ANNALES
POLONICI MATHEMATICI
LXVI (1997)

Selfadjoint operator matrices with finite rows

by JAN JANAS and JAN STOCHEL (Kraków)

Dedicated to the memory of our Professor and Master Włodzimierz Mlak

Abstract. A generalization of the Carleman criterion for selfadjointness of Jacobi matrices to the case of symmetric matrices with finite rows is established. In particular, a new proof of the Carleman criterion is found. An extension of Jørgensen's criterion for selfadjointness of symmetric operators with "almost invariant" subspaces is obtained. Some applications to hyponormal weighted shifts are given.

Introduction. Symmetric Jacobi type matrices with matrix entries appear in several branches of analysis (cf. [2]). In particular, certain unbounded Toeplitz operators in the Segal–Bargmann space are induced by such matrices (cf. [8]). One of the basic questions of the theory of Jacobi matrices is when they induce selfadjoint operators. The classical criterion due to Carleman provides sufficient conditions for their selfadjointness in the scalar case (cf. [2, 3, 12]).

In the present paper we propose a direct approach to the above question. Our method of solving it works not only for Jacobi matrices but also for so called locally band matrices (i.e. matrices with finite rows and columns). We prove that a Carleman type condition is still sufficient for selfadjointness of such martices provided that their nonzero entries are suitably located (cf. Theorem 2.3). In particular, it is sufficient for selfadjointness of band matrices (cf. Corollary 2.5); the latter is implicitly contained in [9] and [10] (see Section 3 for more details).

Band matrices with scalar entries have been exploited in the theory of splines (cf. [5]). Note that a special class of non-symmetric locally band matrices has recently appeared in the context of locally finite decomposition of spline spaces (cf. [4]).

1991 *Mathematics Subject Classification*: Primary 47B25; Secondary 47B37.
Key words and phrases: selfadjoint operator, band matrix, weighted shift.

We should also mention the paper [11] of Professor Mlak devoted to the study of "real" parts of unbounded weighted shifts, where the Carleman criterion has been explicitly used. We extend his results to the case of "real" parts of polynomials of unbounded hyponormal weighted shifts (which are induced by band matrices).

1. Preliminaries. From now on:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_1 = \{1, 2, \ldots\}$, $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$,
- $[a, b) = \{t \in \mathbb{R} : a \leq t < b\}$, $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$, $-\infty < a, b \leq \infty$,
- χ_E is the indicator function of a set E,
- $|E|$ is the cardinal number of a set E.

If $\mathcal{X}$ is a subset of a complex Hilbert space $\mathcal{H}$, we denote by $\mathrm{LIN}\,\mathcal{X}$ the linear span of $\mathcal{X}$. If T is a linear operator in $\mathcal{H}$, we denote by $\mathcal{D}(T)$ its domain and by $\mathcal{N}(T)$ its kernel; $\overline{T}$ and T^* stand for the closure and the adjoint of T, respectively. We say that a symmetric operator is *essentially selfadjoint* if its closure is selfadjoint.

Let $\mathcal{H} = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n$ be the orthogonal sum of complex Hilbert spaces $\mathcal{H}_n$, $n \in \mathbb{N}$. Assume that we are given bounded linear operators $A_{i,j}$ acting from $\mathcal{H}_j$ into $\mathcal{H}_i$, $i, j \in \mathbb{N}$. Define $A_{i,j}^{+} = A_{j,i}^{*}$, $i, j \in \mathbb{N}$. An operator matrix $[A_{i,j}]$ is said to be *symmetric* if $A_{i,j} = A_{i,j}^{+}$ for all $i, j \in \mathbf{N}$. We can associate the operator $\mathbf{A}$ in $\mathcal{H}$ with the operator matrix $[A_{i,j}]$ as follows: $f = \sum_{n=0}^{\infty} \oplus f_n \in \mathcal{D}(\mathbf{A})$ if and only if for each $n \in \mathbb{N}$, the series $\sum_{k=0}^{\infty} A_{n,k} f_k$ is weakly convergent and $\sum_{n=0}^{\infty} \| \sum_{k=0}^{\infty} A_{n,k} f_k \|^2 < \infty$; for such f we define $\mathbf{A} f = \sum_{n=0}^{\infty} \oplus (\sum_{k=0}^{\infty} A_{n,k} f_k)$. Denote by $\mathbf{A}^{+}$ the operator associated with the matrix $[A_{i,j}^{+}]$. In what follows $\mathcal{F}(\mathcal{H})$ stands for the linear space of all vectors $f = \sum_{n=0}^{\infty} \oplus f_n$ such that $f_n = 0$ for all but a finite number of indices n. In case $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{D}(\mathbf{A})$ we set

