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$\mathcal{C}^\infty$-vectors and boundedness

by Jan Stochel and F. H. Szafraniec (Kraków)

Włodzimierz Mlak in memoriam

Abstract. The following two questions as well as their relationship are studied:

(i) Is a closed linear operator in a Banach space bounded if its $\mathcal{C}^\infty$-vectors coincide with analytic (or semianalytic) ones?

(ii) When are the domains of two successive powers of the operator in question equal?

The affirmative answer to the first question is established in case of paranormal operators. All these investigations are illustrated in the context of weighted shifts.

Let $\mathcal{E}$ be a (real or complex) Banach space. By a subspace of $\mathcal{E}$ we always understand a linear subspace; all operators under consideration are assumed to be linear.

Given an operator A in $\mathcal{E}$, we set

$$\mathcal{D}^\infty(A) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$$

where $\mathcal{D}(A)$ stands for the domain of A. The members of $\mathcal{D}^\infty(A)$ are customarily called $\mathcal{C}^\infty$-*vectors* of A. We investigate some subspaces of $\mathcal{C}^\infty$-vectors, which play an essential role in the theory of symmetric, formally normal and subnormal operators as well as generators of continuous semigroups in Banach spaces, in particular. Among them there are the classes of bounded, analytic and semianalytic vectors (see [1, 4, 6, 8, 9, 10 and 16]). Here we work out a common framework for all of them. The problem we deal with is as follows. It is clear that if A is a bounded operator on $\mathcal{E}$, then the class of all its $\mathcal{C}^\infty$-vectors coincides with anyone of the above mentioned. Our main question is the converse: is it true that a closed operator is bounded if its $\mathcal{C}^\infty$-vectors coincide with, say, analytic ones? It turns out that the answer

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is negative, in general. However, it is affirmative for the so called paranormal operators; in the Hilbert space case they form a class which contains hyponormal operators. This class is especially interesting from unbounded operator point of view. We establish necessary and sufficient conditions for paranormal operators to be bounded in terms of properly tempered bounded vectors. This is a fair generalization of our earlier work (see [22, 23, 17, 18 and 19], for instance). Finally, we give examples of closed unbounded operators for which the $\mathcal{C}^\infty$-vectors are the same as the bounded ones.

Preliminaries. Let us start with the definition of bounded vectors. For $a \geq 0$ set

$$\mathcal{B}_a(A) = \{f \in \mathcal{D}^\infty(A) : \exists c > 0 \ \ \forall n \geq 0,\ \|A^n f\| \leq c a^n\}$$

and

$$\mathcal{B}(A) = \bigcup_{a>0} \mathcal{B}_a(A).$$

Call members of $\mathcal{B}(A)$ *bounded vectors* of A. It is clear that the family $\{\mathcal{B}_a(A)\}_{a\geq 0}$ is increasing in a and $\mathcal{B}_0(A) = \mathcal{N}(A)$.

Another way of looking at $\mathcal{B}(A)$ is to realize that it is composed of vectors f for which $\sum_{n=0}^\infty \|A^n f\| t^n < \infty$ with some $t = t_f > 0$. This observation allows us to define the remaining subspaces in a unified way. Denote by $\mathfrak{A}_+$ the family of germs at 0 of holomorphic functions in one complex variable which have positive Taylor's coefficients at 0. Let $\varphi \in \mathfrak{A}_+$. We say that a vector $f \in \mathcal{D}^\infty(A)$ belongs to $\mathcal{A}_\varphi(A)$ if there exists $t = t_f > 0$ such that

$$\sum_{n=0}^\infty \frac{\varphi^{(n)}(0)}{n!} \|A^n f\|\, t^n < \infty.$$

In particular, we have $\mathcal{B}(A) = \mathcal{A}_\varphi(A)$ with $\varphi(z) := 1/(1-z)$, $|z| < 1$. It follows from the Cauchy–Hadamard formula for the radius of convergence of power series that

$$f \in \mathcal{A}_\varphi(A) \Leftrightarrow \limsup_{n\to\infty} \left(\frac{\varphi^{(n)}(0)}{n!}\, \|A^n f\|\right)^{1/n} < \infty. \tag{1}$$

Put $\mathcal{A}(A) := \mathcal{A}_\varphi(A)$ for $\varphi(z) = e^z$ and $\mathcal{A}^{\mathrm{s}}(A) := \mathcal{A}_\psi(A)$ for $\psi(z) = \cosh\sqrt{z}$. Members of $\mathcal{A}(A)$ and $\mathcal{A}^{\mathrm{s}}(A)$ are called *analytic* and *semianalytic vectors* of A, respectively. Notice that $\mathcal{B}_a(A)$, $\mathcal{B}(A)$, $\mathcal{A}(A)$ and $\mathcal{A}^{\mathrm{s}}(A)$ are subspaces of $\mathcal{D}(A)$ which are invariant for A.

We are interested in describing circumstances under which equality in the inclusion $\mathcal{A}_\varphi(A) \subset \mathcal{D}^\infty(A)$ appears. First define, for $n \geq 0$, the following graph norm:

$$\|f\|_{A,n} = \Big(\sum_{i=0}^{n} \|A^i f\|^2\Big)^{1/2}, \quad f \in \mathcal{D}(A^n),$$

and set

$$\|f\|_A = \|f\|_{A,1}.$$

This norm has the advantage of being an inner product norm when $\mathcal{E}$ is a Hilbert space. Denote by $\tau^\infty(A)$ the locally convex topology induced by the family $\{\|\cdot\|_{A,n}\}_{n=1}^\infty$ of norms on $\mathcal{D}^\infty(A)$. It coincides with the locally convex topology induced by the norms $\{\|\cdot\|_{A^n}\}_{n=1}^\infty$.

Our first observation can be deduced straightforwardly by induction.

PROPOSITION 1. *An operator A in $\mathcal{E}$ is closed if and only if $\mathcal{D}(A^n)$ is complete with respect to $\|\cdot\|_{A,n}$ for any $n \geq 1$. Moreover, $\tau^\infty(A)$ is complete provided A is closed.*

The following two propositions are basic for our purpose.

PROPOSITION 2. *Let $n \geq 0$ and $s \geq 1$. Then the following conditions are equivalent*:

(i) *$A|_{\mathcal{D}^\infty(A)}$ is $\|\cdot\|_{A,n}$-bounded*,
(ii) *the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent on $\mathcal{D}^\infty(A)$*,
(iii) *the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+s}$ are equivalent on $\mathcal{D}^\infty(A)$*,
(iv) *$\tau^\infty(A)$ coincides with the topology induced by $\|\cdot\|_{A,n}$*,
(v) *$\tau^\infty(A)$ coincides with the topology induced by $\|\cdot\|_{A,m}$ for any $m \geq n$.*

P r o o f. The proof of (i)⇔(ii) is clear due to the definitions of the norms involved.

