On operators with unitary $\varrho$-dilations

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To the memory of Professor Włodzimierz Mlak

Abstract. We show a polynomially boundend operator $T$ is similar to a unitary operator if there is a singular unitary operator $W$ and an injection $X$ such that $XT = WX$. If, in addition, $T$ is of class $C_\varrho$, then $T$ itself is unitary.

According to Sz.-Nagy and Foiaş [5], a (bounded linear) operator $T$ on a separable Hilbert space $\mathcal{H}$ is said to be of class $C_\varrho$ with $\varrho > 0$ if there exists a unitary operator $U$ on a Hilbert space $\mathcal{K} (\supset \mathcal{H})$ such that $T^n = \varrho P_\mathcal{H} U^n |\mathcal{H}$ for $n = 1, 2, \ldots$, where $P_\mathcal{H}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. For $\varrho = 2$ it is known (see [5, Chapter I, Proposition 11.2]) that $T$ is of class $C_2$ if and only if its numerical radius $w(T) := \sup\{|(Tx, x) : \|x\| \leq 1\}$ is not greater than one. In this paper we show that if $T$ is of class $C_\varrho$ and there exist a singular unitary operator $W$ and an injection $X$ such that $XT = WX$, then $T$ is unitary. Here a unitary operator is singular by definition if its spectral measure is singular with respect to the (linear) Lebesgue measure on the unit circle $\mathbb{T}$. Such a situation occurs in connection with a compact operator $A$, as observed by Watanabe [6], which satisfies $|(Ax, x)| \leq (|A|x, x)$ for all $x$. Our result gives an affirmative answer to a conjecture that such an operator $A$ is normal. Clearly, if $T$ is of class $C_\varrho$, then $T$ is polynomially bounded, i.e., there exists a constant $M$ such that $\|p(T)\| \leq M \max\{|p(z)| : |z| = 1\}$ for every polynomial $p$. In our main result (Theorem 1) an assertion for the case of a polynomially bounded operator $T$ is also included. Though this part can be derived from a result of Mlak [2] (see also [3]), our proof is quite different from Mlak’s.

Let $A(\mathbb{T})$ be the disk algebra, that is, $A(\mathbb{T})$ is the norm closure of polynomials in the algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ with norm

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\[ \|f\| = \sup \{|f(z)| : z \in \mathbb{T} \} \text{ for } f \in C(\mathbb{T}). \]

If \( T \) is a polynomially bounded operator, then there exists a bounded algebra homomorphism, \( f \mapsto f(T) \), from \( A(\mathbb{T}) \) to the uniformly closed algebra generated by \( T \) and \( I \) which maps each polynomial \( p \) to \( p(T) \).

**Theorem 1.** Let \( T \) be an operator on \( \mathcal{H} \), and suppose that there exist a singular unitary operator \( W \) on \( \mathcal{G} \) and an injection \( X : \mathcal{H} \to \mathcal{G} \) such that \( XT = WX \).

(i) If \( T \) is polynomially bounded, then \( T \) is similar to a unitary operator.

(ii) If \( T \) is of class \( C_p \), then \( T \) is unitary.

**Proof.** (i) Let \( W = \int_\mathbb{C} z \, dE(z) \) be the spectral decomposition of \( W \). Let us first show that for every closed set \( \delta (\subset \mathbb{T}) \) of Lebesgue measure zero, there exists an idempotent \( Q \) on \( \mathcal{H} \) such that

\[ XQ = E(\delta)X \quad \text{and} \quad \|Q\| \leq M, \]

where \( M = \sup \{\|h(T)\| : h \in A(\mathbb{T}) \text{ and } \|h\| \leq 1\} \). For such a set \( \delta \), we can take a function \( g \) in \( A(\mathbb{T}) \) such that \( g(z) = 1 \) for \( z \in \delta \) and \( |g(z)| < 1 \) for \( z \in \mathbb{T} \setminus \delta \) (see [1, p. 81]). Then, since \( T \) is polynomially bounded, \( g^n(T) \) is well defined and it follows from the identity \( XT = WX \) that \( Xg^n(T) = g^n(W)X \). Clearly, \( g^n(W) \) converges strongly to \( E(\delta) \), so

\[ \lim_{n \to \infty} (g^n(T)x, X^*y) = (E(\delta)x, y) \quad \text{for } x \in \mathcal{H} \text{ and } y \in \mathcal{G}. \]

Hence, since \( \|g^n(T)\| \leq M \|g^n\| = M \) for \( n = 1, 2, \ldots \) and \( X^* \) has dense range, \( g^n(T) \) converges weakly to an operator \( Q \) such that \( XQ = E(\delta)X \) and \( \|Q\| \leq M \).

