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Circular operators related to some quantum observables

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Abstract. Circular operators related to the operator of multiplication by a homomorphism of a locally compact abelian group and its restrictions are completely characterized. As particular cases descriptions of circular operators related to various quantum observables are given.

1. Introduction. This paper was inspired by the last work of Professor Włodzimierz Mlak, which he carried out in final years of his life and which he announced first in [M1] and subsequently presented in a series of papers [M2]. May this paper be a tribute to his memory—the memory of my Professor, Master, and Friend.

The idea of circular operators is quantum-mechanical and goes back at least to quantum phase "sine" and "cosine" operators. Its physical origins are discussed in [I] and [MS]. Mlak in [M1] was the first to introduce a proper mathematical formulation of these physical properties and began a systematic investigation of circular operators. In [AHHK] circular operators were defined which, in Mlak's terminology, are, actually, $(A,1)$-circular (see below), but the results there concern mostly operators without that property.

$B(H)$ denotes the algebra of all linear, bounded operators in a complex Hilbert space H, which, in general, is assumed to be separable, infinite-dimensional. I_H, or simply I, stands for the identity operator in H. Let A be a densely defined self-adjoint operator in H. Following Mlak, for $\alpha \in \mathbb{R}$ (the set of all real numbers), an operator $T \in B(H)$ is called (A,α)-*circular* if

$$\text{(C)} \qquad e^{-itA} T e^{itA} = e^{it\alpha} T \quad \text{for each } t \in \mathbb{R}.$$

The unitary group $t \to e^{itA}$ is called the *circulating group for T*.

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The roots of this circular relation (C) can, however, be traced to the very foundations of quantum mechanics, namely, to the Heisenberg uncertainty principle expressed by the Canonical Commutation Relation (here in Weyl's form, for one particle)

$$\text{(CCR)} \qquad e^{-itQ}e^{isP}e^{itQ} = e^{its}e^{isP}, \qquad s,t \in \mathbb{R},$$

where Q is the *position operator* in $L^2(\mathbb{R})$ with the Lebesgue measure on $\mathbb{R}$, $(Qf)(x) = xf(x)$, $x \in \mathbb{R}$, P is the *momentum operator*

$$Pf = \frac{1}{i}\frac{d}{dx}f,$$

and the Planck constant $\hbar$ is set to 1. These operators are self-adjoint, defined on suitable dense domains (cf. [Ho] or any text on introductory quantum mechanics).

Moreover, $(e^{isP}f)(x) = f(x+s)$, $f \in L^2(\mathbb{R})$, $x, s \in \mathbb{R}$ (cf. e.g. [Ho], p. 62), i.e., iP is the infinitesimal generator of the unitary group of left translations $s \to L_s$:

$$(L_sf)(x) = f(x+s), \qquad f \in L^2(\mathbb{R}),\ x,s \in \mathbb{R}.$$

Hence (CCR) can be read as follows: L_s is (Q,s)-circular for each $s \in \mathbb{R}$. Circularity is also seen clearly in the "time–energy" uncertainty relation written in the Weyl form—cf. [Ho], Ch. III, 8, (8.9).

In [M2], part III, Mlak found the form of all $(A,-1)$-circular operators with a generalized quantum number operator A. The present paper characterizes certain general circular operators in Theorems (3.1) and (4.3). From these general results circular operators are completely described in several physically important cases, e.g., with number operator (in particular, obtaining the just mentioned Mlak result), with position operator (cf. (CCR)), or with energy time evolution circulating group ("time–energy" uncertainty).

2. Preliminaries. Let (X,μ) be a measure space with a σ-finite measure μ, i.e., X is a countable union of sets of finite measure. Let K be a Hilbert space. $H = L^2(X,K)$ is the Hilbert space of all (equivalence classes of) functions $f : X \to K$ such that $\int_X \|f(x)\|^2\,d\mu(x) < \infty$. An operator B in H *is given by a measurable field* $X \to B(K)$, $x \to B(x)$, if the function $x \to B(x)f(x)$ is measurable for each $f \in H$ and

$$(Bf)(x) = B(x)f(x) \quad \text{for } x \in X \quad \text{and}$$
$$f \in D(B) = \Big\{f \in H : \int\limits_X \|B(x)f(x)\|^2\,d\mu(x) < \infty\Big\}.$$

