ANNALES
POLONICI MATHEMATICI
LXVI (1997)

On the joint spectral radius

by Vladimír Müller (Praha)

Abstract. We prove the ℓ_p-spectral radius formula for n-tuples of commuting Banach algebra elements. This generalizes results of some earlier papers.

Let A be a Banach algebra with the unit element denoted by 1. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of elements of A. Denote by $\sigma(a)$ the *Harte spectrum* of a, i.e. $\lambda = (\lambda_1, \ldots, \lambda_n) \notin \sigma(a)$ if and only if there exist $u_1, \ldots, u_n, v_1, \ldots, v_n \in A$ such that

$$\sum_{j=1}^{n}(a_j - \lambda_j)u_j = \sum_{j=1}^{n} v_j(a_j - \lambda_j) = 1.$$

Let $1 \le p \le \infty$. The (geometric) *spectral radius* of a is defined by

$$r_p(a) = \max\{\|\lambda\|_p : \lambda \in \sigma(a)\},$$

where

$$\|\lambda\|_p = \begin{cases} \max_{1\le j\le n} |\lambda_j| & (p = \infty), \\ (\sum_{j=1}^{n} |\lambda_j|^p)^{1/p} & (1 \le p < \infty); \end{cases}$$

see [10], cf. also [4].

If $\sigma(a)$ is empty we put formally $r_p(a) = -\infty$.

Clearly, $r_p(a)$ depends on p. On the other hand, instead of the Harte spectrum we can take any other reasonable spectrum (e.g. the left, right, approximate point, defect, Taylor etc.) without changing the value of $r_p(a)$; see [4], [9].

For a single Banach algebra element the just defined spectral radius $r_p(a)$ does not depend on p and coincides with the ordinary spectral radius $r(a_1) = \max\{|\lambda_1| : \lambda_1 \in \sigma(a_1)\}$. By the well-known spectral radius formula

1991 *Mathematics Subject Classification*: Primary 46H05, 46J05.

Key words and phrases: Banach algebra, spectrum, spectral radius.

The research was supported by the grant No. 119106 of the Academy of Sciences of the Czech Republic.

we have in this case

$$r(a_1)=\lim_{k\to\infty}\|a_1^k\|^{1/k}=\inf_k\|a_1^k\|^{1/k}.$$

The spectral radius formula for n-tuples of Banach algebra elements was studied by a number of authors, see e.g. [1], [2], [6], [7], [8]. In this paper we generalize results of [6], [7] and [10].

Let $a=(a_1,\dots,a_n)$ be an n-tuple of elements of a Banach algebra A. Instead of powers of a single element it is natural to consider all possible products of $a_1,\dots,a_n$.

Denote by $F(k,n)$ the set of all functions from $\{1,\dots,k\}$ to $\{1,\dots,n\}$. Let further

$$s_{k,p}(a)=\Big(\sum_{f\in F(k,n)}\|a_{f(1)}\dots a_{f(k)}\|^p\Big)^{1/p}\qquad(1\le p<\infty)$$

and

$$s_{k,\infty}(a)=\max_{f\in F(k,n)}\|a_{f(1)}\dots a_{f(k)}\|.$$

Lemma 1. *$s_{k+l,p}\le s_{k,p}(a)\cdot s_{l,p}(a)$.*

Proof. The statement is obvious for $p=\infty$. For $p<\infty$ we have

$$\begin{aligned}[s_{k,p}(a)\cdot s_{l,p}(a)]^p&=\sum_{f\in F(k,n)}\|a_{f(1)}\dots a_{f(k)}\|^p\cdot\sum_{g\in F(l,n)}\|a_{g(1)}\dots a_{g(l)}\|^p\\&\ge\sum_{f,g}\|a_{f(1)}\dots a_{f(k)}a_{g(1)}\dots a_{g(l)}\|^p=[s_{k+l,p}(a)]^p.\end{aligned}$$

It is well known that the above lemma implies that $\lim_{k\to\infty}(s_{k,p}(a))^{1/k}$ exists and it is equal to $\inf_k(s_{k,p}(a))^{1/k}$.