$$\mathbf{A}_0 := \mathbf{A}|_{\mathcal{F}(\mathcal{H})}.$$

The following result, which is known in case of scalar matrices (cf. [12, 14]), will be used in the proof of Theorem 2.3.

THEOREM 1.1. (i) *If $\sum_{j=0}^{\infty} \|A_{j,k}\|^2 < \infty$ for every $k \in \mathbb{N}$, then $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{D}(\mathbf{A})$, $\mathbf{A}^{+}$ is closed and $\mathbf{A}^* \subseteq \mathbf{A}^{+} = \mathbf{A}_0^*$.*

(ii) *If both sums $\sum_{j=0}^{\infty} \|A_{j,k}\|^2$ and $\sum_{j=0}^{\infty} \|A_{k,j}\|^2$ are finite for every $k \in \mathbb{N}$, then $\mathbf{A}$ is closed and densely defined.*

Proof. (i) If $f = f_k \in \mathcal{H}_k$, then $\sum_{j=0}^{\infty} A_{n,j} f_j = A_{n,k} f_k$ and

$$\sum_{n=0}^{\infty} \Big\| \sum_{j=0}^{\infty} A_{n,j} f_j \Big\|^2 \leq \|f_k\|^2 \sum_{n=0}^{\infty} \|A_{n,k}\|^2 < \infty.$$

Thus $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{D}(\mathbf{A})$. To prove that $\mathbf{A}^+ \subseteq \mathbf{A}_0^*$, take $g \in \mathcal{D}(\mathbf{A}^+)$. Then

$$(\mathbf{A}_0 f_k, g) = \sum_{n=0}^{\infty} (A_{n,k} f_k, g_n) = \Big(f_k, \sum_{n=0}^{\infty} A_{n,k}^* g_n\Big) = (f_k, \mathbf{A}^+ g), \qquad f_k \in \mathcal{H}_k.$$

Hence $(\mathbf{A}_0 f, g) = (f, \mathbf{A}^+ g)$ for all $f \in \mathcal{F}(\mathcal{H})$ and $g \in \mathcal{D}(\mathbf{A}^+)$. Consequently, $\mathbf{A}^+ \subseteq \mathbf{A}_0^*$. To prove the opposite inclusion, take $g \in \mathcal{D}(\mathbf{A}_0^*)$. Then

$$(f_k, Q_k \mathbf{A}_0^* g) = (\mathbf{A}_0 f_k, g) = \sum_{n=0}^{\infty} (f_k, A_{n,k}^* g_n), \qquad f_k \in \mathcal{H}_k,$$

where Q_k is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_k$. Therefore $\sum_{n=0}^{\infty} A_{n,k}^* g_n = Q_k \mathbf{A}_0^* g$ (weak convergence) and

$$\sum_{k=0}^{\infty} \Big\| \sum_{n=0}^{\infty} A_{n,k}^* g_n \Big\|^2 = \|\mathbf{A}_0^* g\|^2 < \infty,$$

which means that $g \in \mathcal{D}(\mathbf{A}^+)$ and

$$\mathbf{A}_0^* g = \sum_{k=0}^{\infty} \oplus Q_k \mathbf{A}_0^* g = \mathbf{A}^+ g.$$

Since $\mathbf{A}_0 \subseteq \mathbf{A}$, we get $\mathbf{A}^* \subseteq \mathbf{A}_0^* = \mathbf{A}^+$.