(ii)⇒(iii). It follows from the aforesaid equivalence that $A|_{\mathcal{D}^\infty(A)}$ is $\|\cdot\|_{A,n}$-bounded. This implies that

$$\begin{aligned}\|Af\|_{A,n+1}^2 &= \|Af\|^2 + \|A^2 f\|_{A,n}^2 \\ &\leq \|Af\|_{A,n}^2 + \|A|_{\mathcal{D}^\infty(A)}\|_{A,n}^4 \|f\|_{A,n}^2 \\ &\leq (\|A|_{\mathcal{D}^\infty(A)}\|_{A,n}^2 + \|A|_{\mathcal{D}^\infty(A)}\|_{A,n}^4)\|f\|_{A,n+1}^2, \quad f \in \mathcal{D}^\infty(A).\end{aligned}$$

Thus $A|_{\mathcal{D}^\infty(A)}$ is $\|\cdot\|_{A,n+1}$-bounded. Again the aforesaid equivalence implies that the norms $\|\cdot\|_{A,n+1}$ and $\|\cdot\|_{A,n+2}$ are equivalent on $\mathcal{D}^\infty(A)$. Now induction brings us to (iii).

(iii)⇒(ii) and (iii)⇒(v) come out from monotonicity of the sequence $\{\|\cdot\|_{A,i}\}_{i=1}^\infty$.

(iv)⇒(ii) and (v)⇒(iv) are obvious. ∎

PROPOSITION 3. *Suppose A is closed and $n \geq 0$. Then the following conditions are equivalent*:

(i) *the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent on $\mathcal{D}^\infty(A)$*,
(ii) *$\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-closed in $\mathcal{D}(A^n)$ (⇔ $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete).*

Proof. (i)⇒(ii). Consider a sequence $\{f_k\}_{k=0}^{\infty} \subset \mathcal{D}^{\infty}(A)$ which is $\|\cdot\|_{A,n}$-convergent to $f \in \mathcal{D}(A^n)$. According to Proposition 2, $\{f_k\}_{k=0}^{\infty}$ is a Cauchy sequence in $\tau^{\infty}(A)$. Due to Proposition 1 there is $g \in \mathcal{D}^{\infty}(A)$ such that $\{f_k\}_{k=0}^{\infty}$ is $\tau^{\infty}(A)$-convergent to g. Since the topologies $\tau^{\infty}(A)$ and that of $\|\cdot\|_{A,n}$ coincide (cf. Proposition 2), we have $f = g \in \mathcal{D}^{\infty}(A)$.

(ii)⇒(i). By Proposition 1, $\mathcal{D}^{\infty}(A)$ is $\|\cdot\|_{A,n}$-complete. The restriction $A|_{\mathcal{D}^{\infty}(A)}$ is closed as an operator on the $\|\cdot\|_{A,n}$-complete space $\mathcal{D}^{\infty}(A)$. By the closed graph theorem $A|_{\mathcal{D}^{\infty}(A)}$ is $\|\cdot\|_{A,n}$-bounded. Now the condition (i) follows from the equivalence (i)⇔(ii) of Proposition 2. ■

When $\mathcal{D}^{\infty}(A) = \mathcal{A}_{\varphi}(A)$? This question finds its answer in the following

THEOREM 4. *If A is a closed operator in $\mathcal{E}$ and $\varphi \in \mathfrak{A}_{+}$, then the following conditions are equivalent*:

(i) $\mathcal{D}^{\infty}(A) = \mathcal{A}_{\varphi}(A)$,
(ii) $\mathcal{D}^{\infty}(A) = \mathcal{B}_a(A)$ *for some* $a \geq 0$,
(iii) *there exists* $n \geq 0$ *such that* $\mathcal{D}^{\infty}(A)$ *is* $\|\cdot\|_{A,n}$*-complete*,
(iv) *there exists* $n \geq 0$ *such that* $A|_{\mathcal{D}^{\infty}(A)}$ *is* $\|\cdot\|_{A,n}$*-bounded*,
(v) *there is a norm* ([1]) $\|\cdot\|_*$ *on* $\mathcal{D}^{\infty}(A)$ *such that* $A|_{\mathcal{D}^{\infty}(A)}$ *is* $\|\cdot\|_*$*-bounded and the topology induced by* $\|\cdot\|_*$ *is stronger than that induced by* $\|\cdot\|$.

Proof. (i)⇒(iv). Set $p_k(f) = \|A^k f\| \, \varphi^{(k)}(0)/k!$ for $f \in \mathcal{D}^{\infty}(A)$ and $k \geq 1$. It follows from (1) and from the equality $\mathcal{D}^{\infty}(A) = \mathcal{A}_{\varphi}(A)$ that

$$\mathcal{D}^{\infty}(A) = \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \{f \in \mathcal{D}^{\infty}(A) : p_k(f) \leq j^k\}.$$

Since p_k is $\tau^{\infty}(A)$-continuous on $\mathcal{D}^{\infty}(A)$ and $\mathcal{D}^{\infty}(A)$ is a Fréchet space with respect to $\tau^{\infty}(A)$ (cf. Proposition 1), we conclude (Baire category theorem) that there is $j \geq 1$ such that the $\tau^{\infty}(A)$-interior of $\bigcap_{k=1}^{\infty} \{f \in \mathcal{D}^{\infty}(A) : p_k(f) \leq j^k\}$ is nonempty. But every p_k is a seminorm, so a standard argument allows us to find $n \geq 0$ and $b > 0$ such that $p_k(f) \leq b j^k \|f\|_{A,n}$ for $f \in \mathcal{D}^{\infty}(A)$ and $k \geq 1$. This in turn implies that there is $c > 0$ such that

$$\sum_{k=0}^{n+1} p_k(f) \leq c \|f\|_{A,n}, \qquad f \in \mathcal{D}^{\infty}(A). \tag{2}$$

Since in a finite-dimensional space the ℓ^1-norm and ℓ^2-norm are equivalent, we can find $d > 0$ such that

$$\|f\|_{A,n+1} \leq d \sum_{k=0}^{n+1} p_k(f), \quad f \in \mathcal{D}^{\infty}(A). \tag{3}$$

([1]) The norm can always be chosen to be complete.

It follows from (2) and (3) that the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent on $\mathcal{D}^\infty(A)$, so by Proposition 2, $A|_{\mathcal{D}^\infty(A)}$ is $\|\cdot\|_{A,n}$-bounded.

(iii)$\Leftrightarrow$(iv) is a consequence of Propositions 2 and 3. (iv)$\Rightarrow$(v) is obvious.