Thus we may define

$$r''_p(a)=\lim_{k\to\infty}\Big(\sum_{f\in F(k,n)}\|a_{f(1)}\dots a_{f(k)}\|^p\Big)^{1/(pk)}.$$

Similarly we define

$$r'_p(a)=\limsup_{k\to\infty}\Big(\sum_{f\in F(k,n)}r^p(a_{f(1)}\dots a_{f(k)})\Big)^{1/(pk)}\tag{1}$$

(we write briefly $r^p(x)$ instead of $(r(x))^p$).

In general, the limit in (1) does not exist. The limit exists if $a_1,\dots,a_n$ are mutually commuting. This can be proved analogously as in Lemma 1 by using the submultiplicativity of the spectral radius.

Theorem 2. *Let $a=(a_1,\dots,a_n)$ be an n-tuple of elements of a Banach algebra A. Let $1\le p\le\infty$. Then*

$$r_p(a)\le r'_p(a)\le r''_p(a).$$

Proof. The case $p = \infty$ was proved in [7], Theorem 1.

Let $p < \infty$. The second inequality is clear.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma(a)$. Denote by A_0 the closed subalgebra of A generated by the unit 1 and the elements $a_1, \ldots, a_n$. By [5], Proposition 2, there exists a multiplicative functional $h : A_0 \to \mathbb{C}$ such that $h(a_j) = \lambda_j$ for $j = 1, \ldots, n$. Then

$$\begin{aligned}\sum_{f \in F(k,n)} r^p(a_{f(1)} \ldots a_{f(k)}) &\geq \sum_{f \in F(k,n)} |h(a_{f(1)} \ldots a_{f(k)})|^p \\ &= \sum_{f \in F(k,n)} |\lambda_{f(1)}|^p \ldots |\lambda_{f(k)}|^p \\ &= (|\lambda_1|^p + \ldots + |\lambda_n|^p)^k = \|\lambda\|_p^{pk}.\end{aligned}$$

Thus

$$\sum_{f \in F(k,n)} r^p(a_{f(1)} \ldots a_{f(k)}) \geq r_p^{pk}(a)$$

and $r'_p(a) \geq r_p(a)$.

If $a = (a_1, \ldots, a_n)$ is an n-tuple of mutually commuting elements then a better result can be proved.

We use the standard multiindex notation. Denote by $\mathbb{Z}_+$ the set of all non-negative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $m \in \mathbb{Z}_+$ define $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $a^\alpha = a_1^{\alpha_1} \ldots a_n^{\alpha_n}$ and $m\alpha = (m\alpha_1, \ldots, m\alpha_n)$. If k is an integer, $k \geq |\alpha|$, then let

$$\binom{k}{\alpha} = \frac{k!}{\alpha!(k - |\alpha|)!}$$

(for $n = 1$ this definition coincides with the classical binomial coefficients).

We shall use frequently the following formula (for commuting variables x_i):

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha| = k} \binom{k}{\alpha} x^\alpha.$$

In particular, for $x_1 = \ldots = x_n = 1$ we have $\sum_{|\alpha|=k} \binom{k}{\alpha} = n^k$.

If $a = (a_1, \ldots, a_n)$ is a commuting n-tuple of elements of a Banach algebra A, then the definitions of $r'_p(a)$ and $r''_p(a)$ assume a simpler form (for $1 \leq p < \infty$):

$$r'_p(a) = \lim_{k \to \infty} \left[\sum_{|\alpha| = k} \binom{k}{\alpha} r^p(a^\alpha) \right]^{1/(pk)},$$

$$r''_p(a) = \lim_{k \to \infty} \left[\sum_{|\alpha| = k} \binom{k}{\alpha} \|a^\alpha\|^p \right]^{1/(pk)}.$$

THEOREM 3. *Let $a = (a_1, \ldots, a_n)$ be an n-tuple of mutually commuting elements of a Banach algebra A. Let $1 \le p \le \infty$. Then*

$$r_p(a) = r'_p(a) = r''_p(a).$$

Proof. For $p = \infty$ the first equality was proved in [10] and the second in [7], Theorem 2.