If, in addition, $\sup_X \|B(x)\| < \infty$, then such a B is called *decomposable* and $B \in B(H)$. If $u \in L^\infty(X)$, then the field $X \to B(K)$, $x \to u(x)I_K$, is measurable and the operator

$$(Df)(x) = u(x)f(x), \qquad x \in X,\ f \in H,$$

is called *diagonalizable*. Clearly, each decomposable operator commutes with each diagonalizable operator. Very important and non-trivial is, however, the converse:

(2.1) *Each bounded operator in H that commutes with each diagonalizable operator is decomposable*

— cf. Theorem 1 of [D], Ch. II, 5, where one finds a general direct integral decomposition theory, from which only the necessary facts are recalled here for a constant field of Hilbert spaces $x \to K(x) = K$.

Let G be a locally compact abelian (LCA) group. Choose and fix a Haar measure m in G (cf. e.g. [Ha]). Let K be a Hilbert space. In $H = L^2(G, K)$ the *left translation operator L_g by $g \in G$* is defined by $(L_g f)(x) = f(x + g)$ for $x \in G$, $f \in H$. Since Haar measures are translation-invariant, L_g is a unitary operator in H and $L_g^* = L_g^{-1} = L_{-g}$.

Let H be a Hilbert space. A *projection* always means an orthogonal projection. Let $D(A) \subset H$ be a linear manifold and let $A : D(A) \to H$ be a linear operator. If $B \in B(H)$, then $[A, B] = 0$ means that $BD(A) \subset D(A)$ and $(AB - BA)x = 0$ for $x \in D(A)$.

(2.2) Lemma. *Let $U \in B(H)$ be unitary, $S, T \in B(H)$, and γ, δ be complex numbers. If $U^*SU = \gamma S$ and $U^*TU = \delta T$, then $U^*STU = \gamma\delta ST$.*

Proof. $U^*STU = U^*SUU^*TU = \gamma S \delta T$. ∎

Some elementary properties of circular operators are gathered in the following

(2.3) Proposition. *Let A be a self-adjoint densely defined operator in H, let $\alpha, \beta \in \mathbb{R}$ and $S, T \in B(H)$.*

(i) *If S is (A, α)-circular, T is (A, β)-circular, then ST is $(A, \alpha + \beta)$-circular.*

(ii) *T is $(A, 0)$-circular if and only if $[T, A] = 0$.*

(iii) *If T is (A, α)-circular and $[S, e^{itA}] = 0$ for each $t \in \mathbb{R}$, then TS is (A, α)-circular.*

Proof. (i) follows from Lemma (2.2). (ii) is a consequence of Proposition 1.0 in [M2], Part I. (iii) is obvious. ∎

3. Circular operators in $L^2(G, K)$. Let G be a LCA group with a fixed Haar measure m. From now on dx will mean $dm(x)$ in the integral notation. Let K be a Hilbert space and let $H = L^2(G, K)$. Suppose $a : G \to \mathbb{R}$ is a homomorphism of G into the additive group $\mathbb{R}$, i.e., $a(x+y) = a(x) + a(y)$, $x, y \in G$. Let

$$D(A) = \Big\{ f \in H : \int_X |a(x)|^2 \|f(x)\|^2\, dx < \infty \Big\}$$

and define $A : D(A) \to H$ by $(Af)(x) = a(x)f(x)$, $x \in G$, $f \in D(A)$.

Assume that $D(A)$ is dense in H. A simple argument, e.g., using the Cayley transform, shows that A is self-adjoint. The following theorem describes circular operators with circulating group $t \to e^{itA}$.

(3.1) Theorem. *Let $g \in G$.*

(i) *$L_g B$ is $(A, a(g))$-circular for each decomposable operator $B \in B(H)$.*

(ii) *Suppose additionally that G is σ-compact* (*i.e., is a countable union of compact subsets*) *and that $L^\infty(G) = \{u \circ a : u \in L^\infty(\mathbb{R})\}$. If $T \in B(H)$ is $(A, a(g))$-circular, then there is a decomposable operator $B \in B(H)$ such that $T = L_g B$.*

Proof. (i) First it will be shown that L_g is $(A, a(g))$-circular. Take $x \in X$, $f \in H$. Then $(e^{itA} f)(x) = e^{ita(x)} f(x)$. Hence

$$(L_g e^{itA} f)(x) = e^{ita(x+g)} f(x+g) = e^{it(a(x)+a(g))} f(x+g).$$

Finally, $(e^{-itA} L_g e^{itA} f)(x) = e^{-ita(x)} e^{ita(x)} e^{ita(g)} f(x+g) = e^{ita(g)} (L_g f)(x)$. Let $B \in B(H)$ be a decomposable operator. Since e^{itA} is a diagonalizable operator, $[B, e^{itA}] = 0$ for each $t \in \mathbb{R}$. By Proposition (2.3)(iii), $L_g B$ is $(A, a(g))$-circular.

