On a property of weak resolvents and its application to a spectral problem

by Yoichi Uetake (Poznań)

Abstract. We show that the poles of a resolvent coincide with the poles of its weak resolvent up to their orders, for operators on Hilbert space which have some cyclic properties. Using this, we show that a theorem similar to the Mlak theorem holds under milder conditions, if a given operator and its adjoint have cyclic vectors.

1. Introduction. For a linear bounded operator $A : X \to X$, where $X$ is a Hilbert space, we define a complex-valued function $\varphi(z) = \langle c, (zI - A)^{-1}b \rangle$, which we call a weak resolvent, due to Fong, Nordgren, Radjavi, and Rosenthal (cf. [3], [15]). Here $b, c \in X$, and $\langle f, g \rangle$ denotes the scalar product of the vectors $f$ and $g$. Nordgren et al. considered this function in the study of the invariant subspace problem. Earlier, in the 1960’s, in the model theory of operators, Sz.-Nagy and Foiaş introduced this kind of functions (cf. [17]). Also, in the study of the spectral problem, Mlak proved the following theorem, which also concerns model theory. See also Lebow [12] and Nikol’skiı [14].

Theorem 1 ([13]). If, for every $b, c$ in $X$, $z^{-1}\varphi(z^{-1}) = \langle c, (I - zA)^{-1}b \rangle \in H^1$, then $\rho(A) < 1$. Here $\rho(A)$ is the spectral radius of $A$. 

Janas [7] and Jakóbczak and Janas [6] have extended the above theorem to several commuting operators.

During the 1960’s, Lax and Phillips developed a scattering theory (cf. [11]). Meanwhile, during the same period, engineers developed independently a control theory, initiated by, among others, Kalman (cf. [9], [8]). Surprisingly enough, the above kind of abstract operator theory and these two theories have been shown to be related to one another by Adamyan and Arov (see references in [11]) and Helton ([4], [5]). The weak resolvent cor-

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responds to a scattering matrix in scattering theory and a transfer function (or a frequency response function) in control theory, respectively. The above Mlak theorem is also related to the input-output stability of control systems (cf. [8]).

In our paper, we show that if both \( A \) and its adjoint have cyclic vectors, then the poles of the resolvent of \( A \) and their orders exactly coincide with those of the weak resolvent of \( A \). Next, using this result, we show that such operators, a result similar to Mlak’s theorem holds under milder conditions.

Notations which we use are as follows:

\[
\mathbb{D} = \{ z : |z| < 1 \} \quad \text{(open unit disc in the complex plane)},
\]

\[
\overline{\mathbb{D}} = \{ z : |z| \leq 1 \} \quad \text{(closed unit disc in the complex plane)},
\]

\[
T = \{ z : |z| = 1 \} \quad \text{(unit circle in the complex plane)},
\]

\[
H^1 = \{ f(z) \text{ analytic in } \mathbb{D} : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta = \|f\|_1 < +\infty \}
\]

(\( \text{Hardy space with } p = 1 \)).

2. Main theorems

**Definition.** We say that \( b \) is cyclic for \( A \) iff

\[
\bigcup_{n=0}^{\infty} \text{Span}_{k=0}^{n} \{ A^k b \} = X.
\]

The following lemma is known as the Popov–Belevich–Hautus–Rosenbrock test in control theory for the finite-dimensional case (cf. [8]). To make the paper self-contained, we include the result with a proof for the infinite-dimensional case.

**Lemma 1.** If \( b \) is cyclic for \( A \) and \( x \) is an eigenvector of \( A^* \), i.e., for some \( z_0 \in \mathbb{C} \), \( A^* x = z_0 x \), \( x \neq 0 \), then \( \langle b, x \rangle \neq 0 \).

**Proof.** Suppose \( \langle b, x \rangle = 0 \). Then

\[
\langle A b, x \rangle = \langle b, A^* x \rangle = z_0 \langle b, x \rangle = 0,
\]

\[
\langle A^2 b, x \rangle = \langle A b, A^* x \rangle = z_0 \langle A b, x \rangle = 0,
\]

\[
\vdots
\]

\[
\langle A^k b, x \rangle = \langle A^{k-1} b, A^* x \rangle = z_0 \langle A^{k-1} b, x \rangle = 0 \quad (k = 1, 2, \ldots),
\]

Thus \( x \not\in \bigcup_{n=0}^{\infty} \text{Span}_{k=0}^{n} \{ A^k b \} = X \). However, this contradicts the assumption that \( b \) is cyclic for \( A \). This completes the proof.

In the following theorem and its proof, a pole is an isolated (not accumulating) pole.