We assume in the following $p < \infty$.

Recall that the number of all partitions of the set $\{1, \ldots, k\}$ into n parts is equal to $\binom{k+n-1}{n-1} \le (k+n-1)^{n-1}$.

We have

$$\max_{|\alpha|=k} \binom{k}{\alpha} \|a^\alpha\|^p \le \sum_{|\alpha|=k} \binom{k}{\alpha} \|a^\alpha\|^p \le \binom{k+n-1}{n-1} \max_{|\alpha|=k} \binom{k}{\alpha} \|a^\alpha\|^p.$$

Note that

$$\lim_{k\to\infty} \binom{k+n-1}{n-1}^{1/k} = 1.$$

Thus

$$r''_p(a) = \lim_{k\to\infty} \Big[\sum_{|\alpha|=k} \binom{k}{\alpha} \|a^\alpha\|^p \Big]^{1/(kp)} = \lim_{k\to\infty} \max_{|\alpha|=k} \Big[\binom{k}{\alpha} \|a^\alpha\|^p \Big]^{1/(kp)}.$$

Similarly,

$$r'_p(a) = \lim_{k\to\infty} \max_{|\alpha|=k} \Big[\binom{k}{\alpha} r^p(a^\alpha) \Big]^{1/(kp)}.$$

We now prove the inequality $r'_p(a) \le r_p(a)$:

Choose k and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = k$. Let $\mu \in \sigma(a^\alpha)$ satisfy $|\mu| = r(a^\alpha)$. By the spectral mapping property there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma(a)$ such that $\mu = \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$. Then

$$\begin{aligned} \binom{k}{\alpha} r_p^p(a^\alpha) &= \binom{k}{\alpha} |\mu|^p = \binom{k}{\alpha} |\lambda_1|^{\alpha_1 p} \ldots |\lambda_n|^{\alpha_n p} \\ &\le \sum_{|\beta|=k} \binom{k}{\beta} |\lambda_1|^{\beta_1 p} \ldots |\lambda_n|^{\beta_n p} \\ &= (|\lambda_1|^p + \ldots + |\lambda_n|^p)^k = \|\lambda\|_p^{pk} \le r_p^{pk}(a). \end{aligned}$$

Thus

$$r'_p(a) = \lim_{k\to\infty} \max_{|\alpha|=k} \Big[\binom{k}{\alpha} r^p(a^\alpha) \Big]^{1/(kp)} \le r_p(a).$$

The remaining inequality $r''_p(a) \le r'_p(a)$ will be proved by induction on n.

For $n = 1$, Theorem 3 reduces to the well-known spectral radius formula for a single element.

Let $n \geq 2$ and suppose that the inequality $r''_p \leq r'_p$ is true for all commuting $(n-1)$-tuples.

For each k there is $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = k$, such that

$$\binom{k}{\alpha}\|a^\alpha\|^p = \max_{|\beta|=k}\binom{k}{\beta}\|a^\beta\|^p.$$

Using the compactness of $[0,1]^n$ we can choose a sequence

$$\{\alpha(i)\}_{i=1}^\infty = \{(\alpha_1(i),\ldots,\alpha_n(i))\}_{i=1}^\infty \subset \mathbb{Z}^n_+$$

such that $\lim_{i\to\infty}|\alpha(i)| = \infty$,

$$\binom{|\alpha(i)|}{\alpha(i)}\|a^{\alpha(i)}\|^p = \max_{|\beta|=|\alpha(i)|}\binom{|\alpha(i)|}{\beta}\|a^\beta\|^p \qquad (i=1,2,\ldots) \tag{2}$$

and the sequences $\{\alpha_j(i)/|\alpha(i)|\}_{i=1}^\infty$ are convergent for $j=1,\ldots,n$. Define $k(i) = |\alpha(i)|$ and

$$t_j = \lim_{i\to\infty}\frac{\alpha_j(i)}{k(i)} \in [0,1] \qquad (j=1,\ldots,n).$$

