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Equivalence of analytic and rational functions

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Abstract. We give a criterion for a real-analytic function defined on a compact nonsingular real algebraic set to be analytically equivalent to a rational function.

Throughout this paper M denotes a compact nonsingular algebraic subset of $\mathbb{R}^n$. As usual, by a *polynomial function* on M we mean the restriction to M of a polynomial function from $\mathbb{R}^n$ into $\mathbb{R}$. A function $r : A \to \mathbb{R}$, defined on a subset A of M, is said to be *regular on* A if there exist polynomial functions $p : M \to \mathbb{R}$ and $q : M \to \mathbb{R}$ such that $q^{-1}(0) \cap A = \emptyset$ and $r(x) = p(x)/q(x)$ for all x in A (in other words, r is regular on A if it is a rational function on M, whose denominator is nonzero at each point of A). Of course, every polynomial function is regular, and every regular function on M is (real-) analytic. Two analytic functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are called *analytically equivalent* if $g = f \circ \sigma$ for some analytic diffeomorphism $\sigma : M \to M$. In the present paper we are concerned with the following problem: What conditions have to be imposed on an analytic function $f : M \to \mathbb{R}$ in order for it to be analytically equivalent to a polynomial or a regular function on M?

The reader may consult [1, 2, 5, 6] for earlier results related to this problem. Let us consider now an analytic function $f : M \to \mathbb{R}$ with isolated critical points. It is known that f is not necessarily analytically equivalent to a regular function on M [1, pp. 416, 417], and therefore one has to impose some extra conditions (let us recall, however, that for each nonnegative

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integer k one can find a C^k diffeomorphism $\sigma : M \to M$ such that $f \circ \sigma$ is a polynomial function on M [2, Theorem 8.2]). Denote by $M_{\mathbb{C}}$ the smallest complex algebraic subset of $\mathbb{C}^n$ containing M (we view $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$). Obviously, every point of M is a nonsingular point of $M_{\mathbb{C}}$. Furthermore, every analytic function $g : M \to \mathbb{R}$ has a unique extension $g_{\mathbb{C}} : (M_{\mathbb{C}}, M) \to \mathbb{C}$ to a holomorphic function-germ at M. If each critical point of f is an isolated critical point of the holomorphic function-germ $f_{\mathbb{C}} : (M_{\mathbb{C}}, M) \to \mathbb{C}$, then f is analytically equivalent to a polynomial function on M [1, Theorem 5].

We always consider M endowed with the usual metric topology induced from $\mathbb{R}^n$. Given a point x in M and a function $g : U \to \mathbb{R}$ defined in a neighborhood of x, we denote by g_x the germ of g at x. Two analytic function-germs $\varphi : (M, x) \to \mathbb{R}$ and $\psi : (M, x) \to \mathbb{R}$ are said to be *analytically equivalent* if there exists a local analytic diffeomorphism $\tau : (M, x) \to (M, x)$ such that $\psi = \varphi \circ \tau$. If $h : M \to \mathbb{R}$ is a differentiable function, then Σ_h will denote the set of critical points of h.

CONJECTURE 1. *Let $f : M \to \mathbb{R}$ be an analytic function with isolated critical points. Assume that for each point x in Σ_f the germ f_x is analytically equivalent to the germ at x of a regular function defined in a neighborhood of x. Then f is analytically equivalent to a regular function on M.*

We shall prove a somewhat weaker result than Conjecture 1, but first we need some preparation. Denote by $\mathcal{E}_x$ the ring of all C^∞ function-germs $(M, x) \to \mathbb{R}$ at a point x in M. Given φ in $\mathcal{E}_x$, we define $\triangle(\varphi)$ to be the ideal of $\mathcal{E}_x$ generated by

$$\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_m},$$

where $m = \dim M$ and $(x_1, \ldots, x_m)$ is a local C^∞ coordinate system in a neighborhood of x in M. Assume that x is a critical point of φ, that is, $\triangle(\varphi) \neq \mathcal{E}_x$. Although we shall not use it later on, let us observe, in order to motivate the definition given below, that the following conditions are equivalent:

(a) x is an isolated critical point of φ;

(b) there exists λ in $\triangle(\varphi)$ such that $\lambda^{-1}(0) = \{0\}$ as set-germs;

(c) there exists a C^∞ function $u : M \to \mathbb{R}$ such that $u^{-1}(0) = \{x\}$ and the germ u_x belongs to $\triangle(\varphi)$.