**Theorem 2.** Let \((zI - A)^{-1}\) be meromorphic in an open neighborhood of \( z_0 \). Further, let \( b \) be cyclic for \( A \) and \( c \) be cyclic for \( A^* \). Then the weak resolvent \( \varphi(z) = \langle c, (zI - A)^{-1} b \rangle \) has a pole of order \( m \) at \( z = z_0 \) if and only if the resolvent \((zI - A)^{-1}\) has a pole of order \( m \) at \( z = z_0 \).
Proof. “If”. Write \((zI - A)^{-1}\) in the following form:
\[
(zI - A)^{-1} = B_{-m}(z - z_0)^{-m} + B_{-(m-1)}(z - z_0)^{-(m-1)} + \ldots + B_0
+ B_1(z - z_0) + \ldots
\]
From this,
\[
(z_0I - A)B_{-m} = 0.
\]
Suppose \(B_{-m}b \neq 0\). Then, since \(A^*\) is cyclic for \(c\) by assumption, we have \(\langle c, B_{-m}b \rangle \neq 0\) by Lemma 1, and thus the weak resolvent \(\varphi(z)\) has a pole of order \(m\) at \(z = z_0\). Now we show \(B_{-m}b \neq 0\). Define the Riesz projection \(E(z_0) : X \rightarrow X\) by
\[
E(z_0) = \frac{1}{2\pi i} \int_C (zI - A)^{-1} \, dz,
\]
where \(C\), the path of integration, is a small circle about \(z_0\) containing no other spectral point of \(A\). As is well known, \(E(z_0)\) is a projection from \(X\) onto \(X(z_0) = E(z_0)X\) and \(A\) commutes with \(E(z_0)\). For the spectral theory used in this proof see, e.g., [1], [2], [10]. For each \(n = 0, 1, 2, \ldots\) and \(z \in \mathbb{C}\) define a subspace \(R(z; n)\) of \(X\) by
\[
R(z; n) = \{x : (zI - A)^n x = 0\}.
\]
For each \(z \in \mathbb{C}\) define the index \(v(z)\) to be the least integer such that \(R(z; v(z)) = R(z; v(z) + 1)\). Then obviously
\[
\{0\} = R(z; 0) \subset R(z; 1) \subset \ldots \subset R(z; v(z)) = R(z; v(z) + 1).
\]
From this we see that
\[
\dim R(z_0; v(z_0)) \geq v(z_0).
\]
It is known that \(X(z_0) = R(z_0, v(z_0))\) and that if \(z_0\) is a pole of \((zI - A)^{-1}\) of order \(m\) then \(v(z_0) = m\). Therefore
\[
\dim X(z_0) \geq m.
\]
It is also known that
\[
B_{-k} = (A - z_0I)^{k-1}E(z_0), \quad k \geq 1.
\]
Recalling that \(A\) commutes with \(E(z_0)\), we get
\[
B_{-m}b = (A - z_0I)^{m-1}E(z_0)b = E(z_0)(A - z_0I)^{m-1}b.
\]
If \(B_{-m}b = 0\), then \(E(z_0)A^{m-1}b\) is a linear combination of \(E(z_0)b, E(z_0)Ab, \ldots, E(z_0)A^{m-2}b\) and thus so is \(E(z_0)A^k b\) for \(k \geq m\). Since \(b\) is cyclic for \(A\),
\[
\dim X(z_0) = \dim E(z_0)X = \dim E(z_0) \bigcup_{n=0}^{\infty} \text{Span}_{k=0}^n \{A^k b\} = X \leq m - 1.
\]
However, this contradicts the fact that \(\dim X(z_0) \geq m\).
“Only if”. Let 
\[(zI - A)^{-1} = B_{-k}(z - z_0)^{-k} + B_{-(k-1)}(z - z_0)^{-(k-1)} + \ldots \]

Since 
\[\varphi(z) = \langle c, \langle zI - A \rangle^{-1} b \rangle = \langle c, B_{-k}b \rangle(z - z_0)^{-k} + \langle c, B_{-(k-1)}b \rangle(z - z_0)^{-(k-1)} + \ldots ,\]
k must be \(\geq m\). Now suppose \(k \geq m + 1\). Then by the previous discussion deriving the “if” part, \(\varphi(z)\) has a pole \(z_0\) of order \(k\), which is greater than \(m\). However, this contradicts the fact that \(\varphi(z)\) has a pole \(z_0\) of order \(m\). Thus \(k\) must be equal to \(m\).

Remark. An analogous theorem holds for finite-dimensional linear systems (cf. [8]). In [5], Helton has proved an analogous theorem for the finite-dimensional continuous time case by embedding a (continuously controllable and observable) system into a Lax–Phillips scattering model and using the result of the Lax–Phillips scattering theory.

We now show a result similar to Mlak’s theorem (Theorem 3 below). We need the following lemma.