(v)$\Rightarrow$(ii). If $a := \|A|_{\mathcal{D}^\infty(A)}\|_*$, then

$$\|A^n f\| \le c\|A^n f\|_* \le a^n c\|f\|_*, \qquad f \in \mathcal{D}^\infty(A), n \ge 0,$$

for some $c > 0$. This shows that $\mathcal{D}^\infty(A) = \mathcal{B}_a(A)$.

The implication (ii)$\Rightarrow$(i), which completes the proof, is straightforward. ∎

In fact, we have proved that if $\varphi \in \mathfrak{A}_+$ and $\mathcal{A}_\varphi(A) \ne \mathcal{D}^\infty(A)$, then $\mathcal{A}_\varphi(A)$ is of the first Baire category (with respect to $\tau^\infty(A)$) in $\mathcal{D}^\infty(A)$. On the other hand, the equality $\mathcal{A}_\varphi(A) = \mathcal{D}^\infty(A)$ holds for every $\varphi \in \mathfrak{A}_+$ provided it does for at least one. If this happens, then there is a smallest nonnegative integer n such that $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete (and, by Propositions 2 and 3, $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,k}$-complete for every $k \ge n$). This n depends on A but not on φ (see Example 22).

There are two classes of unbounded operators relevant here. A densely defined operator A in $\mathcal{E}$ is said to be *nilpotent* or *idempotent* if $A^2 = 0|_{\mathcal{D}(A)}$ or $A^2 = A$, respectively. Notice that, referring to Theorem 4, closed idempotents and nilpotents satisfy $\mathcal{D}(A) = \mathcal{D}^\infty(A) = \mathcal{B}_1(A)$, though they may not be bounded (cf. [12, 13]). They are always $\|\cdot\|_{A,n}$-bounded for $n \ge 1$.

Note also that if A is a closed unbounded operator in $\mathcal{E}$ such that $\mathcal{D}^\infty(A) = \mathcal{A}_\varphi(A)$ for some $\varphi \in \mathfrak{A}_+$, then its restriction to $\mathcal{D}^\infty(A)$ is the generator of an analytic group over $(\mathbb{C}, +)$ of unbounded operators in $\overline{\mathcal{D}^\infty(A)}$. Indeed, according to Theorem 4 there is $n \ge 0$ such that $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete and the operator $A_\infty := A|_{\mathcal{D}^\infty(A)}$ is $\|\cdot\|_{A,n}$-bounded. Thus we can define the required group as

$$T(z) = \sum_{k=0}^{\infty} A_\infty^k \frac{z^k}{k!}, \qquad z \in \mathbb{C},$$

where the series is $\|\cdot\|_{A,n}$-convergent. Since the topology induced by $\|\cdot\|_{A,n}$ is stronger than that induced by $\|\cdot\|$, the group $T(\cdot)$ is analytic with respect to $\|\cdot\|$ (i.e. the function $\mathbb{C} \ni z \mapsto T(z)f \in \mathcal{E}$ is entire for every $f \in \mathcal{D}^\infty(A)$) and its generator calculated with respect to $\|\cdot\|$ is the same as the one calculated with respect to $\|\cdot\|_{A,n}$, the latter being equal to A_∞. To illustrate this phenomenon, consider an unbounded closed idempotent A in $\mathcal{E}$. Then clearly $T(z) = I + (e^z - 1)A$ for $z \in \mathbb{C}$. In case A is an unbounded closed nilpotent we have $T(z) = I + zA$ for $z \in \mathbb{C}$.

Paranormal operators. Following [5] we call an operator A in $\mathcal{E}$ *paranormal* if

$$\|Af\|^2 \le \|f\| \cdot \|A^2 f\|, \qquad f \in \mathcal{D}(A^2). \tag{4}$$

Though the definition of paranormality is rather simple it has far reaching consequences. We begin with some inequalities that have been proved in [24].

Proposition 5. *If A is a paranormal operator in $\mathcal{E}$ and $n \geq 0$, then for any $f \in \mathcal{D}(A^{n+1})$ the following inequalities hold*:

$$\|A^n f\| \leq \|f\|^{1/(n+1)} \|A^{n+1} f\|^{n/(n+1)}, \tag{5}$$

$$\|Af\| \leq \|f\|^{n/(n+1)} \|A^{n+1} f\|^{1/(n+1)}. \tag{6}$$

The following result is a consequence of the above inequalities.

Proposition 6. *If A is a paranormal operator in $\mathcal{E}$ and $n \geq 0$, then*

(i) *A^n is paranormal*,
(ii) *A^{-1} is paranormal provided the kernel of A is trivial*,
(iii) *the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A^n}$ are equivalent*,
(iv) *A^n is closed provided A is closed.*

Proof. (i) Notice first that paranormality is equivalent to

$$\frac{\|Af\|}{\|f\|} \leq \frac{\|A^2 f\|}{\|Af\|}, \qquad f \in \mathcal{D}(A^2),\ Af \neq 0.$$

Replacing in the above f by Af and using an induction argument we get

$$\frac{\|Af\|}{\|f\|} \leq \frac{\|A^2 f\|}{\|Af\|} \leq \ldots \leq \frac{\|A^n f\|}{\|A^{n-1} f\|}, \qquad f \in \mathcal{D}(A^n), \quad A^{n-1} f \neq 0. \tag{7}$$

By the inequality (6), we have, for $j \geq 1$,

$$f \in \mathcal{D}(A^j),\ A^j f = 0 \Rightarrow Af = 0. \tag{8}$$

Take $f \in \mathcal{D}(A^{2n})$ such that $A^n f \neq 0$. By (8) we have $A^j f \neq 0$ for $j = 0, \ldots, 2n-1$. This and (7) give us

$$\frac{\|A^n f\|}{\|f\|} = \frac{\|A^n f\|}{\|A^{n-1} f\|} \cdot \ldots \cdot \frac{\|Af\|}{\|f\|} \leq \frac{\|A^{2n} f\|}{\|A^{2n-1} f\|} \cdot \ldots \cdot \frac{\|A^{n+1} f\|}{\|A^n f\|} = \frac{\|A^{2n} f\|}{\|A^n f\|},$$

so A^n is paranormal.

(ii) Suppose A is invertible. Take $g \in \mathcal{D}(A^{-2})$ and put $f = A^{-2} g$ in (4). What we get is just (4) for A^{-1}.

(iii) It is clear that

$$\|f\|_{A^n}^2 \leq \|f\|_{A,n}^2, \qquad f \in \mathcal{D}(A^n).$$

Consider $\{f_k\}_{k=0}^{\infty} \subset \mathcal{D}(A^n)$ such that $\|f_k\|_{A^n} \to 0$ as $k \to \infty$. In particular, $\|A^n f_k\| \to 0$ as $k \to \infty$. It follows from (5) that $\|A^{n-1} f_k\| \to 0$ as $k \to \infty$. Repeating the argument we deduce that $\|A^j f_k\| \to 0$ as $k \to \infty$ for $j = n, n-1, \ldots, 1$ or, equivalently, $\|f_k\|_{A,n} \to 0$ as $k \to \infty$.

