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Wiener's type regularity criteria on the complex plane

by JÓZEF SICIAK (Kraków)

Abstract. We present a number of Wiener's type necessary and sufficient conditions (in terms of divergence of integrals or series involving a condenser capacity) for a compact set $E \subset \mathbb{C}$ to be regular with respect to the Dirichlet problem. The same capacity is used to give a simple proof of the following known theorem [2, 6]: If E is a compact subset of $\mathbb{C}$ such that $d(t^{-1}E \cap \{|z-a| \le 1\}) \ge \text{const} > 0$ for $0 < t \le 1$ and $a \in E$, where $d(F)$ is the logarithmic capacity of F, then the Green function of $\mathbb{C} \setminus E$ with pole at infinity is Hölder continuous.

Introduction. Let r and R be real numbers with $1 < r < R-2$. Given a subset E of the disk $B \equiv B(a,R) := \{|z-a| < R\}$, let $h(z) \equiv h(z,E,B)$ be defined by the formula $h(z) := \sup\{u(z) : u$ is a subharmonic function in B such that $u \le 0$ on E and $u < 1$ in $B\}$.

Then h is the unique subharmonic function in B such that: $0 \le h \le 1$ in B; h is harmonic in $B \setminus E$; $h = 0$ quasi-almost everywhere on E; and $\lim_{z\to\zeta} h(z) = 1$ if $|\zeta - a| = R$.

One can check (see e.g. [10]) that the set function

$$c(E) \equiv c(E; B(a,R), \overline{B}(a,r)) := 1 - \sup_{|z-a|=r} h(z,E,B), \qquad E \subset B,$$

is a Choquet capacity with the property that a subset E of B is polar with respect to subharmonic functions iff $c(E)=0$. The set function $c(E)$ is called a *capacity of the condenser* $(E, B(a,R))$ with respect to the disk $\overline{B}(a,r)$, or *condenser capacity* of E with respect to the disks $B(a,R)$ and $\overline{B}(a,r)$.

In the sequel E denotes a polynomially convex compact subset of $\mathbb{C}$. Given a point a of E, we define

$$(*) \qquad c(a,t) := c(a + t^{-1}(E \cap \overline{B}(a,t) - a)), \qquad d(a,t) := d(E \cap \overline{B}(a,t))$$

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for all t with $0 \le t \le 1$, where $d(F)$ denotes the transfinite diameter (logarithmic capacity) of a compact set F. Observe that $a + t^{-1}(E \cap \overline{B}(a,t) - a)$ is the portion of E contained in $\overline{B}(a,t)$ scaled t^{-1} times.

If $d(E) > 0$, let V_E denote the Green function of the unbounded component of $\mathbb{C} \setminus E$ with pole at ∞ (we put $V_E = 0$ in the bounded components of $\mathbb{C} \setminus E$).

Let m and ϱ_n be real numbers such that $m \ge 1$ and $0 < \varrho_{n+1} < \varrho_n < \varrho_0 = 1$ $(n \ge 1)$. Put

$$\delta(a, \varrho_n) := d(E \cap \{\varrho_{n+1} \le |z - a| \le \varrho_n\}).$$

The aim of this paper is to prove the following theorems:

I. *V_E is continuous at a iff*

$$I := \int_0^1 \frac{dt}{t \log \dfrac{1}{d(a,t)}} = \infty$$

iff

$$J := \int_0^1 \frac{dt}{t \log \dfrac{mt}{d(a,t)}} = \infty$$

iff

$$\mathcal{K} := \int_0^1 \frac{c(a,t)}{t}\, dt = \infty.$$

II. *If $1 < A \le \varrho_n/\varrho_{n+1} \le B < \infty$ $(n \ge 1)$ then V_E is continuous at a iff*

$$S_4 := \sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{d(a, \varrho_n)}} = \infty$$

iff

$$S_5 := \sum_{n=1}^{\infty} \frac{1}{\log \dfrac{m\varrho_n}{d(a, \varrho_n)}} = \infty$$

iff

$$S_6 := \sum_{n=1}^{\infty} c(a, \varrho_n) = \infty.$$

III. *If $1 < A \le (\log \varrho_{n+1})/(\log \varrho_n) \le B < \infty$ $(n \ge 1)$ then V_E is continuous at a iff*

$$S_7 := \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{\delta(a, \varrho_n)}} = \infty$$

iff

$$S_8 := \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(a,\varrho_n)}} = \infty$$

iff

$$S_9 := \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{m\varrho_n}{\delta(a,\varrho_n)}} = \infty$$

iff

$$S_{10} := \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{m\varrho_n}{d(a,\varrho_n)}} = \infty$$

iff

$$S_{11} := \sum_{n=1}^{\infty} c(a,\varrho_n) \log \frac{1}{\varrho_n} = \infty.$$

IV. *If* $d(a,t) \geq \alpha t$ (*resp.* $c(a,t) \geq \beta$) $(0 \leq t \leq 1)$, *where* α *and* β *are positive constants*, *then there exist positive constants* κ *and* μ *such that*

$$V_E(z) \leq \kappa \delta^{\mu}, \qquad |z-a| \leq \delta \leq 1,$$

i.e. V_E *is Hölder continuous at* a. *Moreover*, κ (*resp.* μ) *depends only on* α (*resp. on* β) *and* E (*but not on* a).

Let

$$w(E) := \frac{1}{\log \dfrac{1}{d(E)}}$$

denote the *Wiener capacity* of $E \subset B(a,R)$. Then the integral J can be written in the form $J = \int_0^1 w(a,t)\frac{dt}{t}$, where $w(a,t) := w(a + (mt)^{-1}(E \cap \overline{B}(a,t) - a))$ is the Wiener capacity of the portion $E \cap \overline{B}(a,t)$ of E scaled $(mt)^{-1}$ times.

The divergence of the integral I (and of each of the series S_4, S_7 or S_8) is Wiener's well known necessary and sufficient condition for E to be regular at a (see e.g. [5]). Our proof of Theorems I–IV is based on the following estimates involving the condenser capacity c defined by $(*)$:

$$(**) \qquad \frac{\log \dfrac{R+1}{r-1}}{\log \dfrac{t(R+1)}{d(a,t)}} \leq c(a,t) \leq \frac{\log \dfrac{R-1}{r+1}}{\log \dfrac{t(R-1)}{d(a,t)}}, \quad 0 < t \leq 1,$$

and

$$h(z, E\cap \overline{B}(a,\varrho_n), B(a,\varrho_n R)) \le e^{-c_n-\ldots-c_{n+k}},\qquad |z-a|\le r\varrho_{n+k},\ n\ge 1,\ k\ge 0,$$

where $c_n := c(a,\varrho_n)$.