Since \( X \) is injective, the identity \( XQ = E(\delta)X \) shows that \( Q \) is idempotent.

Now we prove that \( T \) is invertible and \( \|T^k\| \leq M^2 \) for \( k = 0, \pm 1, \pm 2, \ldots \).

Then it follows from the theorem of Sz.-Nagy [4] that \( T \) is similar to a unitary operator. Since \( W \) is singular, there exists a sequence \( \{\delta_n\} \) of closed sets with Lebesgue measure zero such that \( E(\delta_n) \) converges to the identity \( I \) as \( n \to \infty \). Applying the fact shown above to \( \delta = \delta_n \), we obtain an idempotent \( Q_n \) such that \( XQ_n = E(\delta_n)X \) and \( \|Q_n\| \leq M \). For each \( k \) and \( n = 1, 2, \ldots \), take an \( h_{k,n} \in A(\mathbb{T}) \) such that \( h_{k,n}(z) = z^{-k} \) for \( z \in \delta_n \) and \( \|h_{k,n}\| \leq 1 \) (see [1, p. 81]). Then we have

\[ Xh_{k,n}(T)Q_n = h_{k,n}(W)E(\delta_n)X = W^{*k}E(\delta_n)X. \]

For each \( k \), \( W^{*k}E(\delta_n) \) converges strongly to \( W^{*k} \) as \( n \to \infty \) and, for \( n = 1, 2, \ldots \), \( \|h_{k,n}(T)Q_n\| \leq M^2 \). Therefore we can conclude that \( h_{k,n}(T)Q_n \) converges weakly to an operator \( S_k \) such that \( XS_k = W^{*k}X \) and \( \|S_k\| \leq M^2 \). Since

\[ XS_kT^k = W^{*k}XT^k = W^{*k}W^kX = X \quad \text{and} \quad XT^kS_k = W^kW^{*k}X = X, \]

the injectivity of \( X \) shows that \( T \) is invertible and \( T^{-k} = S_k \) for \( k = 1, 2, \ldots \), so that \( \|T^k\| \leq M^2 \) for all \( k = 0, \pm 1, \pm 2, \ldots \).
(ii) Let $U$ be a unitary $g$-dilation of $T$ on $\mathcal{K}$, i.e., a unitary operator such that $T^n = gP_\mathcal{K}U^n|\mathcal{K}$ for $n = 1, 2, \ldots$, and let $U = \int_T z \, dF(z)$ be the spectral decomposition of $U$. For any closed set $\delta$ with Lebesgue measure zero, let $g$ and $Q$ be as in the proof of (i). Since the idempotent $Q$ is a weak limit of $g^n(T)$ and
\[ g^n(T) = P_\mathcal{K}[gg^n(U) + (1 - g)g^n(0)1_\mathcal{K}]|\mathcal{H}, \]
we have $Q = gP_\mathcal{K}F(\delta)|\mathcal{K}$ because $g^n(0) \to 0$ as $n \to \infty$, so that $Q$ is self-adjoint. Hence it follows from $XQ = E(\delta)X$ that $E(\delta)XX^* = XQX^*$ and so $XX^*$ commutes with $E(\delta)$. Then, since $W$ is singular, $XX^*$ commutes with $E(\delta)$ for any Borel set $\alpha$ and so commutes with $W$. Thus $W|\text{ran } X)^-$ is unitary and, using the polar decomposition of $X^*$, we can conclude that $T$ is unitarily equivalent to $W|\text{ran } X)^-$, so $T$ itself is unitary. This completes the proof.