Indeed, if (a) is satisfied, then so is (b) with $\lambda = \big(\frac{\partial \varphi}{\partial x_1}\big)^2 + \ldots + \big(\frac{\partial \varphi}{\partial x_m}\big)^2$. If (b) holds, then, using partition of unity, one easily constructs a C^∞ function $u : M \to \mathbb{R}$ such that $u^{-1}(0) = \{x\}$ and $u_x = \lambda^2$, which implies (c). It is clear that (a) follows from (c).

Furthermore, if φ is an analytic function-germ, then, applying the theory of coherent real analytic sheaves, one can prove that (a), (b), (c) are equivalent to (c′) there exists an analytic function $v : M \to \mathbb{R}$ such that $v^{-1}(0) = \{x\}$ and the germ v_x belongs to $\triangle(\varphi)$.

Definition 2. We say that the critical point x of φ is *algebraically isolated* in M if there exists a regular (or, equivalently, a polynomial) function $r : M \to \mathbb{R}$ such that $r^{-1}(0) = \{x\}$ and the germ r_x belongs to $\triangle(\varphi)$.

Denote by $\mathcal{E}(M)$ the ring of all C^∞ functions on M. Let $X_1, \ldots, X_d$ be C^∞ vector fields on M generating the $\mathcal{E}(M)$-module of all C^∞ vector fields on M. Given a C^∞ function $f : M \to \mathbb{R}$, we denote by $\triangle(f)$ the ideal of $\mathcal{E}(M)$ generated by $X_1 f, \ldots, X_d f$; clearly, $\triangle(f)$ does not depend on the choice of the generators $X_1, \ldots, X_d$. By construction, for each point x in M, we have $\triangle(f)\mathcal{E}_x = \triangle(f_x)$.

We shall need the following fact.

Example 3. Let $g : M \to \mathbb{R}$ be a regular function on M. Assume that $\Sigma_g \cap g^{-1}(g(x)) = \{x\}$ for some point x in M. We claim that x is a critical point of g_x algebraically isolated in M. In order to prove the claim, we choose polynomial vector fields $X_1, \ldots, X_d$ on M generating the $\mathcal{E}(M)$-module of all C^∞ vector fields on M, and set

$$r = (X_1 g)^2 + \ldots + (X_d g)^2 + (g - g(x))^{2m},$$

where $m = \dim M$ (a vector field X on M is said to be a *polynomial vector field* if Xp is a polynomial function on M for every polynomial function $p : M \to \mathbb{R}$). Obviously, r is a regular function on M. Since Σ_g is equal to the set of zeros of $s = (X_1 g)^2 + \ldots + (X_d g)^2$ and since $\Sigma_g \cap g^{-1}(g(x)) = \{x\}$, we obtain $r^{-1}(0) = \{x\}$. It is clear that s belongs to $\triangle(g)$. By [3], the germ $(g_x - g(x))^m$ is in $\triangle(g_x)$, and therefore r_x belongs to $\triangle(g_x)$. Thus the claim is proved. ∎

Recall that a local C^1 diffeomorphism $\tau : (M, x) \to (M, x)$ is said to be *orientation preserving* if $\det(D_x \tau) > 0$, where $D_x \tau : T_x M \to T_x M$ is the derivative of τ at x.

Example 4. Let $\varphi : (M, x) \to \mathbb{R}$ be an analytic function-germ and let $\varphi_{\mathbb{C}} : (M_{\mathbb{C}}, x) \to \mathbb{C}$ be the unique extension of φ to a holomorphic function-germ. Assume that x is an isolated critical point of $\varphi_{\mathbb{C}}$. It is well known that there exists a local orientation preserving analytic diffeomorphism $\tau : (M, x) \to (M, x)$ such that $\varphi \circ \tau$ is the germ at x of a polynomial function on M [7, p. 170, Proposition 4.2, p. 59, Théorème 4.2].