By (2) we have

$$r''^p_p(a) = \lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/(k(i)p)}.$$

We distinguish two cases:

(a) $t_j = 0$ for some j, $1 \leq j \leq n$. Without loss of generality we may assume that $t_n = 0$. Define $a' = (a_1,\ldots,a_{n-1})$, $\alpha'(i) = (\alpha_1(i),\ldots,\alpha_{n-1}(i)) \in \mathbb{Z}^{n-1}_+$ and $k'(i) = |\alpha'(i)| = k(i) - \alpha_n(i)$. Clearly $\lim_{i\to\infty}k'(i)/k(i) = 1$. We have $\|a^{\alpha(i)}\| \leq \|a'^{\alpha'(i)}\|\cdot\|a_n\|^{\alpha_n(i)}$. Then

$$r''^p_p(a') \geq \limsup_{i\to\infty}\left[\binom{k'(i)}{\alpha'(i)}\|a'^{\alpha'(i)}\|^p\right]^{1/k'(i)} \geq L_1\cdot L_2\cdot L_3,$$

where

$$L_1 = \limsup_{i\to\infty}\left[\binom{k'(i)}{\alpha'(i)}\Big/\binom{k(i)}{\alpha(i)}\right]^{1/k'(i)},$$

$$L_2 = \lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k'(i)}$$

and

$$L_3 = \lim_{i\to\infty}\|a_n\|^{-\alpha_n(i)p/k'(i)}.$$

Since $\lim_{i\to\infty}\alpha_n(i)/k'(i) = 0$, we have $L_3 = 1$.

Further,

$$L_2 = \lim_{i\to\infty}\left[\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k(i)}\right]^{k(i)/k'(i)} = r_p''^p(a).$$

Finally,

$$L_1 = \limsup_{i\to\infty}\left[\frac{k'(i)!\cdot\alpha_n(i)!}{k(i)!}\right]^{1/k'(i)} \ge \limsup_{i\to\infty}\left[\frac{(\alpha_n(i)/3)^{\alpha_n(i)}}{k(i)^{\alpha_n(i)}}\right]^{1/k'(i)}$$
$$= \limsup_{i\to\infty}\left(\frac{\alpha_n(i)}{3k(i)}\right)^{(\alpha_n(i)/k(i))\cdot(k(i)/k'(i))} = 1$$

since $\lim_{i\to\infty} k(i)/k'(i) = 1$ and

$$\lim_{i\to\infty}\left(\frac{\alpha_n(i)}{3k(i)}\right)^{\alpha_n(i)/k(i)} = \lim_{x\to 0_+}\left(\frac{x}{3}\right)^x = \lim_{x\to 0_+} x^x = \lim_{x\to 0_+} e^{x\ln x} = 1.$$

Thus $r_p''(a') \ge r_p''(a)$.

By the induction assumption $r_p''(a') = r_p'(a') = r_p(a')$ and by the definition $r_p(a') \le r_p(a) = r_p'(a)$. Hence $r_p''(a) \le r_p'(a)$.