Clearly, a polynomially bounded operator $T$ is power-bounded, that is, $\sup\{\|T^n\| : n = 1, 2, \ldots\} < \infty$. When an operator $T$ is power-bounded, by requiring compactness of the intertwining operator $X$ in Theorem 1(i) we can obtain a similar conclusion.

**Theorem 2.** Let $T$ be a power-bounded operator on $\mathcal{H}$ and let $V$ be an isometry on $\mathcal{G}$. If there exists a compact injection $K : \mathcal{H} \to \mathcal{G}$ having dense range such that $KT = VK$, then $V$ is a singular unitary operator, and $T$ is similar to a unitary operator.

**Proof.** Since $T$ is power-bounded, we can take a subsequence $\{T^n(k)\}$ of $\{T^n\}$ which converges weakly to an operator $S$ as $k \to \infty$. Then, since $KT^n(k) = V^n(k)K$ for $k = 1, 2, \ldots$ and $K$ is compact, $V^n(k)K$ converges strongly to $KS$. But $K$ has dense range, so it follows that $V^n(k)$ converges strongly to an isometry $W$. Considering the Wold decomposition of $V$ (see [5, Chapter I, Theorem 1.1]) and the decomposition of its unitary part into the sum of the singular and absolutely continuous summands, we see that $V$ is singular unitary because $U^n \to 0$ weakly as $n \to \infty$ for an isometry $U$ whose unitary part is absolutely continuous. Next, for integers $j, l$ and $k$ with $n(k) > n(l) + j$, we have
\[ KT^n(k) - n(l) - j = V^{*(j+n(l))}V^n(k)K \]
and $V^{*(j+n(l))}V^n(k)$ converges weakly to $V^{*(j+n(l))}W$ as $k \to \infty$. Hence, since $T$ is power-bounded and $K^*$ has dense range by the injectivity of $K$, $T^n(k) - n(l) - j$ converges weakly to an operator $S_{j,l}$ as $k \to \infty$, which satisfies
\[ KS_{j,l} = V^{*(j+n(l))}WK \quad \text{and} \quad \|S_{j,l}\| \leq M, \]
where $M = \sup\{\|T^n\| : n = 0, 1, 2, \ldots\}$. Also, for each $j$, $V^{*(j+n(l))}W$ converges weakly to $V^*$ as $l \to \infty$ (because $W$ is isometric). So, letting $l \to \infty$ in the identity $KS_{j,l} = V^{*(j+n(l))}WK$, we get an operator $S_j$ such that
KS_j = V^*j K \text{ and } \|S_j\| \leq M. \text{ Thus, as in the proof of Theorem 1(i), it follows that } T \text{ is invertible and } \|T^{-j}\| \leq M \text{ for } j = 1, 2, \ldots, \text{ and by the theorem of Sz.-Nagy [4], } T \text{ is similar to a unitary operator.}

Theorem 1(ii) and Theorem 2 can give an affirmative answer to the question posed in [6]:

**Corollary 3.** If $A$ is a compact operator and satisfies $|\langle Ax, x \rangle| \leq (\|A\|_x, x)$ for all $x$, then $A$ is normal.

**Proof.** Let $A = V|A|$ be the polar decomposition. By [6, Theorem 2.1] there exists an operator $T$ with $w(T) \leq 1$ such that $V|A|^{1/2} = |A|^{1/2}T$. Let $\mathcal{M} = (\text{ran }|A|)^{-}$. The identity $V|A|^{1/2} = |A|^{1/2}T$ implies that $\mathcal{M}$ is invariant for $V$. Let $V_1 = V|\mathcal{M}$ and $T_1 = P_M T|\mathcal{M}$. Then $V_1$ is isometric and $T_1$ belongs to the class $C_2$. Also, the operator $X = |A|^{1/2}|\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}$ is a compact injection with dense range and satisfies $XT_1 = V_1 X$. So, by Theorem 2 and the proof of Theorem 1(ii), $V_1$ is a unitary operator which commutes with $|A||\mathcal{M}$. Hence it follows that $A$ is normal.

**References**


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