(ii) Moreover, if for every $i \in \mathbb{N}$ $\sum_{j=0}^{\infty} \|A_{i,j}\|^2 < \infty$, then $\sum_{j=0}^{\infty} \|A_{j,i}^+\|^2 < \infty$. Applying (i) to the operator $\mathbf{A}^+$, we get $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{D}(\mathbf{A}^+)$ and $\mathbf{A} = (\mathbf{A}^+|_{\mathcal{F}(\mathcal{H})})^*$. Hence $\mathbf{A}$ is closed and densely defined, as $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{D}(\mathbf{A})$. ∎

Corollary 1.2. *If the matrix $[A_{i,j}]$ is symmetric and $\sum_{m=0}^{\infty} \|A_{m,n}\|^2 < \infty$ for every $n \in \mathbb{N}$, then the following conditions are equivalent*:

(i) $\mathbf{A}_0$ *is essentially selfadjoint*,
(ii) $\overline{\mathbf{A}}_0 = \mathbf{A}$,
(iii) $\mathbf{A}$ *is selfadjoint*,
(iv) $\mathbf{A}$ *is symmetric.*

In case $\{e_n\}_{n=0}^{\infty}$ is a fixed orthonormal basis of $\mathcal{H}$ and $[a_{i,j}]$ is a scalar matrix, we denote by $\mathbf{A}$ the operator associated with the operator matrix $[A_{i,j}]$ (via the orthogonal decomposition $\mathcal{H} = \sum_{n=0}^{\infty} \oplus \mathbb{C} \cdot e_n$) defined by

$$A_{i,j}(e_j) := a_{i,j} \cdot e_i, \qquad i, j \in \mathbb{N}.$$

Similarly we define $\mathbf{A}_0$. We do note indicate the dependence of $\mathbf{A}$ on the orthonormal basis $\{e_n\}_{n=0}^{\infty}$, hoping no confusion can arise.

2. Main result. In this section we present some sufficient conditions for a symmetric locally band matrix with operator entries to be essentially selfadjoint.

Given a matrix $[A_{i,j}]$, we write $A_j(n) := A_{n,j+n}$ for $j, n \in \mathbb{N}$, and

$$\begin{aligned}\gamma_N = \gamma_N(A) &= \sup\{\|A_j(n)\| : j \geq 1,\ N - j + 1 \leq n \leq N\} \\ &= \sup\{\|A_{i,j}\| : 0 \leq i \leq N,\ j \geq N + 1\}\end{aligned}$$

for $N \geq 0$; here we adopt the convention that $A_j(n) = 0$ for $n < 0$. In other words, $\{A_n(k)\}_{k=0}^{\infty}$ is the nth upperdiagonal of the matrix $[A_{i,j}]$.

DEFINITION 2.1. A matrix $[A_{i,j}]$ is said to be *locally band* if for each $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $A_{i,j} = A_{j,i} = 0$ for every $j \geq k$; $[A_{i,j}]$ is said to be a *band matrix* (of width $s \in \mathbb{N}$) if $A_{i,j} = 0$ for all $i, j \in \mathbb{N}$ such that $|i - j| > s$.

We say that $\omega : \mathbb{N}_1 \to \mathbb{N} \cup \{\infty\}$ is a *barrier* of a symmetric matrix $[A_{i,j}]$ if

$$A_j(n) = 0, \qquad n < \omega(j),\ j \in \mathbb{N}_1. \tag{2.1}$$

Note that a symmetric matrix $[A_{i,j}]$ is locally band if and only if it has a barrier ω such that $\lim_{n\to\infty} \omega(n) = \infty$. If this is the case, one can always find a barrier ω of $[A_{i,j}]$ of the form

$$\omega(j) = \sum_{k=0}^{\infty} \omega_k \chi_{[\tau_k, \tau_{k+1})}(j), \qquad j \in \mathbb{N}_1, \tag{2.2}$$

where $\{\tau_k\}_{k=0}^{\infty}$ and $\{\omega_k\}_{k=0}^{\infty}$ are strictly increasing sequences in $\mathbb{N}$ with $\tau_0 = 1$. It is convenient to call any such pair $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ a *barrier* of $[A_{i,j}]$.

The problem of selfadjointness of symmetric matrices splits into two disjoint cases: either $\gamma_N = 0$ for infinitely many N's or there exists $N_0 \geq 0$ such that $\gamma_N > 0$ for $N \geq N_0$. In the first case we have the following result.