(b) There remains the case $t_j > 0$ $(j = 1,\ldots,n)$, with $t_j = \lim_{i\to\infty}\alpha_j(i)/k(i)$. Choose $\varepsilon > 0$, $\varepsilon < \min_{1\le j\le n} t_j/n$. For i sufficiently large we have

$$t_j - \frac{\varepsilon}{4} \le \frac{\alpha_j(i)}{k(i)} \le t_j + \frac{\varepsilon}{4}.$$

We approximate $t_1,\ldots,t_n$ by rational numbers. Fix positive integers $c_1,\ldots,c_n,d$ such that

$$t_j - \frac{\varepsilon}{2} \le \frac{c_j}{d} \le t_j - \frac{\varepsilon}{4} \quad (j = 1,\ldots,n).$$

Let $\gamma = (c_1,\ldots,c_n) \in \mathbb{Z}_+^n$ and $u = a^\gamma = a_1^{c_1}\ldots a_n^{c_n}$. For each i write $k(i) = m(i)d + z(i)$, where $0 \le z(i) \le d-1$. So, for i sufficiently large, we have

$$\frac{c_j}{d} \le \frac{\alpha_j(i)}{k(i)}, \qquad \frac{\alpha_j(i)}{k(i)} - \frac{c_j}{d} \le \frac{3\varepsilon}{4}$$

and

$$\alpha_j(i) - m(i)c_j = \alpha_j(i) - \frac{k(i)-z(i)}{d}\cdot c_j = k(i)\left[\frac{\alpha_j(i)}{k(i)} - \frac{c_j}{d}\right] + \frac{z(i)c_j}{d}.$$

Thus $\alpha_j(i) - m(i)c_j \ge 0$ $(1 \le j \le n)$ and

$$k(i) - m(i)|\gamma| = \sum_{j=1}^n(\alpha_j(i) - m(i)c_j) \le k(i)\cdot\frac{3\varepsilon n}{4} + \sum_{j=1}^n\frac{z(i)c_j}{d} \le \varepsilon n k(i)$$

for i large enough. We have

$$\begin{aligned}\|a^{\alpha(i)}\| &\le \|a_1^{m(i)c_1}\dots a_n^{m(i)c_n}\|\cdot\|a_1\|^{\alpha_1(i)-m(i)c_1}\dots\|a_n\|^{\alpha_n(i)-m(i)c_n}\\ &\le \|u^{m(i)}\|\cdot K^{n\varepsilon k(i)},\end{aligned}$$

where $K=\max\{1,\|a_1\|,\dots,\|a_n\|\}$. Then, since $\binom{m(i)|\gamma|}{m(i)\gamma}^{1/(m(i)|\gamma|)}\le n$, we have

$$\begin{aligned} r_p'^p(a) &\ge \limsup_{i\to\infty}\left[\binom{m(i)|\gamma|}{m(i)\gamma} r^p(a^{m(i)\gamma})\right]^{1/(m(i)|\gamma|)}\\ &= \limsup_{i\to\infty}\binom{m(i)|\gamma|}{m(i)\gamma}^{1/(m(i)|\gamma|)}\cdot r(u)^{p/|\gamma|}\\ &= \limsup_{i\to\infty}\left[\binom{m(i)|\gamma|}{m(i)\gamma}\|u^{m(i)}\|^p\right]^{1/(m(i)|\gamma|)} \ge L_1\cdot L_2\cdot L_3,\end{aligned}$$

where

$$\begin{aligned} L_1 &= \liminf_{i\to\infty}\left[\binom{m(i)|\gamma|}{m(i)\gamma}\Big/\binom{k(i)}{\alpha(i)}\right]^{1/(m(i)|\gamma|)},\\ L_2 &= \liminf_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/(m(i)|\gamma|)}\end{aligned}$$

and

$$L_3=\liminf_{i\to\infty}K^{-n\varepsilon pk(i)/(m(i)|\gamma|)}.$$

Since

$$1\le\frac{k(i)}{m(i)|\gamma|}\le\frac{1}{1-n\varepsilon}$$

for i sufficiently large, we have $L_3\ge K^{-n\varepsilon p/(1-n\varepsilon)}$.

Since

$$\lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k(i)}=r_p''^p(a),$$

we have $L_2\ge\min\{r_p''^p(a),(r_p''^p(a))^{1/(1-n\varepsilon)}\}$.