It is well known that the regularity (resp. Hölder Continuity Property, briefly HCP) plays a very important role in the theory of multivariate polynomial approximation of holomorphic (resp. $\mathcal{C}^\infty$) functions on compact subsets of $\mathbb{C}^N$ (see e.g. [18, 8]). Compact sets with HCP also appear in a natural way in complex dynamics [2]. Therefore it is desirable to characterize compact subsets of $\mathbb{C}^N$ (resp. of $\mathbb{R}^N$) with these properties. In the case of $N=1$ we have necessary and sufficient criteria for regularity expressed in terms of capacities. The *condenser capacity*, given by $(*)$, can also be defined for compact subsets of $\mathbb{C}^N$ $(N\ge 1)$, and it permits giving a sufficient Wiener's type condition for a compact subset E of $\mathbb{C}^N$ to be regular [13]. However, if $N\ge 2$, unfortunately we do not know whether the condition is necessary for local regularity.

So far we have no reasonable necessary condition (expressed in terms of a capacity) for the Hölder Continuity Property (even on the complex plane). However, the following theorem is true.

V. *If E is a Cantor set associated with a sequence $\{l_n\}$ such that $l_{n+1}\le \frac{1}{2}l_n$* (see [7] for the definition), *and* $\lim_{n\to\infty} l_{n+1}/l_n$ *exists*, *then the following conditions are equivalent*:

(i) $\inf_{n\ge 1}\sqrt[n]{l_n}>0$;
(ii) $\inf_{n\ge 1} l_{n+1}/l_n>0$;
(iii) $\exists_{c\ge 1}\forall_{\varepsilon\in[0,1]}\forall_{t\in E}\exists_{t'\in E}\ \varepsilon/c\le |t-t'|\le\varepsilon$;
(iv) *$d(a,t)\ge \alpha t$ for $0\le t\le 1$ and $a\in E$ with $\alpha=\mathrm{const}>0$*;
(v) *E has HCP*, *i.e. $V_E(z)\le\kappa\delta^\mu$ if $|z-a|\le\delta\le 1$, $a\in E$, $z\in\mathbb{C}$*;
(vi) *E has the Markov Property*, *i.e. for some $M,\sigma>0$*,

$$\|p'_\nu\|_E\le M\nu^\sigma\|p_\nu\|_E,\qquad \nu=0,1,\ldots,$$

where p_ν is any polynomial of degree $\le\nu$.

Indeed, if $\lim l_{n+1}/l_n$ exists then (i)$\Leftrightarrow$(ii). One can check that (ii)$\Rightarrow$(iii) (see e.g. [12]). By Pommerenke [9], (iii)$\Leftrightarrow$(iv) for every compact set E in $\mathbb{C}$. The implication (iv)$\Rightarrow$(v) follows from IV. The implication (v)$\Rightarrow$(vi) is well known (it follows from Cauchy inequalities). Finally, the implication (vi)$\Rightarrow$(i) is due to W. Pleśniak [7].

Corollary. *The classical triadic Cantor set has all the properties* (i)–(vi).

Let us add that V. Totik [14] has recently shown that (i)$\Leftrightarrow$(v)$\Leftrightarrow$(vi) for all Cantor sets associated with $\{l_n\}$ under the only assumption that $l_{n+1} \le \frac{1}{2} l_n$.

I would like to thank Professor Ch. Pommerenke for having informed me about a simple proof of HCP for uniformly perfect sets communicated to him by José Fernandez (in his letter dated June 28, 1994). Our method of proof of the implication (iv)$\Rightarrow$(v) was inspired by that of Fernandez. Fernandez's proof was based on the Lemma of [3]. Estimates ($**$) may be considered as a modified version of that Lemma.

1. Preliminaries

1.1. Let $\mathrm{SH}(\Omega)$ denote the set of all subharmonic functions in an open subset Ω of $\mathbb{C}$. We say that a property $\mathcal{P}$ holds q.a.e. (*quasi-almost everywhere*) on E if there exist a subset A of E and $W \in \mathrm{SH}(\mathbb{C})$ such that $W = -\infty$ on A, and the property $\mathcal{P}$ holds at each point of $E \setminus A$.

1.2. Given a compact subset E of $\mathbb{C}$, define

$$\Phi_E(z) := \sup_{\nu \ge 1} \Phi_\nu^{1/\nu}(z) \equiv \lim_{\nu \to \infty} \Phi_\nu^{1/\nu}(z)$$

for all $z \in \mathbb{C}$, where

$$\Phi_\nu(z) := \sup\{|p(z)| : p \text{ is a polynomial of degree} \le \nu \text{ with } \|p\|_E \le 1\}.$$

The following theorem is well known [10, 15]:

1.3. THEOREM. (i) *$V_E(z) \equiv \log \Phi_E(z)$ for all $z \in \mathbb{C}$, where*

$$V_E(z) := \sup\{u(z) : u \in \mathrm{SH}(\mathbb{C}),\ u \le 0 \text{ on } E,\ \sup_{\zeta \in \mathbb{C}}\{u(\zeta) - \log(1+|\zeta|)\} < \infty\}.$$

(ii) *If $d(E) > 0$ (where $d(E)$ is the transfinite diameter (logarithmic capacity) of E), then V_E^* is the unique function $u \in \mathrm{SH}(\mathbb{C})$ with the following properties*:

(1) *$u(z) \ge 0$ in $\mathbb{C}$, $u(z) = 0$ q.a.e. on E*;
(2) *u is harmonic in $\mathbb{C} \setminus E$*;
(3) *$\lim_{z \to \infty}[u(z) - \log|z|] = \log(1/d(E))$.*

(iii) *If $d(E) > 0$, then*

$$V_E^*(z) = \int \log \frac{|z-a|}{d(E)}\, d\mu(a), \qquad z \in \mathbb{C},$$

where μ is a positive Borel measure such that $\operatorname{supp}\mu \subset E$ *and* $\mu(E) = 1$ (*μ is called* the equilibrium measure *of E*).

1.4. We say that a compact set E is *regular* at a point $a \in E$ (or a is a *regular point* of E) if $V_E^*(a) = 0$. It is clear that E is regular at a if and

only if $\lim_{\delta\to 0}\omega_E(a,\delta)=0$, where
$$\omega_E(a,\delta) := \sup_{|z-a|\le\delta} V_E(z)$$
is the *modulus of continuity* of E at a. In other words, E is regular at a iff V_E^* is continuous at a. In particular, if $D = D(E)$ is the unbounded component of $\mathbb{C}\setminus E$, then by the Bouligand criterion D is regular at $a \in \partial D$ with respect to the classical Dirichlet problem if and only if E is regular at a.