PROPOSITION 2.2. *If $[A_{i,j}]$ is a symmetric matrix such that $\gamma_N = 0$ for infinitely many N's, then $[A_{i,j}]$ is locally band, $\mathbf{A} = \overline{\mathbf{A}}_0$ and $\overline{\mathbf{A}}_0$ is selfadjoint.*

It is easy to check that the matrix $[A_{i,j}]$ in Proposition 2.2 must be locally band. Its selfadjointness as well as the main result of the paper (Theorem 2.3) will be proved together.

THEOREM 2.3. *Let $[A_{i,j}]$ be a symmetric matrix with a barrier ω and let $\{\sigma_k\}_{k=0}^{\infty} \subseteq \mathbb{R}_+$ be a decreasing sequence such that* [1]

$$\sum_{k=0}^{\infty} \sigma_k |\omega^{-1}(k)| \sup \omega^{-1}(k) < \infty. \tag{2.3}$$

[1] We adopt the convention that $\sup \emptyset = 0$.

If there exists $N_0 \geq 0$ such that $\gamma_N > 0$ for $N \geq N_0$ and

$$\sum_{N=N_0}^{\infty} \sigma_N/\gamma_N = \infty, \tag{2.4}$$

then $\mathbf{A} = \overline{\mathbf{A}}_0$ and $\mathbf{A}_0$ is essentially selfadjoint.

Proof. It follows from (2.3) that $[A_{i,j}]$ is locally band. Hence, in virtue of Corollary 1.2, it is sufficient to show that

$$(\mathbf{A}f, g) = (f, \mathbf{A}g), \quad f, g \in \mathcal{D}(\mathbf{A}). \tag{2.5}$$

Take $f, g \in \mathcal{D}(\mathbf{A})$. Define $\alpha_N = (\mathbf{A}f, \sum_{j=0}^{N} \oplus g_j)$ and $\beta_N = (\sum_{j=0}^{N} \oplus f_j, \mathbf{A}g)$ for $N \geq 0$. It is clear that

$$|(\mathbf{A}f, g) - (f, \mathbf{A}g)| = \lim_{N\to\infty} |\alpha_N - \beta_N|. \tag{2.6}$$

Using the symmetry of $[A_{i,j}]$, one can check that

$$\alpha_N = \sum_{n=0}^{N} \Big(\sum_{j=1}^{\infty} (A_j(n-j)^* f_{n-j}, g_n) + (A_0(n) f_n, g_n) + \sum_{j=1}^{\infty} (A_j(n) f_{n+j}, g_n) \Big),$$

$$\beta_N = \sum_{n=0}^{N} \Big(\sum_{j=1}^{\infty} (f_n, A_j(n-j)^* g_{n-j}) + (A_0(n) f_n, g_n) + \sum_{j=1}^{\infty} (f_n, A_j(n) g_{n+j}) \Big).$$

Changing the order of summation and regrouping terms suitably, we get

$$\begin{aligned} &\alpha_N - \beta_N \\ &\quad = \sum_{j=1}^{\infty} \sum_{n=N-j+1}^{N} ((A_j(n) f_{n+j}, g_n) - (f_n, A_j(n) g_{n+j})), \quad N \geq 0. \end{aligned} \tag{2.7}$$

By (2.5) and (2.6) we are reduced to proving that $\lim_{N\to\infty} |\alpha_N - \beta_N| = 0$. Suppose, contrary to our claim, that there exist $r > 0$ and $N_0 \geq 1$ such that

$$|\alpha_N - \beta_N| \geq r, \quad N \geq N_0. \tag{2.8}$$

Let $\varDelta_{j,n} = \|f_n\| \cdot \|g_{n+j}\| + \|f_{n+j}\| \cdot \|g_n\|$. It follows from (2.1) and (2.7) that

$$\begin{aligned} |\alpha_N - \beta_N| &\leq \sum_{j=1}^{\infty} \sum_{n=N-j+1}^{N} \chi_{[\omega(j),\infty)}(n) \|A_j(n)\| \varDelta_{j,n} \\ &\leq \gamma_N \sum_{j=1}^{\infty} \sum_{n=N-j+1}^{N} \chi_{[\omega(j),\infty)}(n) \varDelta_{j,n}, \quad N \geq N_0. \end{aligned}$$