To estimate L_1, we use the well-known Stirling formula

$$l! = l^l e^{-l}\sqrt{2\pi l}(1+o(l)).$$

We have

$$\begin{aligned}(1-\varepsilon)\left(\frac{\alpha_j(i)}{e}\right)^{\alpha_j(i)/(m(i)|\gamma|)} &\le (\alpha_j(i)!)^{1/(m(i)|\gamma|)}\\ &\le (1+\varepsilon)\left(\frac{\alpha_j(i)}{e}\right)^{\alpha_j(i)/(m(i)|\gamma|)}\end{aligned}$$

for $j=1,\dots,n$ and for i sufficiently large. Similar estimates can be used for $(m(i)c_j)!$, $(m(i)|\gamma|)!$ and $|\alpha(i)|!$. Thus, for i sufficiently large, we have (to simplify the expressions we write m, k and α instead of $m(i)$, $k(i)$ and $\alpha(i)$)

$$\begin{aligned}
\left[\binom{m|\gamma|}{m\gamma}\Big/\binom{k}{\alpha}\right]^{1/(m|\gamma|)} &= \left[\frac{(m|\gamma|)!\alpha_1!\dots\alpha_n!}{k!(mc_1)!\dots(mc_n)!}\right]^{1/(m|\gamma|)} \\
&\geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+1} \\
&\quad \times \frac{m|\gamma|\cdot\alpha_1^{\alpha_1/(m|\gamma|)}\dots\alpha_n^{\alpha_n/(m|\gamma|)}\cdot e^{k/(m|\gamma|)}\cdot e^{c_1/|\gamma|}\dots e^{c_n/|\gamma|}}{e\cdot e^{\alpha_1/(m|\gamma|)}\dots e^{\alpha_n/(m|\gamma|)}\cdot k^{k/(m|\gamma|)}\cdot(mc_1)^{c_1/|\gamma|}\dots(mc_n)^{c_n/|\gamma|}} \\
&= \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+1}\left(\frac{\alpha_1}{mc_1}\right)^{c_1/|\gamma|}\dots\left(\frac{\alpha_n}{mc_n}\right)^{c_n/|\gamma|} \\
&\quad \times \alpha_1^{(\alpha_1-mc_1)/(m|\gamma|)}\dots\alpha_n^{(\alpha_n-mc_n)/(m|\gamma|)}\cdot\frac{m|\gamma|}{k^{k/(m|\gamma|)}} \\
&\geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+1}\cdot\left(\frac{\alpha_1}{k}\right)^{(\alpha_1-mc_1)/(m|\gamma|)}\dots\left(\frac{\alpha_n}{k}\right)^{(\alpha_n-mc_n)/(m|\gamma|)}\cdot\frac{m|\gamma|}{k}.
\end{aligned}$$

Then

$$L_1 \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+1}(1-n\varepsilon)(t_1\dots t_n)^{\varepsilon/(1-n\varepsilon)}.$$

Hence

$$\begin{aligned}
r_p'^p(a) \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+1}&(1-n\varepsilon)(t_1\dots t_n)^{\varepsilon/(1-n\varepsilon)} \\
&\times K^{-n\varepsilon p/(1-n\varepsilon)}\cdot\min\{r_p''^p(a),(r_p''^p(a))^{1/(1-n\varepsilon)}\}.
\end{aligned}$$

Since ε was an arbitrary positive number, we conclude that $r_p'(a) \geq r_p''(a)$.

Theorem 3 is proved.

We now apply the previous result to the case of n-tuples of operators.