Finally, the condition (iv) follows from Proposition 1 and (iii). ∎

We are now in a position to formulate a version of Theorem 4 for paranormal operators.

THEOREM 7. *If A is a closed paranormal operator in $\mathcal{E}$ and $\varphi \in \mathfrak{A}_+$, then the following conditions are equivalent*:

(i) $\mathcal{D}^\infty(A) = \mathcal{A}_\varphi(A)$,
(ii) $\mathcal{D}^\infty(A) = \mathcal{B}_a(A)$ *for some* $a \geq 0$,
(iii) $\mathcal{D}^\infty(A)$ *is closed in* $\mathcal{E}$,
(iv) $A|_{\mathcal{D}^\infty(A)}$ *is a bounded operator in* $\mathcal{E}$.

Proof. (i)⇒(iii). According to Theorem 4, there is n such that $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete. By Proposition 6(iii), $\mathcal{D}^\infty(A)$ is also $\|\cdot\|_{A^n}$-complete. Thus $A_n = A^n|_{\mathcal{D}^\infty(A)}$ is a closed operator in $\mathcal{E}$. It follows from Proposition 6(i) that A^n is paranormal and consequently so is A_n. Applying Theorem 1 of [11] to A_n, we get the required closedness.

(iii)⇒(iv). This comes out from the closed graph theorem applied to $A|_{\mathcal{D}^\infty(A)}$. The implications (iv)⇒(ii) (with $a = \|A|_{\mathcal{D}^\infty(A)}\|$) and (ii)⇒(i) are clear. ■

The following result provides us with a description of bounded vectors for paranormal operators.

LEMMA 8. *Let A be a paranormal operator in $\mathcal{E}$ and let $a \geq 0$. Then*

(a) $\lim_{n\to\infty} \|A^n f\|^{1/n}$ *exists in* $[0,\infty]$ *for any* $f \in \mathcal{D}^\infty(A)$,
(b) $\mathcal{B}_a(A) = \{f \in \mathcal{D}^\infty(A) : \lim_{n\to\infty} \|A^n f\|^{1/n} \leq a\}$,
(c) $\|Af\| \leq a\|f\|$ *for any* $f \in \mathcal{B}_a(A)$,
(d) *the space $\mathcal{B}_a(A)$ is closed and $A|_{\mathcal{B}_a(A)}$ is a bounded paranormal operator provided A is closed.*

Proof. (a) Take $f \in \mathcal{D}^\infty(A)$ and put $a_n = \|A^n f\|$ for $n \geq 0$. Replacing f in (4) by $A^{n-1}f$ we get

$$a_n^2 \leq a_{n-1}a_{n+1}, \qquad n \geq 1.$$

This implies that the limit $\alpha_A(f) := \lim_{n\to\infty} a_n^{1/n}$ exists.

(b)&(c). Suppose that $\alpha_A(f)$ is finite. By (6), we have

$$\|Af\| \leq \alpha_A(f)\,\|f\|. \tag{9}$$

It follows from (a) that $\alpha_{A^n}(f) = \alpha_A(f)^n$ for $n \geq 0$. Since by Proposition 6(i) the operator A^n is paranormal, we conclude from (9) that

$$\|A^n f\| \leq \alpha_{A^n}(f)\,\|f\| = \alpha_A(f)^n\|f\|, \qquad n \geq 0.$$

This means that f is in $\mathcal{B}_{\alpha_A(f)}(A)$, implying immediately (b) and, consequently, (c).

(d) From (c) we see that the operator $A_a = A|_{\mathcal{B}_a(A)}$ is bounded and $\|A_a\| \leq a$. Since A is closed, we have $\bar{A}_a \subset A$ and, consequently, $\bar{A}_a =$

$A|_{\overline{\mathcal{B}_a(A)}}$. This, in turn, implies that $\overline{\mathcal{B}_a(A)} \subset \mathcal{B}_a(A)$, which shows that $\mathcal{B}_a(A)$ is closed. It is clear that A_a is bounded and paranormal. ■

COROLLARY 9. *Let A be a closable paranormal operator in $\mathcal{E}$. Then* $\mathcal{D}^\infty(A) \cap \overline{\mathcal{B}_a(A)} = \mathcal{B}_a(A)$ *for any* $a \geq 0$.

Proof. Put $B = (A|_{\mathcal{B}_a(A)})^-$. By Lemma 8(c), B is bounded on $\overline{\mathcal{B}_a(A)}$ and $\|B\| \leq a$. This implies that the subspace $\mathcal{L} = \mathcal{D}^\infty(A) \cap \overline{\mathcal{B}_a(A)}$ is invariant for A and $A|_{\mathcal{L}} \subset B$ (because $B \subset \overline{A}$). Take now $f \in \mathcal{L}$. Then $\|A^n f\| = \|B^n f\| \leq \|B\|^n \|f\| \leq a^n \|f\|$ for $n \geq 0$, so $f \in \mathcal{B}_a(A)$. ■

Bounded vectors play an essential role in the following criterion for the boundedness of paranormal operators.

THEOREM 10. *Let A be a paranormal operator in $\mathcal{E}$.*

1° *If* $\mathcal{D}(A) = \operatorname{lin}\{A^n f : f \in \mathcal{X},\ n \geq 0\}$ *for some subset $\mathcal{X}$ of $\mathcal{D}^\infty(A)$, then A is bounded if and only if* $\sup\{\lim_{n\to\infty} \|A^n f\|^{1/n} : f \in \mathcal{X}\}$ *is finite. If this happens, then*

$$\|A\| = \sup\{\lim_{n\to\infty} \|A^n f\|^{1/n} : f \in \mathcal{X}\}. \tag{10}$$

2° *If $A \in \boldsymbol{B}(\mathcal{E})$, then the spectral radius of A is equal to $\|A\|$ and*

$$\|A\| = \max\{\lim_{n\to\infty} \|A^n f\|^{1/n} : f \in \mathcal{E}\}. \tag{11}$$

Proof. 1° The inequality "$\geq$" in (10) is obvious. Denote by a the right hand side of (10) and suppose $a < \infty$. It follows from Lemma 8(b) that $\mathcal{X} \subset \mathcal{B}_a(A)$. However, $\mathcal{B}_a(A)$ is invariant for A, so $\mathcal{D}(A) = \operatorname{lin}\{A^n f : f \in \mathcal{X},\ n \geq 0\} \subset \mathcal{B}_a(A)$. Now, by Lemma 8(c), we have $\|A\| \leq a$.