(ii) Suppose now that T is $(A, a(g))$-circular. As shown in (i), $L_g^* = L_{-g}$ is $(A, -a(g))$-circular, because $a(-g) = -a(g)$. By Proposition (2.3)(i), $B = L_g^* T$ is $(A, 0)$-circular. By Proposition (2.3)(ii), $[B, A] = 0$. Thus $[B, u(A)] = 0$ for each $u \in L^\infty(\mathbb{R})$. Now, $(u(A)f)(x) = (u \circ a)(x) f(x)$, $x \in G$, $f \in H$. The assumption $L^\infty(G) = \{u \circ a : u \in L^\infty(\mathbb{R})\}$ implies that B commutes with each diagonalizable operator in H. Since G is σ-compact, the Haar measure m on G is σ-finite; cf. [Ha], p. 256, (9). By (2.1), B is decomposable. Clearly, $T = L_g B$. ∎

The interpretation of this theorem for two particular cases important in quantum mechanics is now in order.

Case 1: *$G = \mathbb{Z}$ = the group of integers, A is the generalized number operator.* The Haar measure on $\mathbb{Z}$ is the counting measure. $H = L^2(\mathbb{Z}, K)$ can be seen as $\bigoplus_{n=-\infty}^{\infty} K_n$, $K_n = K$, i.e., as $l^2(K)$ = the space of doubly infinite square norm-summable sequences of vectors in K. If $g = 1$, then $L_g = L_1$ is the backward bilateral shift of multiplicity $\dim K$ and $L_n = (L_1)^n$. Let $a : \mathbb{Z} \to \mathbb{R}$ be defined by $a(n) = n$, $n \in \mathbb{Z}$. Then $(Af)(n) = nf(n)$, $n \in \mathbb{Z}$, $f \in H$, is the *generalized number operator*. Further, $D(A) = \{f \in H : \sum_{n=-\infty}^{\infty} n^2 \|f(n)\|^2 < \infty\}$ is dense in H, because it contains all $f \in H$ that vanish off a finite subset of $\mathbb{Z}$. Moreover, $L^\infty(\mathbb{Z}) = l^\infty$ = the space of all doubly infinite bounded complex sequences $= \{u \circ a : u \in L^\infty(\mathbb{R})\}$, the group $\mathbb{Z}$ is σ-compact and the Haar measure is σ-finite. Now Theorem (3.1) reads:

(3.2) COROLLARY. *Let $n \in \mathbb{Z}$. Then $T \in B(H)$ is (A,n)-circular if and only if $T = L_n B$ with a decomposable operator B.*

In the matrix interpretation of operators in $H = \bigoplus_{n=-\infty}^{\infty} K_n,\ K_n = K$, a decomposable operator B given by a measurable field $\mathbb{Z} \to B(K),\ n \to B(n)$, is a diagonal matrix with the diagonal entries $B(n)$. Hence, in particular, $L_{-1}B$ is a bilateral forward weighted shift with weights $B(n)$.

(3.3) COROLLARY. *$T \in B(H)$ is $(A,-1)$-circular if and only if T is a bilateral forward operator weighted shift.*

This is exactly Theorem 2.1 of [M2], Part III.