Let $T = (T_1,\dots,T_n)$ be an n-tuple of bounded operators in a Banach space X. Define

$$\|T\|_p = \sup_{\substack{x\in X\\ \|x\|=1}}\Big(\sum_{j=1}^n\|T_jx\|^p\Big)^{1/p}.$$

Equivalently, $\|T\|_p$ is the norm of the operator $\widetilde{T}: X\to X_p^n$, where X_p^n is the direct sum of n copies of X endowed with the ℓ_p-norm, $\|x_1\oplus\dots\oplus x_n\| = (\sum_{j=1}^n\|x_j\|^p)^{1/p}$, and $\widetilde{T}x = T_1x\oplus\dots\oplus T_nx$ (for $p=\infty$ the definitions are changed in the obvious way). Let $T=(T_1,\dots,T_n)\in B(X)^n$ and $S=(S_1,\dots,S_m)\in B(X)^m$. Denote by TS the mn-tuple

$$TS = (T_1S_1,\dots,T_1S_m,T_2S_1,\dots,T_2S_m,\dots,T_nS_1,\dots,T_nS_m).$$

Further, let $T^2 = TT$ and $T^{k+1} = T \cdot T^k$. With this notation we can state the spectral radius formula in the familiar way:

THEOREM 4. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X, and let $1 \le p \le \infty$. Then*

$$r_p(T) = \lim_{k\to\infty} \|T^k\|_p^{1/k}.$$

Proof. We have

$$\|T^k\|_p = \sup_{\|x\|=1} \Big[\sum_{|\alpha|=k} \binom{k}{\alpha} \|T^\alpha x\|^p\Big]^{1/p}$$

and

$$\begin{aligned}
r_p(T) &= \lim_{k\to\infty} \Big[\sum_{|\alpha|=k} \binom{k}{\alpha} \|T^\alpha\|^p\Big]^{1/(kp)} = \lim_{k\to\infty} \max_{|\alpha|=k} \Big[\binom{k}{\alpha} \|T^\alpha\|^p\Big]^{1/(kp)} \\
&= \lim_{k\to\infty} \max_{|\alpha|=k} \sup_{\|x\|=1} \Big[\binom{k}{\alpha} \|T^\alpha x\|^p\Big]^{1/(kp)} \\
&= \lim_{k\to\infty} \sup_{\|x\|=1} \max_{|\alpha|=k} \Big[\binom{k}{\alpha} \|T^\alpha x\|^p\Big]^{1/(kp)} \\
&= \lim_{k\to\infty} \sup_{\|x\|=1} \Big[\sum_{|\alpha|=k} \binom{k}{\alpha} \|T^\alpha x\|^p\Big]^{1/(kp)} = \lim_{k\to\infty} \|T^k\|_p^{1/k}.
\end{aligned}$$

Remark. For $p = 2$ and Hilbert space operators the previous result was proved in [6]; cf. also [3].

References

[1] M. A. Berger and Y. Wang, *Bounded semigroups of matrices*, Linear Algebra Appl. 166 (1992), 21–27.

[2] J. W. Bunce, *Models for n-tuples of non-commuting operators*, J. Funct. Anal. 57 (1984), 21–30.

[3] M. Cho and T. Huruya, *On the spectral radius*, Proc. Roy. Irish Acad. Sect. A 91 (1991), 39–44.

[4] M. Cho and W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. 21 (1992), 251–258.

[5] C.-K. Fong and A. Sołtysiak, *Existence of a multiplicative linear functional and joint spectra*, Studia Math. 81 (1985), 213–220.

[6] V. Müller and A. Sołtysiak, *Spectral radius formula for commuting Hilbert space operators*, ibid. 103 (1992), 329–333.

[7] P. Rosenthal and A. Sołtysiak, *Formulas for the joint spectral radius of non-commuting Banach algebra elements*, Proc. Amer. Math. Soc. 123 (1995), 2705–2708.

[8] G.-C. Rota and W. G. Strang, *A note on the joint spectral radius*, Indag. Math. 22 (1960), 379–381.
[9] A. Sołtysiak, *On a certain class of subspectra*, Comment. Math. Univ. Carolin. 32 (1991), 715–721.
[10] —, *On the joint spectral radii of commuting Banach algebra elements*, Studia Math. 105 (1993), 93–99.

Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: vmuller@mbox.cesnet.cz

Reçu par la Rédaction le 22.12.1994