Put
$$\omega_E(\delta) := \sup\{\omega_E(a,\delta) : a \in E\}.$$
We say that E has the *Hölder Continuity Property* (HCP) if
$$\omega_E(\delta) \le \kappa\delta^\mu, \quad 0<\delta\le 1,$$
where κ and μ are positive constants.

1.5. Let F be a compact subset of the unit interval $[0,1]$. Let a be a point of a compact subset E of $\mathbb{C}$ such that
$$\forall_{t\in F} \quad \{|z-a|=t\}\cap E \ne \emptyset.$$
Then
$$V_E(z) \le V_F(-|z-a|), \quad z\in\mathbb{C}.$$
(For the proof see e.g. Lemma 3.1 of [11].) Hence $\omega_E(a,\delta)\le\omega_F(0,\delta)$. In particular, if E is a compact subset of $\mathbb{C}$ such that for each component S of E, $\operatorname{diam} S \ge 2r = \text{const} > 0$, then for all $a\in E$, $\omega_E(a,\delta)\le\kappa\delta^{1/2}$, $0<\delta\le 1$, where $\kappa := \frac{2}{r}(1+\sqrt{1+r})$. Therefore E has HCP with exponent $\mu = 1/2$. In particular each nontrivial continuum has HCP with exponent $\mu=1/2$.

1.6. If E is a subset of an open bounded set Ω, we define the *zero-one extremal function* by the formula
$$h(z,E,\Omega) := \sup\{u(z) : u\in \mathrm{SH}(\Omega),\ u\le 0 \text{ on } E,\ u<1 \text{ in } \Omega\}, \quad z\in\Omega.$$

One can easily check (see e.g. [10]) that

(1) $h^*(z,E,\Omega)\equiv 1$ iff E is polar (i.e. $W=-\infty$ on E for some $W\in \mathrm{SH}(\mathbb{C})$);

(2) $h^*(z,E,\Omega)=h(z,E,\Omega)$ in $\Omega\setminus E$ and h is harmonic in $\Omega\setminus E$;

(3) $h^*(z,E,\Omega)=0$ q.a.e. on E;

(4) if F is a fixed regular compact subset of Ω then the set function
$$c(E)\equiv c(E;\Omega,F) := 1-\sup_{z\in F} h(z,E,\Omega), \quad E\subset\Omega,$$
is a Choquet capacity such that $E\subset\Omega$ is polar with respect to subharmonic functions if and only if $c(E;\Omega,F)=0$.

1.7. Lemma. *If*
$$c(a,t) := c(E\cap\overline{B}(a,t); B(a,tR), \overline{B}(a,tr)), \quad 0\le t\le 1,$$

then

$$\text{(i)}\qquad \frac{\log \dfrac{R-1}{r+1}}{\log \dfrac{t(R-1)}{d(a,t)}} \le c(a,t) \quad \text{if } 0 < r < R-2 < \infty,$$

$$\text{(ii)}\qquad c(a,t) \le \frac{\log \dfrac{R+1}{r-1}}{\log \dfrac{t(R+1)}{d(a,t)}} \quad \text{if } 1 < r < R < \infty,\ 0 < t \le 1,$$

and

$$\text{(iii)}\qquad c(a,t) \equiv c(a + t^{-1}(E \cap \overline{B}(a,t) - a); B(a,R), \overline{B}(a,r)).$$

Proof. Put

$$m(a,t,\varrho) := \inf_{\partial B(a,t\varrho)} V_{E\cap \bar{B}(a,t)}, \qquad M(a,t,\varrho) := \sup_{\bar{B}(a,t\varrho)} V_{E\cap \bar{B}(a,t)}.$$

Then

$$\frac{V_{E\cap\bar{B}(a,t)}(z)}{M(a,t,R)} \le h(z, E\cap \overline{B}(a,t), B(a,tR)) \le \frac{V^*_{E\cap\bar{B}(a,t)}(z)}{m(a,t,R)}$$

for all $z \in B(a,tR)$. Hence

$$\frac{m(a,t,r)}{M(a,t,R)} \le 1 - c(a,t) \le \frac{M(a,t,r)}{m(a,t,R)},$$

and consequently

$$\frac{m(a,t,R) - M(a,t,r)}{m(a,t,R)} \le c(a,t) \le \frac{M(a,t,R) - m(a,t,r)}{M(a,t,R)}.$$

From the integral representation of $V^*_{E\cap\bar{B}(a,t)}$ with respect to the equilibrium measure of $E \cap \overline{B}(a,t)$ (see (iii) of Theorem 1.3(iii)) one gets the following inequalities:

$$m(a,t,\varrho) \ge \log \frac{t(\varrho-1)}{d(a,t)}, \quad M(a,t,\varrho) \le \log \frac{t(\varrho+1)}{d(a,t)}, \quad 1 < \varrho \le R,$$

which imply inequalities (i) and (ii).

(iii) follows from the formula

$$h(a + t(z-a), E \cap \overline{B}(a,t), B(a,tR)) \equiv h(z, a + t^{-1}(E\cap \overline{B}(a,t) - a), B(a,R)), \qquad |z-a| \le R,$$

which is a direct consequence of the invariance of subharmonicity under complex linear transformations of coordinates.

1.8. Proposition. *For a compact set $E \subset \mathbb{C}$ the following conditions are equivalent*:

(1) *E is locally regular at a, i.e. for all $\varrho > 0$, $E(a,\varrho) := \widehat{E}\cap\{|z-a| \le \varrho\}$ is regular at a, where $\widehat{E}$ denotes the polynomially convex hull of E*;

(2) *E is regular at a*;

(3) *for every regular* (*with respect to the classical Dirichlet problem*) *open bounded set Ω containing $\widehat{E}$ one has $h^*(a,E,\Omega) = 0$*;

(4) *there exists a regular open bounded set Ω such that $\widehat{E} \subset \Omega$ and $h^*(a,E,\Omega) = 0$.*

Proof. (1)⇒(2). It is sufficient to observe that $V_E \equiv V_{\hat{E}}$ and $V_E \le V_{E(a,\varrho)}$.