**Lemma 2.** If \(h(z) \in H^1\) and is meromorphic in an open set including \(\mathbb{D}\), then \(h(z)\) has no pole in \(\mathbb{D}\).

**Proof.** By the definition of \(H^1\) we see that \(h(z) \in H^1\) is analytic in \(\mathbb{D}\) and thus has no pole in \(\mathbb{D}\). So we prove that \(h(z)\) has no pole on \(\mathbb{T}\). A function \(h(z)\) that is meromorphic in an open set containing \(\mathbb{T}\) may have only a finite number of poles on \(\mathbb{T}\), since if there exist an infinite number of poles on \(\mathbb{T}\), then there exists an accumulating point on \(\mathbb{T}\) since \(\mathbb{T}\) is compact. However, this contradicts the definition of meromorphic functions (see, e.g., [16]). Let the finite number of poles of \(h(z)\) on \(\mathbb{T}\) (and thus in \(\overline{\mathbb{D}}\)) be \(z_1, \ldots, z_m\). Then at each point \(z_i, i = 1, \ldots, m\), \(h(z)\) can be written locally in the form 
\[h(z) = \eta_i(z) + \psi_i(z),\]

where 
\[\eta_i(z) = b_{m_i} (z - z_i)^{-n_i} + \ldots + b_{i1} (z - z_i)^{-1}, \quad b_{m_i} \neq 0, \quad i = 1, \ldots, m,\]
and \(\psi_i(z)\) is analytic in a neighbourhood of \(z_i\). Since \(\eta_i(z), i = 1, \ldots, m,\) is analytic in \(\mathbb{D} - \{z_i\}\), it is easily seen that 
\[\psi(z) = h(z) - \sum_{i=1}^{m} \eta_i(z)\]
is analytic in \(\overline{\mathbb{D}}\). Thus \(h(z)\) can be written as 
\[h(z) = \psi(z) + \sum_{i=1}^{m} \eta_i(z).\]
Multiplying $h(z)$ by
\[ q(z) = (z - z_1)^{n_1-1}(z - z_2)^{n_2} \cdots (z - z_m)^{n_m} = (z - z_1)^{n_1-1}g(z), \]
and noting that $g(z_1) \neq 0$, we obtain
\[ q(z)h(z) = \frac{c}{z - z_1} + p(z) + q(z)\psi(z), \]
where $p(z)$ is a polynomial and $c \neq 0$. Since $p(z)$, $q(z)$ and \( \psi(z) \) are holomorphic in $\overline{D}$ and thus bounded in $\overline{D}$, if $h(z) \in H^1$ then
\[
\sup_{r<1} \int_0^{2\pi} \left| \frac{c}{re^{i\theta} - z_1} \right| \, d\theta \\
\leq \sup_{r<1} \int_0^{2\pi} \left[ |q(re^{i\theta})| \cdot |h(re^{i\theta})| + |p(re^{i\theta})| + |q(re^{i\theta})| \cdot |\psi(re^{i\theta})| \right] \, d\theta \\
< \infty.
\]
Thus $c/(z - z_1) \in H^1$. However, this is obviously impossible, which follows immediately from the celebrated Hardy inequality, i.e.,
\[
\sum_{n=1}^{\infty} \frac{|a_n|}{n+1} \leq \frac{1}{\pi} \|f\|_1
\]
for $f(z) = a_0 + a_1z + a_2z^2 + \ldots \in H^1$. Therefore $h(z)$ has no pole in $\overline{D}$.

We can now prove the following theorem.

**Theorem 3.** Assume that $(I - zA)^{-1}$ is meromorphic in an open set including $\overline{D}$. Let $b$ be cyclic for $A$ and $c$ be cyclic for $A^*$. If $z^{-1}\varphi(z^{-1}) = \langle c, (I - zA)^{-1}b \rangle \in H^1$, then $\varrho(A) < 1$.

**Proof.** By the assumption and Lemma 2, $z^{-1}\varphi(z^{-1})$ has no pole in $\overline{D}$. Thus $\varphi(z) = \langle c, (zI - A)^{-1}b \rangle$ has no pole in $C - D = \{z : |z| \geq 1\}$. Hence, by Theorem 2, $(zI - A)^{-1}$ has no pole in $C - D$. Therefore $\varrho(A) \leq 1$. Now suppose $\varrho(A) = 1$. Then there exists an infinite sequence $z_1, z_2, \ldots$ in $\sigma(A)$, the spectrum of $A$, such that $|z_i| \to 1$. Since the spectrum of $A$ is compact, a subsequence of $\{z_i\}$ has a limit point $z_0 \in \sigma(A)$ on $T$. However, again by Lemma 2 and Theorem 2, this is impossible. Thus $\varrho(A) < 1$.

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References


Department of Mathematics and Computer Science
Adam Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
E-mail: uetake@math.amu.edu.pl

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