sheet of $\mathcal{R}$:

$$\phi(z)=\Phi(z)/F_1(z)-g_1(z),\qquad z\in\overline{D}, \tag{4.2}$$

where

$$g_1(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(\tau)}{F_1(\tau)(\tau-z)}\,d\tau.$$

The function $g(\tau)$ is extended to $\Gamma_0\setminus\Gamma$ in such a way that it is Hölder-continuous. Analogously we introduce a function on the second sheet of $\mathcal{R}$:

$$\phi_1(z)=\Phi(z)/F_2(z)-g_2(z),\qquad z\in\overline{D}. \tag{4.3}$$

It follows from (1.1) and (4.1) that $\phi^+(t)=\phi^-(t)$, $\phi_1^+(t)=\phi_1^-(t)$, $t\in\Gamma$. Let us find $\phi(z)$ and $\phi_1(z)$ on ∂D_k. We use the condition $\Phi^+(p)=\Phi^-(p)$, where p is the local parameter of $\mathcal{R}$ on the circumference: $p=t$ on the first sheet, $p=\overline{t}$ on the second sheet. As a result we obtain the problem

$$\phi(t)=\frac{F_2(\overline{t})}{F_1(t)}\overline{\psi(t)}+\frac{F_2(\overline{t})}{F_1(t)}g_2(\overline{t})-g_1(t),\qquad |t-a_k|=r_k, \tag{4.4}$$

where $\psi(z):=\overline{\phi_1(\overline{z})},\ z\in\overline{D}$.

Let us apply Theorems 1 or 2 to the problem (4.4). If the necessary and sufficient conditions hold and we have $\phi(z)$ and $\psi(z)$ then the function $\Phi(z)$ is found from the relations (4.2) and (4.3):

$$\Phi(z)=\begin{cases}F_1(z)(\phi(z)-g_1(z)) & \text{on the first sheet,}\\ F_2(z)(\overline{\psi(\overline{z})}-g_2(z)) & \text{on the second sheet.}\end{cases}$$

It is easy to verify the identity

$$\operatorname*{ind}_{p\in\Gamma}G(p)=\operatorname*{ind}_{t\in\partial D}F_2(\overline{t})/F_1(t),$$

where on the left hand side the index is calculated on the Riemann surface $\mathcal{R}$, and on the right hand side on the plane $\mathbb{C}$.

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Reçu par la Rédaction le 5.4.1993
Révisé le 10.12.1996