(2)⇒(3) follows from the inequalities

$$(*)\qquad \frac{1}{M(E,\Omega)}V_E^*(z) \le h^*(z,E,\Omega) \le \frac{1}{m(E,\Omega)}V_E^*(z), \qquad z\in\Omega,$$

where $m(E,\Omega) := \inf_{\partial\Omega} V_E^*$ and $M(E,\Omega) := \sup_{\Omega} V_E^*$. In order to show $(*)$ recall that $V_{E^\delta} \uparrow V_E$ in $\mathbb{C}$ and $h(z,E^\delta,\omega) \uparrow h(z,E,\Omega)$ in Ω as $\delta\downarrow 0$, where $E^\delta := \{z : \operatorname{dist}(z,E) \le \delta\}$. Observe that $(*)$ is true for E^δ (by the maximum principle for harmonic functions in the open set $\Omega\setminus E^\delta$). Hence letting $\delta\downarrow 0$, we get $(*)$.

(3)⇒(4) is obvious.

(4)⇒(1). Without loss of generality we may assume that for every $r_0 > 0$ there is $r \in (0,r_0)$ such that $E\cap\{|z-a| = r\} = \emptyset$. Given $\varrho > 0$ choose $r\in(0,\varrho)$ such that $E\cap\{|z-a| = r\} = \emptyset$ and $\overline{B}(a,r)\subset\Omega$. It is clear that $E(a,\varrho)\cap\{|z-a| = r\} = \emptyset$ and $d(E(a,\varrho)) > 0$ (otherwise $h^*(a,E,\Omega) > 0$). Now by the maximum principle

$$V_{E(a,\varrho)}^*(z) \le Mh^*(z,E,\Omega), \qquad |z-a|\le r,$$

where

$$M := \sup_{|z-a|=r} V_{E(a,\varrho)}(z)/\inf_{|z-a|=r} h(z,E,\Omega).$$

Therefore $V_{E(a,\varrho)}^*(a) = 0$.

1.9. Proposition. *Let $\{\varrho_n\}$ be a sequence of real numbers such that $0 < \varrho_{n+1} < \varrho_n < \varrho_0 = 1$ $(n\ge 1)$ and $\lim_{n\to\infty}\varrho_n = 0$. Let a be a fixed point of a compact set $E\subset\mathbb{C}$. Put*

$$d(a,t) := d(E\cap\{|z-a|\le t\}), \qquad 0\le t\le 1,$$
$$\delta(a,\varrho_n) := d(E\cap\{\varrho_{n+1}\le|z-a|\le\varrho_n\})$$

and

$$I \quad := \int_0^1 \frac{dt}{t\log\dfrac{1}{d(a,t)}}.$$

Then the following statements are true:

(1) *We have*

$$\sum_{n=0}^{\infty} \frac{\log \dfrac{\varrho_n}{\varrho_{n+1}}}{\log \dfrac{1}{d(a,\varrho_{n+1})}} \le I = \sum_{n=0}^{\infty} \int_{\varrho_{n+1}}^{\varrho_n} \frac{dt}{t \log \dfrac{1}{d(a,t)}} \le \sum_{n=0}^{\infty} \frac{\log \dfrac{\varrho_n}{\varrho_{n+1}}}{\log \dfrac{1}{d(a,\varrho_n)}}.$$

(2) *If* $1 < A \le \varrho_n/\varrho_{n+1}$ $(n \ge 0)$, *then*

$$\log A \sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{d(a,\varrho_n)}} \le I.$$

(3) *If* $1 < A \le (\log \varrho_{n+1})/(\log \varrho_n)$ $(n \ge 0)$, *then*

$$\left(1 - \frac{1}{A}\right) \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(a,\varrho_n)}} \le I.$$

(4) *If* $\varrho_n/\varrho_{n+1} \le B < \infty$ $(n \ge 0)$, *then*

$$I \le \log B \sum_{n=0}^{\infty} \frac{1}{\log \dfrac{1}{d(a,\varrho_n)}}.$$

(5) *If* $(\log \varrho_{n+1})/(\log \varrho_n) \le B < \infty$ $(n \ge 0)$ *then*

$$I \le \frac{\log \dfrac{1}{\varrho_1}}{\log \dfrac{1}{d(a,1)}} + (B-1) \sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(a,\varrho_n)}}.$$

(6) *If* $1 < A \le (\log \varrho_{n+1})/(\log \varrho_n)$ $(n \ge 1)$ *and*

$$\sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(a,\varrho_n)}} = \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\dfrac{1}{\delta(a,\varrho_n)}} = \infty.$$

(7) (Fundamental Inequality) *If* $0 < r < R < \infty$, $R > 1$ *and*

$$(*) \qquad R\varrho_{n+1} \le r\varrho_n, \qquad n \ge 1,$$

then

$$(\S)\qquad h(z, E\cap \overline{B}(a,\varrho_n), B(a,\varrho_n R)) \le e^{-c_n-\ldots-c_{n+k}},\qquad |z-a|\le r\varrho_{n+k},$$

for all $n\ge 1$ *and* $k\ge 0$, *where* $c_n := c(a,\varrho_n)$.

Proof. Statements (1)–(5) can be easily checked. To show (6) observe that by the subadditivity of the Wiener capacity $w(E) := 1/\log(1/d(E))$ (see [5]) we have

$$(\mathrm{S})\qquad \frac{1}{\log\dfrac{1}{d(a,\varrho_n)}} \le \frac{1}{\log\dfrac{1}{d(a,\varrho_{n+1})}} + \frac{1}{\log\dfrac{1}{\delta(a,\varrho_n)}},\qquad n\ge k,$$

where k is so large that $\varrho_n < 1/2$ for $n\ge k$. It follows from (S) that

$$\frac{\log\dfrac{1}{\varrho_k}}{\log\dfrac{1}{d(a,\varrho_k)}} + \left(1-\frac{1}{A}\right)\sum_{n=k+1}^{\infty}\frac{\log\dfrac{1}{\varrho_n}}{\log\dfrac{1}{d(a,\varrho_n)}} \le \sum_{n=k}^{\infty}\frac{\log\dfrac{1}{\varrho_n}}{\log\dfrac{1}{\delta(a,\varrho_n)}},$$

which implies (6).