We assert that x is a critical point of both φ and $\varphi \circ \tau$ algebraically isolated in M. Of course, it suffices to prove the assertion for φ. Denote by $m(\mathcal{E}_x)$ the unique maximal ideal of $\mathcal{E}_x$; obviously,

$$m(\mathcal{E}_x) = \{\lambda \in \mathcal{E}_x \mid \lambda(x) = 0\}.$$

Since x is an isolated critical point of $\varphi_{\mathbb{C}}$, it follows that some power of $m(\mathcal{E}_x)$, say $m(\mathcal{E}_x)^l$, is contained in $\triangle(\varphi)$ [7, p. 170, Proposition 4.2 and its proof]. Choose a regular function $s : M \to \mathbb{R}$ such that $s^{-1}(0) = \{x\}$ and set $r = s^l$. Then $r^{-1}(0) = \{x\}$ and r_x belongs to $\triangle(\varphi)$, which implies our assertion. ∎

THEOREM 5. *Let $f : M \to \mathbb{R}$ be an analytic function with isolated critical points. Assume that for each point x in Σ_f, there exists a local orientation preserving analytic diffeomorphism $\sigma_x : (M, x) \to (M, x)$ such that $f_x \circ \sigma_x$ is the germ at x of a regular function defined in a neighborhood of x, and x is a critical point of $f_x \circ \sigma_x$ algebraically isolated in M. Then f is analytically equivalent to a regular function on M.*

For the proof of Theorem 5 we shall need the following.

LEMMA 6. *Let x be a point in M and let $\psi : (M, x) \to \mathbb{R}$ be the germ at x of a regular function defined in a neighborhood of x. Assume that x is a critical point of ψ algebraically isolated in M. Then there exist a regular function $g : M \to \mathbb{R}$ and a local orientation preserving analytic diffeomorphism $\tau : (M, x) \to (M, x)$ such that $\Sigma_g \cap g^{-1}(g(x)) = \{x\}$ and $\psi \circ \tau = g_x$.*

Proof. Let $r : M \to \mathbb{R}$ be a regular function such that $r^{-1}(0) = \{x\}$ and r_x belongs to $\triangle(\psi)$. Pick polynomial functions $p : M \to \mathbb{R}$ and $q : M \to \mathbb{R}$ such that $q(x) \neq 0$ and $\psi = p_x/q_x$. Note that the function

$$u = \frac{pq + r^4}{q^2 + r^4}$$

is regular on M.

An obvious modification of [2, Lemma 3.2] implies the existence of a C^∞ function $f : M \to \mathbb{R}$ such that $\Sigma_f \cap f^{-1}(f(x)) = \{x\}$, $f_x = \psi$, and $f|M \setminus \{x\}$ has only nondegenerate critical points. Note that

$$f_x - u_x = \psi - u_x = r_x^4 \frac{p_x - q_x}{q_x(q_x^2 + r_x^4)}.$$

Applying this and the equality $r^{-1}(0) = \{x\}$, we can find a C^∞ function $\alpha : M \to \mathbb{R}$ satisfying

$$f = u + r^4\alpha. \tag{7}$$

Let $\mathcal{V}$ be a neighborhood of 0 in the C^∞ topology on $\mathcal{E}(M)$ and let $\beta : M \to \mathbb{R}$ be a polynomial function such that $\beta - \alpha$ is in $\mathcal{V}$ and $j_y^1(\beta) = j_y^1(\alpha)$ for all y in $\Sigma_f \setminus \{x\}$, where $j_y^1(-)$ stands for the 1-jet at y (β exists in view of [1, Corollary 1]). Then

$$g = u + r^4\beta \tag{8}$$

is a regular function on M.

By construction, r_x^2 belongs to $\triangle(f)^2\mathcal{E}_x = \triangle(\psi)^2$. Furthermore, for each y in $\Sigma_f\backslash\{x\}$, we have $\triangle(f)\mathcal{E}_y = m(\mathcal{E}_y)$, and hence $\beta_y - \alpha_y$ belongs to $\triangle(f)^2\mathcal{E}_y$. Since Σ_f is precisely the set of zeros of the ideal $\triangle(f)$, using (1), (2), and partition of unity, we get