Case 2: *G is the additive group $\mathbb{R}$, A is the position operator.* The Haar measure m is the Lebesgue measure. Let $H = L^2(\mathbb{R}, K)$. For $g = s \in \mathbb{R}$, $(L_s f)(x) = f(x+s)$ is the backward translation operator. Consider a = the identity mapping in $\mathbb{R}$. Then $(Af)(x) = xf(x),\ x \in \mathbb{R},\ f \in H$, is the *position operator*, which will be denoted by Q. Its domain $D(Q) = \{f \in H : \int_{-\infty}^{\infty} |x|^2 \|f(x)\|^2\,dx < \infty\}$ is dense in H, because it contains all functions that vanish outside a finite interval. $\mathbb{R}$ is σ-compact. Hence Theorem (3.1) characterizes circular operators with circulating group $t \to e^{itQ}$ and reads:

(3.4) COROLLARY. *$T \in L^2(\mathbb{R}, K)$ is (Q,s)-circular if and only if there exists a decomposable operator $B \in B(L^2(\mathbb{R}, K))$ such that $T = L_s B$.*

Operators of the form $L_s B$ with a decomposable $B \in B(H)$ are continuous analogues of bilateral operator weighted shifts discussed in Case 1.

4. Restrictions to subsets of G. Let G be a LCA group with a fixed Haar measure m. Let K be a Hilbert space, $H = L^2(G, K)$, as at the beginning of Section 3. Let X be a measurable subset of G. Denote $H_X = L^2(X, K)$, with the measure m restricted to X. Define $J : H_X \to H$ by

$$H_X \ni \varphi \to (J\varphi)(x) = \begin{cases} \varphi(x) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Then J is an isometric embedding of H_X into H and maps unitarily H_X onto $JH_X = H_0$, which consists of all functions $f \in H$ that vanish outside X. The adjoint $J^* : H \to H_X$ is the operator of restriction to X:

$$J^* f = f|_X \quad \text{for } f \in H.$$

Moreover, J^*J is the identity operator in H_X and JJ^* is the projection of H onto H_0:

$$JJ^*f = \chi_X f, \qquad f \in H,$$

where χ_X is the characteristic function of X.

(4.1) LEMMA. *Suppose an operator B in H is given by a measurable field $G \to B(K)$, $x \to B(x)$, on $D(B) = \{f \in H : \int_G \|B(x)f(x)\|^2\,dx < \infty\}$. Then*

(i) $B(H_0 \cap D(B)) \subset H_0$,

(ii) $D(J^*BJ) = \{\varphi \in H_X : \int_X \|B(x)\varphi(x)\|^2\,dx < \infty\}$ *and* $J(D(J^*BJ)) = H_0 \cap D(B)$,

(iii) J^*BJ *in* H_X *is given by the measurable field* $X \to B(K)$, $x \to B_X(x)$, *where* $B_X(x) = B(x)$, $x \in X$,

(iv) *if* $D(B)$ *is dense in* H, *then* $D(J^*BJ)$ *is dense in* H_X.

Proof. $H_0 \cap D(B)$ consists of all functions $f \in D(B)$ that vanish outside X. Hence (i) is clear. To prove (ii) it is enough to notice that

$$\int_G \|B(x)(J\varphi)(x)\|^2\,dx = \int_X \|B(x)\varphi(x)\|^2\,dx \qquad \text{for } \varphi \in H_X,$$

which is understood that if one integral is finite, so is the other, and they are equal. If $\varphi \in D(J^*BJ)$, then $(BJ\varphi)(x) = B(x)(J\varphi)(x) = B(x)\varphi(x)$ for $x \in X$ and 0 otherwise. Thus $(J^*BJ\varphi)(x) = B_X(x)\varphi(x)$, $x \in X$, which proves (iii). Finally, it is elementary that the image of a dense subset of a Hilbert space under a projection is dense in the range of that projection. Hence $JJ^*D(B) = H_0 \cap D(B)$ is dense in H_0. Since J maps unitarily H_X onto H_0, (iv) follows from (ii). ∎

Let $a : G \to \mathbb{R}$ be a group homomorphism. Let $(Af)(x) = a(x)f(x)$, $x \in G$, $f \in D(A) = \{f \in H : \int_G |a(x)|^2 \|f(x)\|^2\,dx < \infty\}$. It follows from Lemma (4.1)(iii) that $(J^*AJ\varphi)(x) = a(x)\varphi(x)$, $x \in X$, $\varphi \in D(J^*AJ)$. Hence

$$(e^{itJ^*AJ}\varphi)(x) = e^{ita(x)}\varphi(x) = (J^*e^{itA}J\varphi)(x) \qquad \text{for } x \in X,\ \varphi \in H_X,\ t \in \mathbb{R}. \tag{4.2}$$

Suppose now that $D(A)$ is dense in H. By Lemma (4.1)(iv), J^*AJ is densely defined. Thus J^*AJ is self-adjoint. The following theorem characterizes circular operators with circulating group $t \to e^{itJ^*AJ}$.