2° It follows from (10) with $\mathcal{X} = \mathcal{E}$ and the Gelfand formula that the spectral radius of A is equal to $\|A\|$. The equality (11) follows from [3] and Lemma 8(a). ■

Remark. In general it is not true that in (10) "sup" can be replaced by "max", as in (11). Take for example the restriction of the multiplication operator by the independent variable on $\mathcal{L}^2([0,1])$ to the dense subspace $\mathcal{X}$ of $\mathcal{L}^2([0,1])$ composed of all continuous functions with closed support in $[0,1)$.

On the other hand, the boundedness criterion 1° in Theorem 10 is no longer true for operators which are not paranormal, even if $\mathcal{X} = \mathcal{D}(A)$, A is closed and the limit $\lim_{n\to\infty} \|A^n f\|^{1/n}$ exists for each $f \in \mathcal{D}(A)$. Indeed, all these conditions are satisfied by unbounded closed nilpotents and idempotents, because in the first case $\lim_{n\to\infty} \|A^n f\|^{1/n} = 0$, $f \in \mathcal{D}(A)$, and in the other $\lim_{n\to\infty} \|A^n f\|^{1/n} = 0$ or 1 depending on whether $Af = 0$ or $Af \neq 0$.

Call an operator A *cyclic* with a *cyclic vector* $f_0 \in \mathcal{D}^\infty(A)$ if

$$\operatorname{lin}\{A^n f_0 : n \geq 0\} = \{p(A) f_0 : p \in \mathbb{C}[Z]\} = \mathcal{D}(A).$$

COROLLARY 11. *If A is a cyclic paranormal operator in $\mathcal{E}$ with a cyclic vector f_0, then A is bounded if and only if $f_0 \in \mathcal{B}(A)$. Moreover*,

$$\|A\| = \lim_{n\to\infty} \|A^n f_0\|^{1/n}.$$

Proof. Apply Theorem 10 to $\mathcal{X} = \{f_0\}$. ∎

Formally normal operators. In this section we deal with a Hilbert space $\mathcal{H}$ instead of $\mathcal{E}$. Recall that a densely defined operator N in $\mathcal{H}$ is said to be *formally normal* if $\mathcal{D}(N) \subset \mathcal{D}(N^*)$ and $\|Nf\| = \|N^*f\|$ for $f \in \mathcal{D}(N)$. Set

$$\mathcal{D}^\infty(N,N^*) = \bigcap_{n=1}^{\infty} \{\mathcal{D}(A_1\cdots A_n) : A_k = N \text{ or } A_k = N^* \text{ for } k = 1,\dots,n\}\,.$$

The space $\mathcal{D}^\infty(N,N^*)$ is the largest subspace of $\mathcal{D}(N)\cap\mathcal{D}(N^*)$ which is invariant for both N and N^*. Our question here is when $\mathcal{D}^\infty(N,N^*)$ is equal to $\mathcal{A}_\varphi(N)$. The answer is in the following

THEOREM 12. *Suppose N is a closed formally normal operator in $\mathcal{H}$. If $\varphi \in \mathfrak{A}_+$, then the following conditions are equivalent*:

(i) $\mathcal{D}^\infty(N,N^*) \subset \mathcal{A}_\varphi(N)$;

(ii) $\mathcal{D}^\infty(N,N^*) \subset \mathcal{B}_a(N)$ *for some* $a \ge 0$;

(iii) $\mathcal{D}^\infty(N,N^*)$ *is closed in* $\mathcal{H}$;

(iv) $N|_{\mathcal{D}^\infty(N,N^*)}$ *is a bounded operator in* $\mathcal{H}$;

(v) $N = N_\infty \oplus N_0$, *where N_∞ is a bounded normal operator and N_0 is a closed formally normal operator with* $\mathcal{D}^\infty(N_0,N_0^*) = \{0\}$.

Inclusion in (i) *becomes equality if and only if the operator N_0 in* (v) *satisfies additionally* $\mathcal{A}_\varphi(N_0) = \{0\}$.

Proof. First we prove that the space $\mathcal{D}^\infty(N,N^*)$ is $\tau^\infty(N)$-closed in $\mathcal{D}^\infty(N)$. Take $\{f_n\}_{n=0}^\infty \subset \mathcal{D}^\infty(N,N^*)$ converging to $f \in \mathcal{D}^\infty(N)$ in $\tau^\infty(N)$. Applying induction we show that for every $k \ge 1$,

$$(\mathrm{C}_k)\quad A_1,\dots,A_k \in \{N,N^*\} \Rightarrow f \in \mathcal{D}(A_k\cdots A_1) \;\&\; \lim_{n\to\infty} A_k\cdots A_1 f_n = A_k\cdots A_1 f,$$

which will force that $f \in \mathcal{D}^\infty(N,N^*)$.

Since the proof of (C_1) is similar to that of $(\mathrm{C}_k)\Rightarrow(\mathrm{C}_{k+1})$, $k \ge 1$, we restrict ourselves to the latter. Take $A_1,\dots,A_{k+1} \in \{N,N^*\}$. It follows from (C_k) that $\lim_{n\to\infty} A_k\cdots A_1 f_n = A_k\cdots A_1 f$. One can check, using the formal normality of N, that $NN^*g = N^*Ng$ for all $g \in \mathcal{D}^\infty(N,N^*)$. Thus, by the $\tau^\infty(N)$-convergence of the sequence $\{f_n\}_{n=0}^\infty$, we have

$$\|A_{k+1}A_k\cdots A_1(f_n - f_m)\| = \|N^{k+1}(f_n - f_m)\| \to 0 \quad \text{as } m,n\to\infty.$$

Hence, there is g such that $g = \lim_{n\to\infty} A_{k+1}(A_k\cdots A_1 f_n)$. Since A_{k+1} is closed, $A_k\cdots A_1 f \in \mathcal{D}(A_{k+1})$ and $g = A_{k+1}(A_k\cdots A_1 f)$, which means that (C_{k+1}) holds.

Now, since N is paranormal, the proof of equivalence of (i)–(iv) is the same as that of Theorem 7 after replacing $\mathcal{D}^\infty(N)$ by $\mathcal{D}^\infty(N,N^*)$.