Now we prove (7). If u is a subharmonic function in $B(a,\varrho_n R)$ such that $u\le 0$ on $E\cap\overline{B}(a,\varrho_n)$ and $u\le 1$ on $B(a,\varrho_n R)$, then $u(z)\le 1-c_n\le e^{-c_n}$ for all $z\in B(a,R\varrho_{n+1})$, because $B(a,R\varrho_{n+1})\subset B(a,r\varrho_n)$. Therefore

$$u(z)\le e^{-c_n} h(z, E\cap\overline{B}(a,\varrho_{n+1}), B(a,R\varrho_{n+1})),\qquad |z-a|<R\varrho_{n+1},$$

as $E\cap\overline{B}(a,\varrho_{n+1})\subset E\cap\overline{B}(a,\varrho_n)$. Since u is arbitrary, we get

$$h(z, E\cap\overline{B}(a,\varrho_n), B(a,R\varrho_n)) \le e^{-c_n} h(z, E\cap\overline{B}(a,\varrho_{n+1}), B(a,R\varrho_{n+1}))$$

for all z with $|z-a|\le R\varrho_{n+1}$, which implies

$$h(z, E\cap\overline{B}(a,\varrho_n), B(a,R\varrho_n)) \le e^{-c_n-c_{n+1}},\qquad |z-a|\le r\varrho_{n+1}.$$

Repeating this procedure k times, we get (§).

2. Sufficient conditions

2.1. Lemma. *Let* $0<r<R<\infty$, $R>1$, *and let* $\{\varrho_n\}$ *be a sequence of positive numbers such that*

$$(*)\qquad R\varrho_{n+1}\le r\varrho_n,\qquad n\ge 1.$$

If a *is a point of a compact set* E *in* $\mathbb{C}$ *such that*

$$(\%)\qquad \sum_{n=1}^{\infty} c(a,\varrho_n) = \infty,$$

then E *is regular at* a.

P r o o f. Given $n \ge 1$, choose M so large that

$$V_E(z) \le Mh(z, E \cap \overline{B}(a, \varrho_n), B(a, R\varrho_n)), \qquad |z-a| < R\varrho_n.$$

Given $\varepsilon > 0$, by (%) we can choose k so large that

$$Me^{-c_n - c_{n+1} - \ldots - c_{n+k}} < \varepsilon.$$

Therefore by the Fundamental Inequality (§),

$$V_E(z) \le \varepsilon, \qquad |z-a| < R\varrho_{n+k},$$

which implies that E is regular at a.

2.2. In the sequel m, r and R are real numbers with $m \ge 1$ and $1 < r < R-2$, and $\{\varrho_n\}$ denotes a sequence of real numbers such that

$$0 < \varrho_{n+1} < \varrho_n < \varrho_0 = 1 \, (n \ge 1) \quad \text{and} \quad \lim_{n\to\infty} \varrho_n = 0.$$

Given a compact set $E \subset \mathbb{C}$ and a point $a \in E$, we define

$$I := \int_0^1 \frac{dt}{t \log \frac{1}{d(a,t)}}, \qquad J := \int_0^1 \frac{dt}{t \log \frac{mt}{d(a,t)}}, \qquad \mathcal{K} := \int_0^1 \frac{c(a,t)}{t}\, dt,$$

$$S_4 := \sum_{n=0}^{\infty} \frac{1}{\log \frac{1}{d(a,\varrho_n)}}, \quad S_5 := \sum_{n=0}^{\infty} \frac{1}{\log \frac{m\varrho_n}{d(a,\varrho_n)}}, \quad S_6 := \sum_{n=0}^{\infty} c(a, \varrho_n),$$

$$S_7 := \sum_{n=0}^{\infty} \frac{\log \frac{1}{\varrho_n}}{\log \frac{1}{\delta(a,\varrho_n)}}, \quad S_8 := \sum_{n=0}^{\infty} \frac{\log \frac{1}{\varrho_n}}{\log \frac{1}{d(a,\varrho_n)}}, \quad S_9 := \sum_{n=0}^{\infty} \frac{\log \frac{1}{\varrho_n}}{\log \frac{m\varrho_n}{d(a,\varrho_n)}},$$

$$S_{10} := \sum_{n=0}^{\infty} \frac{\log \frac{1}{\varrho_n}}{\log \frac{m\varrho_n}{\delta(a,\varrho_n)}}, \qquad S_{11} := \sum_{n=0}^{\infty} c(a,\varrho_n) \log \frac{1}{\varrho_n}.$$

2.3. THEOREM (Sufficient conditions). (i) *If $I = \infty$ (or $\mathcal{K} = \infty$, or $J = \infty$ for some $m \ge 1$) then E is regular at a.*

(ii) *If $1 < A \le \varrho_n/\varrho_{n+1}$ $(n \ge 1)$ and $S_4 = \infty$ (or $S_6 = \infty$, or $S_5 = \infty$ for some $m \ge 1$) then E is regular at a.*

(iii) *If $1 < A \le (\log \varrho_{n+1})/(\log \varrho_n)$ $(n \ge 1)$ and $S_7 = \infty$ (or $S_8 = \infty$, or $S_{11} = \infty$, or $S_{10} = \infty$ for some $m \ge 1$) then E is regular at a.*

P r o o f. (i) It is sufficient to show the following implications:

$$\mathcal{K} = \infty \Rightarrow \exists_{m\ge 1} J = \infty \Rightarrow I = \infty \Rightarrow E \text{ is regular at } a.$$

The first implication follows from Lemma 1.7(ii) by putting $m=R+1$. In order to show the second implication fix λ with $0<\lambda<1$ and put $\varrho_n:=\lambda^{2^n}$ $(n\ge 1)$, $\varrho_0=1$. Then

$$\begin{aligned}\infty=J&=\sum_{n=0}^{\infty}\int_{\varrho_{n+1}}^{\varrho_n}\frac{dt}{t\log\dfrac{mt}{d(a,t)}}\le\sum_{n=0}^{\infty}\frac{\log\dfrac{\varrho_n}{\varrho_{n+1}}}{\log\dfrac{m\varrho_{n+1}}{d(a,\varrho_n)}}\\
&=\frac{\log\dfrac{1}{\varrho_1}}{\log\dfrac{m\varrho_1}{d(a,1)}}+\sum_{n=1}^{\infty}\frac{\log\dfrac{1}{\varrho_n}}{\log\dfrac{m\varrho_{n+1}}{d(a,\varrho_n)}}\\
&=\frac{\log\dfrac{1}{\varrho_1}}{\log\dfrac{m\varrho_1}{d(a,1)}}+\sum_{n=1}^{\infty}\frac{\log\dfrac{1}{\varrho_n}}{\log\dfrac{1}{d(a,\varrho_n)}}\varphi_n,\quad\text{where}\quad\varphi_n:=\frac{\log\dfrac{1}{d(a,\varrho_n)}}{\log\dfrac{m\varrho_{n+1}}{d(a,\varrho_n)}}.\end{aligned}$$

If $\sup_{n\ge1}\varphi_n<\infty$, then $S_8=\infty$ and consequently by Proposition 1.9(3) we get $I=\infty$. If $\sup_{n\ge1}\varphi_n=\infty$, then there exists a subsequence ϱ_{n_k} such that $(\log\varrho_{n_{k+1}})/(\log\varrho_{n_k})\ge 2$ $(k\ge1)$ and $\lim_{k\to\infty}\varphi_{n_k}=\infty$. Observe that

$$d(a,\varrho_{n_k})=(m\varrho_{n_k+1})^{\varphi_{n_k}/(\varphi_{n_k}-1)}\ge(m\varrho_{n_k+1})^2=(m\varrho_{n_k}^2)^2,\qquad k>k_0.$$

Hence

$$\frac{\log\dfrac{1}{\varrho_{n_k}}}{\log\dfrac{1}{d(a,\varrho_{n_k})}}\ge\frac{1}{2\Big[\dfrac{\log(1/m)}{\log(1/\varrho_{n_k})}+2\Big]}\ge\frac{1}{6},\qquad k>k_1,$$

which implies that $S_8=\infty$, and consequently $I=\infty$.

