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# On a property of weak resolvents and its application to a spectral problem

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**Abstract.** We show that the poles of a resolvent coincide with the poles of its weak resolvent up to their orders, for operators on Hilbert space which have some cyclic properties. Using this, we show that a theorem similar to the Mlak theorem holds under milder conditions, if a given operator and its adjoint have cyclic vectors.

**1. Introduction.** For a linear bounded operator $A : X \to X$, where $X$ is a Hilbert space, we define a complex-valued function $\varphi(z) = \langle c, (zI - A)^{-1} b\rangle$, which we call a *weak resolvent*, due to Fong, Nordgren, Radjavi, and Rosenthal (cf. [3], [15]). Here $b, c \in X$, and $\langle f, g\rangle$ denotes the scalar product of the vectors $f$ and $g$. Nordgren *et al.* considered this function in the study of the invariant subspace problem. Earlier, in the 1960's, in the model theory of operators, Sz.-Nagy and Foiaş introduced this kind of functions (cf. [17]). Also, in the study of the spectral problem, Mlak proved the following theorem, which also concerns model theory. See also Lebow [12] and Nikol'skiĭ [14].

THEOREM 1 ([13]). *If, for every $b, c$ in $X$, $z^{-1}\varphi(z^{-1}) = \langle c, (I - zA)^{-1} b\rangle \in H^1$, then $\varrho(A) < 1$. Here $\varrho(A)$ is the spectral radius of $A$.* ∎

Janas [7] and Jakóbczak and Janas [6] have extended the above theorem to several commuting operators.

During the 1960's, Lax and Phillips developed a scattering theory (cf. [11]). Meanwhile, during the same period, engineers developed independently a control theory, initiated by, among others, Kalman (cf. [9], [8]). Surprisingly enough, the above kind of abstract operator theory and these two theories have been shown to be related to one another by Adamyan and Arov (see references in [11]) and Helton ([4], [5]). The weak resolvent cor-

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responds to a scattering matrix in scattering theory and a transfer function (or a frequency response function) in control theory, respectively. The above Mlak theorem is also related to the input-output stability of control systems (cf. [8]).

In our paper, we show that if both $A$ and its adjoint have cyclic vectors, then the poles of the resolvent of $A$ and their orders exactly coincide with those of the weak resolvent of $A$. Next, using this result, we show that such operators, a result similar to Mlak's theorem holds under milder conditions.

Notations which we use are as follows:

$\mathbb{D} = \{z : |z| < 1\}$ (open unit disc in the complex plane),
$\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ (closed unit disc in the complex plane),
$\mathbb{T} = \{z : |z| = 1\}$ (unit circle in the complex plane),
$H^1 = \{f(z)$ analytic in $\mathbb{D} : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta = \|f\|_1 < +\infty\}$ (Hardy space with $p = 1$).

## 2. Main theorems

Definition. We say that $b$ is *cyclic for* $A$ iff $\overline{\bigcup_{n=0}^{\infty} \operatorname{Span}_{k=0}^{n}\{A^k b\}} = X$. ∎

The following lemma is known as the Popov–Belevich–Hautus–Rosenbrock test in control theory for the finite-dimensional case (cf. [8]). To make the paper self-contained, we include the result with a proof for the infinite-dimensional case.

Lemma 1. *If $b$ is cyclic for $A$ and $x$ is an eigenvector of $A^*$, i.e., for some $z_0 \in \mathbb{C}$, $A^*x = z_0 x$, $x \neq 0$, then $\langle b, x\rangle \neq 0$.*

Proof. Suppose $\langle b, x\rangle = 0$. Then

$$\begin{aligned}\langle Ab, x\rangle &= \langle b, A^*x\rangle = z_0\langle b, x\rangle = 0,\\ \langle A^2 b, x\rangle &= \langle Ab, A^*x\rangle = z_0\langle Ab, x\rangle = 0,\\ &\vdots\\ \langle A^k b, x\rangle &= \langle A^{k-1}b, A^*x\rangle = z_0\langle A^{k-1}b, x\rangle = 0 \qquad (k = 1, 2, \ldots),\end{aligned}$$

Thus $x \notin \overline{\bigcup_{n=0}^{\infty} \operatorname{Span}_{k=0}^{n}\{A^k b\}} = X$. However, this contradicts the assumption that $b$ is cyclic for $A$. This completes the proof. ∎

In the following theorem and its proof, a *pole* is an isolated (not accumulating) pole.