(v)⇒(i) in both versions. This follows from

$$\begin{aligned}\mathcal{D}^\infty(N,N^*) &= \mathcal{D}^\infty(N_\infty,N_\infty^*)\oplus\{0\}\\ &= \mathcal{A}_\varphi(N_\infty)\oplus\{0\}\begin{cases}\subset \mathcal{A}_\varphi(N),\\ = \mathcal{A}_\varphi(N) \quad \text{provided } \mathcal{A}_\varphi(N_0)=\{0\}.\end{cases}\end{aligned}$$

(i)⇒(v) in both versions. Employing the implications (i)⇒(iii) and (i)⇒(iv) as well as the fact that $N_\infty := N|_{\mathcal{D}^\infty(N,N^*)}$ is formally normal, we see that N_∞ is a bounded normal operator on $\mathcal{D}^\infty(N,N^*)$. According to Corollary 1 of [20], $\mathcal{D}^\infty(N,N^*)$ reduces N. Set $N_0 = N|_{\mathcal{H}\ominus\mathcal{D}^\infty(N,N^*)}$. Since $\mathcal{D}^\infty(N_0,N_0^*)\subset\mathcal{D}^\infty(N,N^*)\cap(\mathcal{H}\ominus\mathcal{D}^\infty(N,N^*))$, we must have $\mathcal{D}^\infty(N_0,N_0^*) = \{0\}$. If $\mathcal{D}^\infty(N,N^*) = \mathcal{A}_\varphi(N)$, then $\mathcal{A}_\varphi(N_\infty)\oplus\mathcal{A}_\varphi(N_0) = \mathcal{A}_\varphi(N) = \mathcal{D}^\infty(N,N^*) = \mathcal{D}^\infty(N_\infty,N_\infty^*)\oplus\{0\} = \mathcal{A}_\varphi(N_\infty)\oplus\{0\}$, so $\mathcal{A}_\varphi(N_0)=\{0\}$. ■

Thus, if N is normal, the spectral theorem implies that $\mathcal{D}^\infty(N,N^*) = \mathcal{D}^\infty(N)$ and, consequently, $\mathcal{D}^\infty(N)=\mathcal{A}_\varphi(N)$ if and only if N is bounded.

COROLLARY 13. *If A is a closed symmetric operator in $\mathcal{H}$ and $\varphi\in\mathfrak{A}_+$, then $\mathcal{D}^\infty(A)=\mathcal{A}_\varphi(A)$ if and only if $A = A_\infty\oplus A_0$, where A_∞ is a bounded selfadjoint operator and A_0 is a closed symmetric operator with $\mathcal{D}^\infty(A_0) = \{0\}$.*

Note that there are closed symmetric operators having $\mathcal{D}^\infty(A)=\{0\}$ (cf. [2]).

Equality of successive domains. In this section we give necessary and sufficient conditions for the domains of two successive powers of an operator to be equal. Let us start with an observation which is true in a more general setting.

PROPOSITION 14. *If $n\ge 0$ and $s\ge 1$, then the following conditions are equivalent*:

(i) $\mathcal{D}(A^n)=\mathcal{D}(A^{n+s})$,
(ii) $A^s\mathcal{D}(A^n)\subset\mathcal{D}(A^n)$,
(iii) $\mathcal{D}(A^n)=\mathcal{D}(A^{n+1})$,
(iv) $\mathcal{D}(A^n)=\mathcal{D}^\infty(A)$.

Proof. (i)⇒(ii). If $f\in\mathcal{D}(A^n)=\mathcal{D}(A^{n+s})$, then $A^s f\in\mathcal{D}(A^n)$.

(ii)⇒(i). If $f\in\mathcal{D}(A^n)$, then $A^s f\in\mathcal{D}(A^n)$ and, consequently, $f\in\mathcal{D}(A^{n+s})$.

(iii)⇒(iv). Since (i)⇒(ii) for $s = 1$, we get $A\mathcal{D}(A^n) \subset \mathcal{D}(A^n)$ and, consequently, $A^i\mathcal{D}(A^n) \subset \mathcal{D}(A^n)$, $i \geq 1$. Since (ii)⇒(i), we get $\mathcal{D}(A^n) = \mathcal{D}(A^m)$ for any $m \geq n$ and, consequently, (iv) holds.

(i)⇒(iii) and (iv)⇒(i) are trivial. ■

Notice that if A is closed, then, by Proposition 1 and the closed graph theorem, $\mathcal{D}(A^n) = \mathcal{D}(A^{n+1})$ if and only if the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent on $\mathcal{D}(A^{n+1})$ and $\mathcal{D}(A^{n+1})$ is $\|\cdot\|_{A,n}$-dense in $\mathcal{D}(A^n)$. This fact can be strengthened as follows:

Proposition 15. *Let A be a closed operator in $\mathcal{E}$ and let $n \geq 0$. If $\mathcal{D}(A^{i+1})$ is $\|\cdot\|_{A,i}$-dense in $\mathcal{D}(A^i)$ for every $i \geq n$, then the following conditions are equivalent*:

(i) $\mathcal{D}(A^n) = \mathcal{D}(A^{n+1})$,

(ii) *the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent on $\mathcal{D}^\infty(A)$*,

(iii) *$\tau^\infty(A)$ coincides with the topology induced by $\|\cdot\|_{A,n}$*,

(iv) *$\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete.*

Proof. (i)⇒(ii). Applying Proposition 1, under our circumstances, we infer that the spaces $(\mathcal{D}(A^n), \|\cdot\|_{A,n})$ and $(\mathcal{D}(A^n), \|\cdot\|_{A,n+1})$ are complete. So, by the closed graph theorem, the norms $\|\cdot\|_{A,n}$ and $\|\cdot\|_{A,n+1}$ are equivalent.

(ii)⇔(iii). This is a direct consequence of Proposition 2.

(ii)⇒(iv). This is a consequence of Proposition 3.

(iv)⇒(i). The Mittag-Leffler approximation theorem [15, Lemma 1.1.2] implies that $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-dense in $\mathcal{D}(A^n)$. Since $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-closed, the condition (i) holds. ■

As the next result shows, equality of the domains of two successive powers of a closed operator always implies that $\mathcal{D}^\infty(A) = \mathcal{B}(A)$. It would be interesting to know whether the converse implication holds.

Corollary 16. *If A is a closed operator in $\mathcal{E}$ and $\varphi \in \mathfrak{A}_+$, then the following conditions are equivalent*:

(i) *$\mathcal{D}(A^k) = \mathcal{D}(A^{k+1})$ for some $k \geq 0$*,

(ii) *$\mathcal{D}^\infty(A) = \mathcal{A}_\varphi(A)$ and there exists $m \geq 0$ such that $\mathcal{D}(A^{j+1})$ is $\|\cdot\|_{A,j}$-dense in $\mathcal{D}(A^j)$ for every $j \geq m$.*

Proof. (i)⇒(ii). Combine Propositions 14, 15 and Theorem 4.