In order to show the last implication put $r=2$, $R=5$, $\varrho_n=(r/R)^n$ $(n\ge1)$, $\varrho_0=1$. By Lemma 1.7(i) we have

$$\frac{\log\dfrac{4}{3}}{\log\dfrac{1}{d(a,\varrho_n)}}\le\frac{\log\dfrac{4}{3}}{\log\dfrac{4\varrho_n}{d(a,\varrho_n)}}\le c(a,\varrho_n),\qquad n>n_0.$$

Hence by Proposition 1.9(4) we get $S_6=\infty$. Therefore by Lemma 2.1 the set E is regular at a.

(ii) First let us prove the following implications:

$$S_4=\infty\Rightarrow\forall_{m\ge1}S_5=\infty\Rightarrow S_6=\infty\Rightarrow\exists_{m\ge1}S_5=\infty.$$

The first implication is obvious, the last one (resp. the second one) is a direct consequence of Lemma 1.7(ii) (resp. (i)). Now by (i) it remains to prove the implication

$$(\varrho_n/\varrho_{n+1} \geq A > 1 \ (n \geq 1) \ \& \ S_5 = \infty) \Rightarrow I = \infty.$$

By Proposition 1.9(2),

$$I \geq \log A \sum_{n=1}^{\infty} \frac{1}{\log \dfrac{1}{d(a, \varrho_n)}} = \log A \sum_{n=1}^{\infty} \frac{1}{\log \dfrac{m\varrho_n}{d(a, \varrho_n)}} \psi_n,$$

where

$$\psi_n := \frac{\log \dfrac{m\varrho_n}{d(a, \varrho_n)}}{\log \dfrac{1}{d(a, \varrho_n)}}.$$

Consider two cases: $\varepsilon := \inf_{n\geq 1} \psi_n > 0$ and $\varepsilon = 0$. In the first case we get $I \geq (\log A) S_5 \varepsilon = \infty$, which implies that $I = \infty$. In the second case choose a subsequence $\{\varrho_{n_k}\}$ with $(\log \varrho_{n_{k+1}})/(\log \varrho_{n_k}) \geq 2$ $(k \geq 1)$ and $\lim_{k\to\infty} \psi_{n_k} = 0$. Then

$$d(a, \varrho_{n_k}) = (m\varrho_{n_k})^{1/(1-\psi_{n_k})} \geq (m\varrho_{n_k})^2, \quad k \geq k_0,$$

whence $S_8 = \infty$, and consequently by Proposition 1.9(3), we get $I = \infty$.

(iii) First we check the following implications:

$$(\alpha) \qquad S_7 = \infty \Rightarrow \exists_{m\geq 1} S_{10} = \infty \Rightarrow \exists_{m\geq 1} S_9 = \infty \Rightarrow S_8 = \infty.$$

The first two are obvious. In order to show the third, observe that

$$\frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{m\varrho_n}{d(a, \varrho_n)}} = \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(a, \varrho_n)}} \varphi_n, \quad \text{where} \quad \varphi_n := \frac{\log \dfrac{1}{d(a, \varrho_n)}}{\log \dfrac{m\varrho_n}{d(a, \varrho_n)}}.$$

It is clear that if $\sup_{n\geq 1} \varphi_n < \infty$, then $S_8 = \infty$. If $\lim_{k\to\infty} \varphi_{n_k} = \infty$, then

$$d(a, \varrho_{n_k}) = (m\varrho_{n_k})^{\varphi_{n_k}/(\varphi_{n_k}-1)} \geq (m\varrho_{n_k})^2, \quad k > k_0,$$

which again implies that $S_8 = \infty$.

By Lemma 1.7(ii),

$$(\beta) \qquad S_{11} = \infty \Rightarrow \exists_{m\geq 1} S_9 = \infty.$$

By Proposition 1.9(3) we get the implication

$$(\gamma) \qquad \left(\frac{\log \varrho_{n+1}}{\log \varrho_n} \geq A > 1 \ (n \geq 1) \ \& \ S_8 = \infty \right) \Rightarrow I = \infty.$$

Statement (iii) now follows from (α), (β), (γ) and (i). The proof of Theorem 2.3 is complete.

2.4. Corollary. *If* $\liminf_{t\downarrow 0} d(a,t)/t > 0$, *or* $\liminf_{t\downarrow 0} c(a,t) > 0$, *or* $\limsup_{t\downarrow 0} t^{-q} d(a,t) > 0$ *for some* $q > 0$, *or* $\limsup_{t\downarrow 0} c(a,t)\log(1/t) > 0$, *then* E *is regular at* a.

In the first two cases, we get $I = \infty$ (resp. $\mathcal{K} = \infty$), so by (i), E is regular at a. In the remaining two cases we can find $\varepsilon > 0$ and $\{\varrho_n\}$ such that $(\log \varrho_{n+1})/(\log \varrho_n) \ge 2$ $(n \ge 1)$ and $\varrho_n^{-q} d(a,\varrho_n) \ge \varepsilon$ (resp. $c(a,\varrho_n)\log(1/\varrho_n) \ge \varepsilon$) $(n \ge 1)$. Hence $S_8 = \infty$ (resp. $S_{11} = \infty$), which by (iii) implies the regularity of E at a.

3. Necessary conditions

3.1. Lemma [4, 16]. *If a polynomially convex compact subset* E *of* $\mathbb{C}$ *is regular at* a *and*

$$\frac{\log \varrho_{n+1}}{\log \varrho_n} \le B < \infty \qquad (n \ge 1)$$

then $S_7 = \infty$.