(4.3) THEOREM. *Fix $g \in G$.*

(i) J^*L_gJC *is* $(J^*AJ, a(g))$*-circular for each decomposable operator* $C \in B(H_X)$.

(ii) *Suppose G is σ-compact and $L^\infty(G)=\{u\circ a : u\in L^\infty(\mathbb{R})\}$. If $S\in B(H_X)$ is $(J^*AJ, a(g))$-circular, then there exists a decomposable operator $C\in B(H_X)$ such that $S=J^*L_gJC$.*

P r o o f. To prove (i) it is enough to show that J^*L_gJ is $(J^*AJ, a(g))$-circular, because, from (4.2), e^{itJ^*AJ} is a diagonalizable operator for $t\in\mathbb{R}$, thus it commutes with each decomposable operator. Proposition (2.3)(iii) will then finish the proof.

Now notice that

$$(L_g\psi)(x)=\begin{cases}\psi(x+g) & \text{if } x+g\in X,\\ 0 & \text{otherwise,}\end{cases}$$

for each $\psi\in H_0,\ x\in G$. Since $JH_X=H_0$, for $\varphi\in H_X,\ x\in X$, one has $(J^*L_gJ\varphi)(x)=\varphi(x+g)$ if $x+g\in X$ and 0 if $x+g\notin X$. Take $x\in X$ such that $x+g\in X$. Then

$$\begin{aligned}(e^{-itJ^*AJ}J^*L_gJe^{itJ^*AJ}\varphi)(x)&=e^{-ita(x)}e^{ita(x+g)}\varphi(x+g)\\&=e^{ita(g)}\varphi(x+g)=(e^{ita(g)}J^*L_gJ\varphi)(x).\end{aligned}$$

If $x\in X$ is such that $x+g\notin X$, then the first and the last expressions in this equality are 0. Hence J^*L_gJ is $(J^*AJ, a(g))$-circular.

(ii) Let now $S\in B(H_X)$ be $(J^*AJ, a(g))$-circular. First it will be shown that $JSJ^*\in B(H)$ is $(A, a(g))$-circular, i.e., that

$$(e^{-itA}JSJ^*e^{itA}f)(x)=e^{ita(g)}(JSJ^*f)(x)\quad \text{for } f\in H,\ x\in G.$$

If $x\notin X$, then both sides equal 0, because $S\in B(H_X)$ and $(J\varphi)(x)=0$ for $\varphi\in H_X,\ x\notin X$. Hence it remains to check this equality for $f\in H_0$. If $f\in H_0$, then $f=JJ^*f=J\varphi$, where $\varphi=J^*f\in H_X$. Take $x\in X$, and using (4.2), compute

$$\begin{aligned}(e^{-itA}JSJ^*e^{itA}f)(x)&=(J^*e^{-itA}JSJ^*e^{itA}J\varphi)(x)\\&=(e^{-itJ^*AJ}Se^{itJ^*AJ}\varphi)(x)=(e^{ita(g)}S\varphi)(x)\\&=(e^{ita(g)}SJ^*f)(x)=(e^{ita(g)}JSJ^*f)(x).\end{aligned}$$

Hence JSJ^* is $(A, a(g))$-circular. By Theorem (3.1), there exists a decomposable operator $B\in B(H)$ such that $JSJ^*=L_gB$. Hence $S=J^*L_gBJ$. It follows from Lemma (4.1)(i) that $BJH_X=BH_0\subset H_0$. Hence $BJ=JJ^*BJ$ and $S=(J^*L_gJ)(J^*BJ)$. By Lemma (4.1)(iii), $J^*BJ\in B(H_X)$ is a decomposable operator. ■

Now, as in Section 3, the interpretation of Theorem (4.3) will be given for two cases.

C a s e 1: $G=\mathbb{Z}$, *m is the counting measure on $\mathbb{Z}$, $H=L^2(\mathbb{Z},K)$, $a:\mathbb{Z}\to\mathbb{R}$, $a(n)=n$ and $(Af)(n)=nf(n)$, $n\in\mathbb{Z}$, is the generalized number operator.* First take $X=\mathbb{Z}_+=$ the set of all non-negative integers.