(ii)⇒(i). It follows from Theorem 4 that $\mathcal{D}^\infty(A)$ is $\|\cdot\|_{A,n}$-complete for some $n \geq 0$. Combining Propositions 1–3 we conclude that it is $\|\cdot\|_{A,j}$-complete for $j \geq n$. Now Proposition 15 implies (i) with $k = \max\{m, n\}$. ■

The proof of the next corollary follows essentially the same lines as that of Corollary 16 with one exception: we use Theorem 7 instead of Theorem 4 in the proof of (ii)⇒(i).

Corollary 17. *If A is a closed paranormal operator in $\mathcal{E}$, $\varphi \in \mathfrak{A}_+$ and $k \geq 0$, then the following conditions are equivalent*:

(i) $\mathcal{D}(A^k) = \mathcal{D}(A^{k+1})$,

(ii) $\mathcal{D}^\infty(A) = \mathcal{A}_\varphi(A)$ *and* $\mathcal{D}(A^{j+1})$ *is* $\|\cdot\|_{A,j}$-*dense in* $\mathcal{D}(A^j)$ *for any* $j \geq k$.

Proposition 6(iii) implies that $\mathcal{D}(A^{j+1})$ is $\|\cdot\|_{A,j}$-dense in $\mathcal{D}(A^j)$ if and only if $\mathcal{D}(A^{j+1})$ is a core of A^j. In this connection, notice that Theorem 4.5 of [14] provides us with a possibility of constructing a closed symmetric operator (which is apparently paranormal) for which the domains of its higher powers may or may not be a core for lower ones, depending on our wish in a sense.

Boundedness of closed paranormal operators can also be described in terms of their successive domains as follows.

Corollary 18. *Let A be a closed paranormal operator in $\mathcal{E}$ and let $n \geq 2$ be such that $\mathcal{D}(A^n)$ is dense in $\mathcal{D}(A)$. Then the following conditions are equivalent*:

(i) $\mathcal{D}(A^n) = \mathcal{D}(A^{n+1})$,

(ii) $\mathcal{D}(A) = \mathcal{D}^\infty(A)$,

(iii) A *is bounded.*

Proof. (i)⇒(iii). Due to Proposition 14, we have $A(\mathcal{D}(A^n)) \subset \mathcal{D}(A^n)$. By Proposition 6, A^n is paranormal and closed. According to Theorem 1 of [11], the space $\mathcal{D}(A^n)$ is closed. It follows from the denseness of $\mathcal{D}(A^n)$ in $\mathcal{D}(A)$ that $\mathcal{D}(A) = \mathcal{D}(A^n)$ and, consequently, the space $\mathcal{D}(A)$ is closed as well. Hence, due to the closed graph theorem, A is bounded.

(iii)⇒(ii). It follows from Proposition 6(iv) that $\mathcal{D}(A^n)$ is closed in $\mathcal{H}$, so by the denseness of $\mathcal{D}(A^n)$ in $\mathcal{D}(A)$, we conclude that $\mathcal{D}(A) = \mathcal{D}(A^n)$. Applying Proposition 14 we get (ii). The implication (ii)⇒(i) is clear. ■

Concluding this section we show how our considerations imply a criterion for boundedness of symmetric operators. Consider a closed symmetric operator A having one of its defect indices finite. According to Theorem 1.9 of [14], $\mathcal{D}^\infty(A)$ is a core for A^n, $n \geq 0$. Applying Corollary 17 (or Corollary 13) we arrive at the following

Corollary 19. *Suppose $\varphi \in \mathfrak{A}_+$. A closed symmetric Hilbert space operator A is bounded if and only if one of its defect indices is finite and $\mathcal{D}^\infty(A) = \mathcal{A}_\varphi(A)$.*

Illustrating by weighted shifts. From now on $\mathcal{E}$ is a separable Hilbert space $\mathcal{H}$ with inner product $\langle\cdot\,,-\rangle$. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis in $\mathcal{H}$ and let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers. The operator S in $\mathcal{H}$ defined by

$$\mathcal{D}(S)=\Big\{f\in\mathcal{H}:\sum_{k=0}^\infty|\langle f,e_k\rangle|^2\lambda_k^2<\infty\Big\},$$

$$Sf=\sum_{k=0}^\infty\langle f,e_k\rangle\lambda_k e_{k+1},\qquad f\in\mathcal{D}(S),$$

is called a *weighted shift* with weights $\{\lambda_n\}_{n=0}^\infty$. The operator so defined is closed and $\mathcal{D}:=\operatorname{lin}\{e_n\}_{n=0}^\infty$ is its core, that is, $S=(S|_{\mathcal{D}})^-$ (cf. [7, 21]). Notice that a weighted shift S is paranormal if and only if the sequence $\{\lambda_n\}_{n=0}^\infty$ is increasing. This, in turn, is equivalent to S being hyponormal.

Our first observation is that never $\mathcal{D}=\mathcal{D}^\infty(S)$ (indeed, otherwise $\mathcal{D}^\infty(S)$ would be a Fréchet space with respect to $\tau^\infty(S)$ having a countable basis) though always $\mathcal{D}\subset\mathcal{D}^\infty(S)$. The domains of powers of S and the graph norms of S can be described explicitly as follows (²):

$$\mathcal{D}(S^n)=\Big\{f\in\mathcal{H}:\sum_{j=1}^n\sum_{k=0}^\infty|\langle f,e_k\rangle|^2\lambda_k^2\cdots\lambda_{k+j-1}^2<\infty\Big\},\tag{12}$$

$$\|f\|_{S,n}^2=\sum_{j=0}^n\sum_{k=0}^\infty|\langle f,e_k\rangle|^2\lambda_k^2\cdots\lambda_{k+j-1}^2,\qquad f\in\mathcal{D}(S^n),\ n\ge 1.\tag{13}$$

This immediately implies that

$$\lim_{k\to\infty}\|f-P_kf\|_{S,n}=0,\qquad f\in\mathcal{D}(S^n),\ n\ge 0,$$

where P_k is the orthogonal projection of $\mathcal{H}$ onto $\operatorname{lin}\{e_0,\dots,e_k\}$. Consequently, for any $n\ge 0$, $\mathcal{D}(S^{n+1})$ is $\|\cdot\|_{S,n}$-dense in $\mathcal{D}(S^n)$. Hence, by Proposition 15, Theorem 4 and Corollary 17, we have

PROPOSITION 20. *If $\varphi\in\mathfrak{A}_+$, then the following conditions are equivalent*:

(i) $\mathcal{D}^\infty(S)=\mathcal{A}_\varphi(S)$,
(ii) $\mathcal{D}^\infty(S)=\mathcal{B}_a(S)$ *for some* $a>0$,
(iii) $\mathcal{D}(S^n)=\mathcal{D}(S^{n+1})$ *for some* $n\ge 0$.

Moreover, if S is paranormal, then $\mathcal{D}^\infty(S)=\mathcal{A}_\varphi(S)$ if and only if S is bounded.