Proof. Put $E_n := E \cap \{\varrho_{n+1} \le |z-a| \le \varrho_n\}$ and $E^n := E \cap \{|z-a| \le \varrho_n\}$. By Proposition 1.8 it is enough to show that if the series S_7 is convergent then E^N is not regular at a for all N sufficiently large. Put $\beta_n := \sup_D V^*_{E_n}$, where $D := \{|z-a| < 1/2\}$. Fix $N \ge 1$ so large that $E^N \subset D$. The function

$$u_N(z) := 1 + \sum_{n=N}^{\infty} \frac{V^*_{E_n}(z) - \beta_n}{\beta_n}$$

is either subharmonic or identically $-\infty$ in D, because each term of the last series is a nonpositive subharmonic function in D. But it easily follows from Theorem 1.3(iii) that

$$u_N(a) \ge 1 - \sum_{n=N}^{\infty} \frac{\log \dfrac{1}{\varrho_{n+1}}}{\log \dfrac{1}{d(E_n)}} \ge 1 - B \sum_{n=N}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(E_n)}} > -\infty,$$

as $V^*_{E_n}(a) \ge \log(\varrho_{n+1}/d(E_n))$ and $\beta_n \le \log(1/d(E_n))$. Therefore u_N is subharmonic in D. Moreover, $u_N(z) \le 0$ q.a.e. on E_n for all $n \ge N$.

It is also clear that $u_N(z) \le 1$ in D. Hence $u_N(z) \le h^*(z, E^N, D)$ for all $z \in D$. In particular,

$$h^*(a, E^N, D) \ge 1 - \varepsilon_N, \quad \text{where} \quad \varepsilon_N := B \sum_{n=N}^{\infty} \frac{\log \dfrac{1}{\varrho_n}}{\log \dfrac{1}{d(E_n)}},$$

which implies that $h^*(a, E^N, D) > 1/2$ if N is sufficiently large. Therefore E^N is not regular at a, and consequently E is not regular at a.

3.2. THEOREM (Necessary conditions). *Let E be a polynomially convex compact set regular at a. Then*:

(I) $I = \infty$, $\mathcal{K} = \infty$, *and* $J = \infty$ *for all* $m \geq 1$.

(II) *If* $\varrho_n/\varrho_{n+1} \leq B < \infty$ $(n \geq 1)$, *then the series* S_4–S_6 *are divergent.*

(III) *If* $(\log \varrho_{n+1})/(\log \varrho_n) \leq B < \infty$ $(n \geq 1)$ *then the series* S_7–S_{11} *are divergent.*

Proof. (I) It is sufficient to show the following implications:

$$E \text{ is regular at } a \Rightarrow I = \infty \Rightarrow \forall_{m\geq 1} J = \infty \Rightarrow \mathcal{K} = \infty.$$

The first is a direct consequence of Lemma 3.1 and of Proposition 1.9(3). The second is obvious, and the third follows from Lemma 1.7(i).

(II) We know by (I) that $I = \infty$. Hence by Proposition 1.9(4) we get $S_4 = \infty$. It is clear that $S_4 = \infty \Rightarrow \forall_{m\geq 1} S_5 = \infty$. Finally, the implication $S_4 = \infty \Rightarrow S_6 = \infty$ follows from Lemma 1.7(i).

(III) By Lemma 3.1 the series S_7 is divergent, which implies that so are S_8 and S_9, S_{10} (for all $m \geq 1$). Finally, if S_8 is divergent then by Lemma 1.7(i) the series S_{11} is divergent for all $m \geq 1$.

4. Hölder Continuity Property

4.1. THEOREM (Capacity Scale Condition). *Let $1 \leq r < R < \infty$ and let $\{\varrho_n\}$ be a sequence of real numbers such that $0 < \varrho_n < 1$ and*

$$\frac{R}{r} \leq \frac{\varrho_n}{\varrho_{n+1}} \leq B < \infty, \quad n \geq 1. \tag{1}$$

If a is a point of a compact subset E of $\mathbb{C}$ such that $c(a, \varrho_n) \geq m > 0$, $(n \geq 1)$, then for every $\varrho > 0$ the function $V_{E\cap\bar{B}(a,\varrho)}$ is Hölder continuous at a with exponent $\mu = m/\log B$:

$$V_{E\cap\bar{B}(a,\varrho)}(z) \leq M\delta^{m/\log B} \qquad \text{if } |z - a| \leq \delta \leq 1,$$

where $M = M(\varrho, r, R, m, B)$ depends only on ϱ, r, R, m and B.

Proof. Given $\varrho > 0$ take n so large that $\varrho_n \leq \varrho$. By Proposition 1.9(7),

$$h(z, E \cap \overline{B}(a, \varrho_n), B(a, R\varrho_n)) \leq e^{-m(k+1)}, \qquad |z - a| \leq r\varrho_{n+k},\ k \geq 1. \tag{2}$$

Given δ with $0 < \delta \leq \min\{1, r\varrho_{n+1}\}$, choose k such that $r\varrho_{n+k+1} \leq \delta \leq r\varrho_{n+k}$. Then $B^{-k-1}r\varrho_n \leq \delta$ and consequently $-(k + 1)\log B \leq \log \frac{\delta}{r\varrho_n}$, whence

$$-m(k + 1) \leq \log \left(\frac{\delta}{r\varrho_n}\right)^{m/\log B},$$

which by (2) gives

(2a) $$h(z, E\cap \overline{B}(a,\varrho_n), B(a,R\varrho_n)) \le \left(\frac{1}{r\varrho_n}\right)^{m/\log B} \delta^{m/\log B}$$

for all z with $|z-a|\le \delta \le \min\{1, r\varrho_{n+1}\}$. There is $M_1 = M_1(r,R,\varrho) > 0$ such that

$$V_{E\cap \bar{B}(a,\varrho)}(z) \le M_1 h(z, E\cap \overline{B}(a,\varrho_n), B(a,R\varrho_n)), \quad |z-a| < R\varrho_n,$$

which by (2a) gives the required result.

4.2. COROLLARY. (i) (Capacity Scale Condition) *If* $\inf_{0<t\le 1} c(a,t) > 0$, *then E has local HCP at a.*

(ii) (Uniform Capacity Scale Condition I) *If there exists a positive constant m such that*

$$c(a,t) \ge m, \quad a\in E,\ 0<t\le 1,$$

then E has HCP with exponent

(3) $$\mu = \frac{m}{\log \dfrac{R}{r}}.$$

(iii) (Uniform Capacity Scale Condition II) *Let $\{\varrho_n\}$ be a sequence satisfying* (1). *If there exists a positive constant m such that*

$$c(a,\varrho_n) \ge m, \quad a\in E,\ n\ge 1,$$

then E has HCP with exponent $\mu = m/\log B$.