Then $L^2(\mathbb{Z}_+, K) = l^2_+(K) =$ the Hilbert space of all square norm-summable sequences $(\varphi(n))_0^\infty$, $\varphi(n) \in K$, and $(J^*L_{-1}J\varphi)(n) = \varphi(n-1)$ if $n \geq 1$, $(J^*L_{-1}J\varphi)(1) = 0$. Hence $J^*L_{-1}J$ is the forward unilateral shift of multiplicity $\dim K$. If $H_X = L^2(\mathbb{Z}_+, K)$ is treated as $\bigoplus_{n=0}^\infty K_n$, $K_n = K$, then a decomposable operator $C \in B(H_X)$ given by a measurable field $\mathbb{Z}_+ \to B(K)$, $n \to C(n)$, is a diagonal matrix with diagonal entries $C(n)$ and $L_{-1}C$ is a forward unilateral operator weighted shift with weights $C(n)$. Finally, $(J^*AJ\varphi)(n) = n\varphi(n)$, $n \in \mathbb{Z}_+$, $\varphi \in D(J^*AJ)$, i.e., J^*AJ is precisely the *quantum number operator* N (cf. e.g. [Ho], Ch. III, Section 9). Hence Theorem (4.3) reads:

(4.4) Corollary. *$T \in B(L^2(\mathbb{Z}_+, K))$ is $(N, -1)$-circular if and only if T is a forward unilateral operator weighted shift.*

This is exactly Theorem 2.0 of [M2], Part III.

Now take $X_k = \{1, \ldots, k\}$, $k \in \mathbb{Z}$. Then $H_{X_k} = L^2(X_k, K) = \bigoplus_{n=1}^k K_n$, $K_n = K$. The operator $J^*L_{-1}J$ is a forward truncated shift of multiplicity $\dim K$. A decomposable operator $C \in B(H_{X_k})$ given by a measurable field $X \to B(K)$, $n \to C(n)$, is a diagonal matrix with diagonal entries $C(n)$ and $L_{-1}C$ is a truncated operator weighted shift with weights $C(n)$. The operator $N_k = J^*AJ$ has the form $(N_k\varphi)(n) = n\varphi(n)$, $n \in X_k$, $\varphi \in D(N_k)$, and thus deserves to be called a *truncated number operator*.

(4.5) Corollary. *$T \in B(L^2(X_k, K))$ is $(N_k, -1)$-circular if and only if T is a truncated operator weighted shift.*

Case 2: *G is the additive group $\mathbb{R}$, m is the Lebesgue measure, $H = L^2(\mathbb{R}, K)$, $a : \mathbb{R} \to \mathbb{R}$, $a(x) = x$, $x \in \mathbb{R}$, Q is the position operator.* First let $X = \mathbb{R}_+ =$ the set of all non-negative real numbers, $H_X = L^2(\mathbb{R}_+, K)$. For $s \in \mathbb{R}$, $L_s^+ = J^*L_sJ$ is the unilateral translation operator $(L_s^+\varphi)(x) = \varphi(x+s)$ if $x + s \geq 0$ and 0 otherwise, for $x \in \mathbb{R}_+$. Let $(Q_+\varphi)(x) = x\varphi(x)$, $x \in \mathbb{R}_+$, $\varphi \in D(Q_+) \subset H_X$. In particular, Q_+ can be treated as the Hamiltonian—energy operator—for the time evolution group $t \to e^{-itQ_+}$ and L_{-s}^+ $(s > 0)$ as the energy shift operator in the "time–energy" uncertainty relation [Ho], Ch. III, 8.

(4.6) Corollary. *$T \in B(L^2(\mathbb{R}_+, K))$ is (Q_+, s)-circular if and only if there exists a decomposable operator $B \in B(L^2(\mathbb{R}_+, K))$ such that $T = L_s^+B$.*

Operators of the form L_s^+B are continuous analogues of unilateral operator weighted shifts.

If X is an interval, then Theorem (4.3) gives a similar characterization of (Q_X, s)-circular operators, where $(Q_X\varphi)(x) = x\varphi(x)$, $x \in X$, $\varphi \in D(Q_X) \subset L^2(X, K)$, in terms of continuous analogues of truncated operator weighted shifts.

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