(²) We set $\lambda_k^2\cdots\lambda_{k+j-1}^2:=1$ for $j=0$ and $k\ge 0$.

Proposition 15 can be strengthened in case of weighted shifts as follows:

PROPOSITION 21. *If $n \geq 0$, then the following conditions are equivalent*:

(i) $\mathcal{D}(S^n) = \mathcal{D}(S^{n+1})$,

(ii) *there is $c > 0$ such that*

$$\lambda_k^2 \cdots \lambda_{k+n}^2 \leq c \sum_{j=0}^{n} \lambda_k^2 \cdots \lambda_{k+j-1}^2, \qquad k \geq 0,$$

(iii) *$\tau^\infty(S)$ coincides with the topology induced by $\|\cdot\|_{S,n}$ on $\mathcal{D}$.*

Proof. According to (12) and (13), the norms $\|\cdot\|_{S,n}$ and $\|\cdot\|_{S,n+1}$ are equivalent on $\mathcal{D}^\infty(S)$ if and only if (ii) holds. On the other hand, if the topology $\tau^\infty(S)$ coincides with that induced by the norm $\|\cdot\|_{S,n}$ on $\mathcal{D}$, then the norms $\|\cdot\|_{S,n}$ and $\|\cdot\|_{S,n+1}$ are equivalent on $\mathcal{D}$, which implies (ii). Thus the conclusion follows from Proposition 15. ∎

We are now in a position to state two examples related to the main question of the paper.

EXAMPLE 22. For any $m \geq 0$, we construct a weighted shift S such that $\mathcal{D}(S^n) = \mathcal{D}(S^{n+1})$ for $n \geq m+1$ and $\mathcal{D}(S^n) \neq \mathcal{D}(S^{n+1})$ for $n \leq m$. For $\vartheta > 1$ we set

$$\lambda_{j(m+2)+l}^2 = \begin{cases} \vartheta^j, & 0 \leq l \leq m, \\ \vartheta^{-j(m+1)}, & l = m+1, \end{cases} \qquad j \geq 0.$$

One can check that for all $j \geq 0$ and $0 \leq l \leq m+1$,

$$\lambda_k^2 \cdots \lambda_{k+m+1}^2 = \vartheta^l \leq \vartheta^l \sum_{j=0}^{m+1} \lambda_k^2 \cdots \lambda_{k+j-1}^2, \qquad k = j(m+2)+l.$$

Thus, by Proposition 21, we have $\mathcal{D}(S^{m+1}) = \mathcal{D}(S^{m+2})$. Consequently, $\mathcal{D}(S^n) = \mathcal{D}(S^{n+1})$ for $n \geq m+1$ (cf. Proposition 14). On the other hand, the following inequalities hold for $j \geq 0$:

$$\frac{\lambda_k^2 \cdots \lambda_{k+m}^2}{\sum_{j=0}^m \lambda_k^2 \cdots \lambda_{k+j-1}^2} \geq \frac{\vartheta^{j(m+1)}}{\lambda_k^2 \cdots \lambda_{k+m-1}^2} = \vartheta^j, \quad k = j(m+2).$$

Hence, once more by Proposition 21, we get $\mathcal{D}(S^m) \neq \mathcal{D}(S^{m+1})$, which in turn implies $\mathcal{D}(S^n) \neq \mathcal{D}(S^{n+1})$ for $n \leq m$ (cf. Proposition 14). ∎

EXAMPLE 23. We construct a weighted shift S such that $\overline{\mathcal{B}_a(S)} = \mathcal{H}$ for some $a > 0$ and $\mathcal{B}_b(S) \neq \mathcal{D}^\infty(S)$ for every $b > 0$ (consequently, S must be unbounded and $\mathcal{A}_\varphi(S) \neq \mathcal{D}^\infty(S)$ for every $\varphi \in \mathfrak{A}_+$). Note that this could not happen for closable paranormal operators (cf. Corollary 9). Set

$$\lambda_n = \begin{cases} 1, & n \notin \{m! : m \geq 1\}, \\ m, & n = m!,\ m \geq 1, \end{cases} \qquad n \geq 0.$$

Since $\|S^n e_0\| = \lambda_0 \cdots \lambda_{n-1}$ for $n \geq 1$, we get

$$1 \leq \|S^n e_0\|^{1/n} = (m!)^{1/n} \leq (m!)^{1/m!}, \quad m! + 1 \leq n < (m+1)! + 1,\ m \geq 1,$$

so $\lim_{n\to\infty} \|S^n e_0\|^{1/n} = 1$. This implies that (3) $e_0 \in \mathcal{B}_2(S)$ and, consequently, $\mathcal{D} = \operatorname{lin}\{S^n e_0 : n \geq 0\} \subset \mathcal{B}_2(S)$, which in turn yields $\overline{\mathcal{B}_2(S)} = \mathcal{H}$. We claim that $\mathcal{B}_b(S) \neq \mathcal{D}^\infty(S)$ for every $b > 0$. Otherwise, by Proposition 20, $\mathcal{D}(S^n) = \mathcal{D}(S^{n+1})$ for some $n \geq 0$. It follows from Proposition 21 that there is $c > 0$ such that

$$\lambda_k^2 \cdots \lambda_{k+n}^2 \leq c \sum_{j=0}^{n} \lambda_k^2 \cdots \lambda_{k+j-1}^2, \quad k \geq 0. \tag{14}$$

Let $m_0 \geq 1$ be such that $m_0!\, m_0 > n$. Take $m \geq m_0$. Then there is $k \geq 1$ such that $m! < k, k+1, \ldots, k+n = (m+1)!$. Hence we have $\lambda_k^2 \cdots \lambda_{k+j-1}^2 = 1$ for $j = 0, \ldots, n$ and $\lambda_k^2 \cdots \lambda_{k+n}^2 = (m+1)^2$, which contradicts (14). ■

We conclude the paper with an example of a paranormal (in fact subnormal) weighted shift S for which $\mathcal{A}^{\mathrm{s}}(S) = \{0\}$. Take the weighted shift S with weights $\lambda_n = \exp(n \cdot n!)$. Then $e_0 \notin \mathcal{A}^{\mathrm{s}}(S)$, so by Proposition 5 of [21] we have $\mathcal{A}^{\mathrm{s}}(S) = \{0\}$; in this particular case even the space of Stieltjes vectors of S is trivial (Proposition 5 of [21] is easily seen to be true for semianalytic as well as for Stieltjes vectors).

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(3) In fact, $e_0 \in \mathcal{B}_{1+\varepsilon}(S)$ for every $\varepsilon > 0$.

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: stochel@im.uj.edu.pl
fhszafra@im.uj.edu.pl

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