(iv) (Uniform Logarithmic Capacity Scale Condition) *If E is uniformly perfect in the sense of Pommerenke* [9], *i.e.*

$$t^{-1}d(a,t) \equiv d(t^{-1}E\cap \overline{B}(a,t)) \ge m = \text{const} > 0, \quad 0\le t\le 1,\ a\in E,$$

then E has HCP.

Observe that by Lemma 1.7 a compact set E is uniformly perfect if and only if $c(a,t)\ge m>0$ for $a\in E$ and $0<t\le 1$. We assume here that $1<r<R-2$.

4.3. Remark. The condition of (ii) (resp. (iii)) of Corollary 4.2 means that for each t (resp. for each n) the portion of E contained in the disk $\overline{B}(a,t)$ (resp. in the disk $\overline{B}(a,\varrho_n)$) scaled $1/t$ times (resp. $1/\varrho_n$ times) has the condenser capacity with respect to the balls $B(a,R)$ and $\overline{B}(a,r)$ (see Lemma 1.7(iii)) larger than a positive constant m. From the point of view of the condenser capacity a compact set satisfying the Uniform Capacity Scale Condition is "self-similar" at each of its points. Analogously, from the point of view of the transfinite diameter (equivalently: Wiener capacity) a compact set E is uniformly perfect iff it is self-similar at each of its points.

4.4. Given a compact set E in $\mathbb{C}$, consider the following conditions:

(a) E satisfies the *Local Markov Inequality*, i.e. for every $\nu \geq 1$ there exists c_ν such that

$$\|p_\nu'\|_{E\cap\bar{B}(a,\delta)} \leq \frac{c_\nu}{\delta}\|p_\nu\|_{E\cap\bar{B}(a,\delta)}, \qquad a \in E,\ 0 < \delta \leq 1,$$

where p_ν is any polynomial of degree $\leq \nu$;

(b) E is *uniformly perfect*, i.e. $\exists_{c\geq 1}\forall_{\varepsilon\in(0,1]}\forall_{z\in E}\exists_{z'\in E}\, \varepsilon/c \leq |z-z'| \leq \varepsilon$;

(c) E satisfies the *Uniform Logarithmic Capacity Scale Condition*, i.e.

$$d(t^{-1}E \cap \{|z-a| \leq t\}) \geq m = \text{const} > 0, \qquad a \in E,\ 0 < t \leq 1;$$

(d) E has the *Hölder Continuity Property*, i.e. for some $\kappa, \mu > 0$,

$$V_E(z) \leq \kappa|z-a|^\mu, \qquad a \in E,\ z \in \mathbb{C},\ |z-a| \leq 1;$$

(e) E satisfies the *Markov Inequality*, i.e. for some $M, \sigma > 0$,

$$\|p_\nu'\|_E \leq M\nu^\sigma\|p_\nu\|_E, \qquad \nu \geq 1.$$

It is known that

$$\text{(a)}\Leftrightarrow\text{(b)}\Leftrightarrow\text{(c)}\Rightarrow\text{(d)}\Rightarrow\text{(e)}.$$

The equivalence (a)⇔(b) is due to Wallin and Wingren [17](see also [6]). The equivalence (b)⇔(c) is due to Pommerenke [9]. As already observed in 4.2 the implication (c)⇒(d) follows from Theorem 4.1. Other proofs of this implication were earlier given by Lithner [6] and José Fernandez (in a letter dated June 28, 1994). The present author does not know who was the first to prove the implication (c)⇒(d). In the book [2] (pages 64 and 138) this implication is stated without proof as if it were well known.

The last implication (d)⇒(e) is known since a long time; it follows from the Cauchy integral formula (or from the Cauchy inequalities).

As already mentioned in the introduction, for Cantor sets associated with a sequence $\{l_n\}$ such that $l_{n+1} \leq \frac{1}{2}l_n$ and the limit $\lim_{n\to\infty} l_{n+1}/l_n$ exists, all the above conditions are equivalent.

Question. Which (if any) of the implications (e)⇒(d), (e)⇒(c), or (d)⇒(c) is true for all compact sets E in $\mathbb{C}$?

4.5. Example. Let $\{l_n\}$ be a sequence of positive real numbers with $l_{n+1} \leq \frac{1}{2}l_n$ $(n \geq 0)$, $l_0 = 1$. Put $F := \{0\} \cup \bigcup_{n=0}^\infty [l_n - l_{n+1}, l_n]$.

(i) If $\sum_{n=0}^\infty \frac{1}{\log(1/l_n)} = \infty$ (or $\limsup_{n\to\infty} l_n^{-q} l_{n+1} > 0$ for some $q > 0$), then F is regular.

(ii) If

$$\inf_{n\geq 1} l_{n+1}/l_n \geq \alpha = \text{const} > 0,$$

then

$$d(a,t) := d(F \cap \{|z-a| \le t\}) \ge \frac{\alpha^2}{8} t, \quad 0 \le t \le 1, \ a \in F.$$

Proof. (i) follows from Theorem 2.3(ii) (resp. from Corollary 2.4).

(ii) First we shall show that

$$(*) \qquad d(0,t) \ge \frac{\alpha^2}{4} t, \quad 0 \le t \le 1.$$

Indeed, given t with $0 \le t \le 1$, there exists n such that $l_{n+1} < t \le l_n$. Therefore

$$d(0,t) \ge d(0,l_{n+1}) \ge \frac{1}{4} l_{n+2} \ge \frac{\alpha^2}{4} l_n \ge \frac{\alpha^2}{4} t.$$

Given $a \in F$, there exists n such that $a \in [l_n - l_{n+1}, l_n]$. It is clear that $d(a,t) \ge \frac{t}{4}$, when $0 \le t \le \frac{1}{2} l_{n+1}$. If $\frac{1}{2} l_{n+1} < t \le l_n$, then $d(a,t) \ge d\big(a, \frac{1}{2} l_{n+1}\big) \ge \frac{1}{8} l_{n+1} \ge \frac{\alpha}{8} l_n \ge \frac{\alpha}{8} t$. Finally, if $l_n < t \le 1$, then $d(a,t) \ge d(0,t)$, which by $(*)$ gives $d(a,t) \ge \frac{\alpha^2}{4} t$. It is clear that $\alpha \le \frac{1}{2}$. Therefore $d(a,t) \ge \frac{\alpha^2}{8} t$ for $a \in F$ and $0 < t \le 1$.

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: siciak@im.uj.edu.pl

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