Theorem 2. *Let $(zI - A)^{-1}$ be meromorphic in an open neighborhood of $z_0$. Further, let $b$ be cyclic for $A$ and $c$ be cyclic for $A^*$. Then the weak resolvent $\varphi(z) = \langle c, (zI - A)^{-1}b\rangle$ has a pole of order $m$ at $z = z_0$ if and only if the resolvent $(zI - A)^{-1}$ has a pole of order $m$ at $z = z_0$.*

Proof. "If". Write $(zI-A)^{-1}$ in the following form:

$$(zI-A)^{-1} = B_{-m}(z-z_0)^{-m} + B_{-(m-1)}(z-z_0)^{-(m-1)} + \ldots + B_0 + B_1(z-z_0) + \ldots$$

From this,

$$(z_0 I - A)B_{-m} = 0.$$

Suppose $B_{-m}b \neq 0$. Then, since $A^*$ is cyclic for $c$ by assumption, we have $\langle c, B_{-m}b\rangle \neq 0$ by Lemma 1, and thus the weak resolvent $\varphi(z)$ has a pole of order $m$ at $z = z_0$. Now we show $B_{-m}b \neq 0$. Define the Riesz projection $E(z_0) : X \to X$ by

$$E(z_0) = \frac{1}{2\pi i}\oint_C (zI-A)^{-1}\,dz,$$

where $C$, the path of integration, is a small circle about $z_0$ containing no other spectral point of $A$. As is well known, $E(z_0)$ is a projection from $X$ onto $X(z_0) = E(z_0)X$ and $A$ commutes with $E(z_0)$. For the spectral theory used in this proof see, e.g., [1], [2], [10]. For each $n = 0, 1, 2, \ldots$ and $z \in \mathbb{C}$ define a subspace $R(z; n)$ of $X$ by

$$R(z; n) = \{x : (zI-A)^n x = 0\}.$$

For each $z \in \mathbb{C}$ define the index $\upsilon(z)$ to be the least integer such that $R(z; \upsilon(z)) = R(z; \upsilon(z)+1)$. Then obviously

$$\{0\} = R(z; 0) \subsetneq R(z; 1) \subsetneq \ldots \subsetneq R(z; \upsilon(z)) = R(z; \upsilon(z)+1).$$

From this we see that

$$\dim R(z_0; \upsilon(z_0)) \geq \upsilon(z_0).$$

It is known that $X(z_0) = R(z_0, \upsilon(z_0)$ and that if $z_0$ is a pole of $(zI-A)^{-1}$ of order $m$ then $\upsilon(z_0) = m$. Therefore

$$\dim X(z_0) \geq m.$$

It is also known that

$$B_{-k} = (A - z_0 I)^{k-1}E(z_0), \qquad k \geq 1.$$

Recalling that $A$ commutes with $E(z_0)$, we get

$$B_{-m}b = (A - z_0 I)^{m-1}E(z_0)b = E(z_0)(A - z_0 I)^{m-1}b.$$

If $B_{-m}b = 0$, then $E(z_0)A^{m-1}b$ is a linear combination of $E(z_0)b, E(z_0)Ab, \ldots, E(z_0)A^{m-2}b$ and thus so is $E(z_0)A^k b$ for $k \geq m$. Since $b$ is cyclic for $A$,

$$\dim X(z_0) = \dim E(z_0)X = \dim E(z_0)\overline{\bigcup_{n=0}^{\infty} \operatorname{Span}_{k=0}^{n}\{A^k b\}} = X \leq m-1.$$

However, this contradicts the fact that $\dim X(z_0) \geq m$.

"Only if". Let

$$(zI-A)^{-1} = B_{-k}(z-z_0)^{-k} + B_{-(k-1)}(z-z_0)^{-(k-1)} + \ldots$$

Since

$$\begin{aligned}\varphi(z) &= \langle c, (zI-A)^{-1}b\rangle \\ &= \langle c, B_{-k}b\rangle(z-z_0)^{-k} + \langle c, B_{-(k-1)}b\rangle(z-z_0)^{-(k-1)} + \ldots,\end{aligned}$$

$k$ must be $\geq m$. Now suppose $k \geq m+1$. Then by the previous discussion deriving the "if" part, $\varphi(z)$ has a pole $z_0$ of order $k$, which is greater than $m$. However, this contradicts the fact that $\varphi(z)$ has a pole $z_0$ of order $m$. Thus $k$ must be equal to $m$. ∎

Remark. An analogous theorem holds for finite-dimensional linear systems (cf. [8]). In [5], Helton has proved an analogous theorem for the infinite-dimensional continuous time case by embedding a (continuously controllable and observable) system into a Lax–Phillips scattering model and using the result of the Lax–Phillips scattering theory. ∎

We now show a result similar to Mlak's theorem (Theorem 3 below). We need the following lemma.