$$g - f = r^4(\beta - \alpha) \in r^2\triangle(f)^2.$$

It also follows from the observations recorded in this paragraph and from [7, p. 119, Corollaire 1.6] that $\triangle(f)^2$ is a closed ideal (in the C^∞ topology) of $\mathcal{E}(M)$. Taking $\mathcal{V}$ sufficiently small and applying [2, Theorem 2.1] (with $G = \{1\}$ = the trivial subgroup of $\mathbb{R}\setminus\{0\}$), we obtain a C^∞ diffeomorphism $\sigma : M \to M$ isotopic to the identity and such that $g = f \circ \sigma$, $\sigma(x) = x$. Hence $\Sigma_g \cap g^{-1}(g(x)) = \{x\}$ and $g_x = f_x \circ \sigma_x = \psi \circ \sigma_x$. So, by construction, $\sigma_x : (M, x) \to (M, x)$ is a local orientation preserving C^∞ diffeomorphism. By [7, p. 59, Théorème 4.2], there also exists a local analytic diffeomorphism $\tau : (M, x) \to (M, x)$ such that $g_x = \psi \circ \tau$ and $j_x^1(\tau) = j_x^1(\sigma_x)$. It follows from the last equality that τ is orientation preserving. Thus the lemma is proved. ■

P r o o f o f T h e o r e m 5. Let $\Sigma_f = \{x_1, \ldots, x_k\}$. By Lemma 6, for each $i = 1, \ldots, k$ there exist a regular function $g_i : M \to \mathbb{R}$ and a local orientation preserving analytic diffeomorphism $\tau_i : (M, x_i) \to (M, x_i)$ such that $\Sigma_{g_i} \cap g_i^{-1}(g_i(x_i)) = \{x_i\}$ and $f_{x_i} \circ \tau_i = g_{ix_i}$. Let $\sigma : M \to M$ be a C^∞ diffeomorphism satisfying $\sigma_{x_i} = \tau_i$ for all $i = 1, \ldots, k$. Then $f \circ \sigma = g_i$ in a neighborhood of x_i.

Choose a regular function $r_i : M \to \mathbb{R}$ such that $r_i^{-1}(0) = \{x_i\}$ and r_{ix_i} belongs to $\triangle(g_{ix_i})$ (cf. Example 3). Then $s = r_1^2 \ldots r_k^2$ is a regular function on M, $s^{-1}(0) = \Sigma_f = \Sigma_{f\circ\sigma}$, and s belongs to $\triangle(f \circ \sigma)^2$ (the last property follows by applying partition of unity). Note that

$$u = \Big(\sum_{i=1}^{k} \Big(\prod_{j\neq i} r_j^2\Big) g_i\Big) \Big/ \Big(\sum_{i=1}^{k} \Big(\prod_{j\neq i} r_j^2\Big)\Big)$$

is a regular function on M and

$$(f \circ \sigma)_{x_l} - u_{x_l} = g_{lx_l} - u_{x_l} = s_{x_l} v_l,$$

where v_l is the germ at x_l of the regular function

$$\Big(\sum_{i=1}^{k} \Big(\prod_{\substack{j\neq i \\ j\neq l}} r_j^2\Big)(g_l - g_i)\Big) \Big/ \Big(\Big(\sum_{i=1}^{k} \prod_{j\neq i} r_j^2\Big)\Big(\prod_{j\neq l} r_j^2\Big)\Big)$$

on $(M \setminus \{x_1, \ldots, x_k\}) \cup \{x_l\}$. It follows that we can find a C^∞ function $\alpha : M \to \mathbb{R}$ satisfying $f \circ \sigma = u + s\alpha$. Let $\beta : M \to \mathbb{R}$ be a regular function and let $g = u + s\beta$. Then

$$g - f \circ \tau = (\beta - \alpha)s \in \triangle(f \circ \sigma)^2.$$

Let $\sigma^* : \mathcal{E}(M) \to \mathcal{E}(M)$ be the ring isomorphism defined by $\sigma^*(h) = h \circ \sigma$ for h in $\mathcal{E}(M)$. Clearly, $\triangle(f \circ \sigma)^2 = \sigma^*(\triangle(f)^2)$, and therefore, in view of [7, p. 119, Corollaire 1.6], the ideal $\triangle(f \circ \sigma)^2$ of $\mathcal{E}(M)$ is closed in the C^∞ topology on $\mathcal{E}(M)$. By [2, Theorem 2.1], if β is sufficiently close to α in the C^∞ topology, then there exists a C^∞ diffeomorphism $\eta : M \to M$ such that $g = f \circ \sigma \circ \eta$. It follows from the last equality and [4, Theorem 8.4] that f and g are analytically equivalent. Since g is a regular function on M, the proof is complete. ■

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