LEMMA 2. *If $h(z) \in H^1$ and is meromorphic in an open set including $\overline{\mathbb{D}}$, then $h(z)$ has no pole in $\overline{\mathbb{D}}$.*

Proof. By the definition of $H^1$ we see that $h(z) \in H^1$ is analytic in $\mathbb{D}$ and thus has no pole in $\mathbb{D}$. So we prove that $h(z)$ has no pole on $\mathbb{T}$. A function $h(z)$ that is meromorphic in an open set containing $\mathbb{T}$ may have only a finite number of poles on $\mathbb{T}$, since if there exist an infinite number of poles on $\mathbb{T}$, then there exists an accumulating point on $\mathbb{T}$ since $\mathbb{T}$ is compact. However, this contradicts the definition of meromorphic functions (see, e.g., [16]). Let the finite number of poles of $h(z)$ on $\mathbb{T}$ (and thus in $\overline{\mathbb{D}}$) be $z_1, \ldots, z_m$. Then at each point $z_i$, $i = 1, \ldots, m$, $h(z)$ can be written locally in the form

$$h(z) = \eta_i(z) + \psi_i(z),$$

where

$$\eta_i(z) = b_{in_i}(z-z_i)^{-n_i} + \ldots + b_{i1}(z-z_i)^{-1}, \qquad b_{in_i} \neq 0,\ i = 1, \ldots, m,$$

and $\psi_i(z)$ is analytic in a neighbourhood of $z_i$. Since $\eta_i(z)$, $i = 1, \ldots, m$, is analytic in $\overline{\mathbb{D}} - \{z_i\}$, it is easily seen that

$$\psi(z) = h(z) - \sum_{i=1}^{m} \eta_i(z)$$

is analytic in $\overline{\mathbb{D}}$. Thus $h(z)$ can be written as

$$h(z) = \psi(z) + \sum_{i=1}^{m} \eta_i(z).$$

Multiplying $h(z)$ by

$$q(z) = (z - z_1)^{n_1 - 1}(z - z_2)^{n_2} \dots (z - z_m)^{n_m} = (z - z_1)^{n_1 - 1} g(z),$$

and noting that $g(z_1) \neq 0$, we obtain

$$q(z)h(z) = \frac{c}{z - z_1} + p(z) + q(z)\psi(z),$$

where $p(z)$ is a polynomial and $c \neq 0$. Since $p(z)$, $q(z)$ and $\psi(z)$ are holomorphic in $\overline{\mathbb{D}}$ and thus bounded in $\overline{\mathbb{D}}$, if $h(z) \in H^1$ then

$$\begin{aligned} \sup_{r<1} \int_0^{2\pi} \left| \frac{c}{re^{i\theta} - z_1} \right| d\theta \\ &\le \sup_{r<1} \int_0^{2\pi} [|q(re^{i\theta})| \cdot |h(re^{i\theta})| + |p(re^{i\theta})| + |q(re^{i\theta})| \cdot |\psi(re^{i\theta})|]\, d\theta \\ &< \infty. \end{aligned}$$

Thus $c/(z - z_1) \in H^1$. However, this is obviously impossible, which follows immediately from the celebrated Hardy inequality, i.e.,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n+1} \le \frac{1}{\pi} \|f\|_1$$

for $f(z) = a_0 + a_1 z + a_2 z^2 + \dots \in H^1$. Therefore $h(z)$ has no pole in $\overline{\mathbb{D}}$. ∎

We can now prove the following theorem.

THEOREM 3. *Assume that* $(I - zA)^{-1}$ *is meromorphic in an open set including* $\overline{\mathbb{D}}$. *Let* $b$ *be cyclic for* $A$ *and* $c$ *be cyclic for* $A^*$. *If* $z^{-1}\varphi(z^{-1}) = \langle c, (I - zA)^{-1} b \rangle \in H^1$, *then* $\varrho(A) < 1$.

Proof. By the assumption and Lemma 2, $z^{-1}\varphi(z^{-1})$ has no pole in $\overline{\mathbb{D}}$. Thus $\varphi(z) = \langle c, (zI - A)^{-1} b \rangle$ has no pole in $\mathbb{C} - \mathbb{D} = \{z : |z| \ge 1\}$. Hence, by Theorem 2, $(zI - A)^{-1}$ has no pole in $\mathbb{C} - \mathbb{D}$. Therefore $\varrho(A) \le 1$. Now suppose $\varrho(A) = 1$. Then there exists an infinite sequence $z_1, z_2, \dots$ in $\sigma(A)$, the spectrum of $A$, such that $|z_i| \to 1$. Since the spectrum of $A$ is compact, a subsequence of $\{z_i\}$ has a limit point $z_0 \in \sigma(A)$ on $\mathbb{T}$. However, again by Lemma 2 and Theorem 2, this is impossible. Thus $\varrho(A) < 1$. ∎

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