The above inequalities and (2.8) imply

$$0 < r \leq \gamma_N \sum_{j=1}^{\infty} \sum_{n=N-j+1}^{N} \chi_{[\omega(j),\infty)}(n) \varDelta_{j,n}, \quad N \geq N_0. \tag{2.9}$$

Now there are two possibilities: either $\gamma_N = 0$ for infinitely many N's, which contradicts (2.9), or $\gamma_N > 0$ for N large enough, say for $N \geq N_0$. The latter and (2.9) imply, via the monotonicity of $\{\sigma_k\}_{k=0}^{\infty}$, that

$$\begin{aligned}
r \sum_{N=N_0}^{\infty} \frac{\sigma_N}{\gamma_N} &\leq \sum_{N=N_0}^{\infty} \sum_{j=1}^{\infty} \sum_{n=N-j+1}^{N} \sigma_N \chi_{[\omega(j),\infty)}(n) \Delta_{j,n} \\
&\leq \sum_{j=1}^{\infty} \sum_{N=0}^{\infty} \sum_{n=N-j+1}^{N} \sigma_N \chi_{[\omega(j),\infty)}(n) \Delta_{j,n} \\
&= \sum_{k=0}^{\infty} \sum_{j\in\omega^{-1}(k)} \sum_{N=0}^{\infty} \sum_{n=N-j+1}^{N} \sigma_N \chi_{[\omega(j),\infty)}(n) \Delta_{j,n} \\
&= \sum_{k=0}^{\infty} \sum_{j\in\omega^{-1}(k)} \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} \sigma_N \chi_{[k,\infty)}(n) \chi_{[N-j+1,N]}(n) \Delta_{j,n} \\
&= \sum_{k=0}^{\infty} \sum_{j\in\omega^{-1}(k)} \sum_{n=k}^{\infty} \sum_{N=0}^{\infty} \sigma_N \chi_{[n,n+j-1]}(N) \Delta_{j,n} \\
&= \sum_{k=0}^{\infty} \sum_{j\in\omega^{-1}(k)} \sum_{n=k}^{\infty} \sum_{N=n}^{n+j-1} \sigma_N \Delta_{j,n} \\
&\leq \sum_{k=0}^{\infty} \sum_{j\in\omega^{-1}(k)} \sum_{n=k}^{\infty} j \sigma_n \Delta_{j,n} \\
&\leq \sum_{k=0}^{\infty} \Big(\sum_{j\in\omega^{-1}(k)} j \Big) \sigma_k \sum_{n=k}^{\infty} (\|f_n\| \cdot \|g_{n+j}\| + \|f_{n+j}\| \cdot \|g_n\|) \\
&\leq \sum_{k=0}^{\infty} \Big(\sum_{j\in\omega^{-1}(k)} j \Big) \sigma_k \Big(\sqrt{\sum_{n=k}^{\infty} \|f_n\|^2} \sqrt{\sum_{n=k}^{\infty} \|g_{n+j}\|^2} \\
&\qquad\qquad + \sqrt{\sum_{n=k}^{\infty} \|f_{n+j}\|^2} \sqrt{\sum_{n=k}^{\infty} \|g_n\|^2} \Big) \\
&\leq 2\|f\| \cdot \|g\| \sum_{k=0}^{\infty} \sigma_k |\omega^{-1}(k)| \sup \omega^{-1}(k).
\end{aligned}$$

The next to last inequality is a consequence of the Cauchy–Schwarz inequality in ℓ^2. Hence, by (2.3), the series $\sum_{N=N_0}^{\infty} \sigma_N/\gamma_N$ is convergent, which contradicts our assumption (2.4). This completes the proof. ∎

Corollary 2.4. *Let $[A_{i,j}]$ be a symmetric matrix with a barrier $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ and let $\{\sigma_k\}_{k=0}^{\infty} \subseteq \mathbb{R}_+$ be a decreasing sequence such that*

$$\sum_{k=0}^{\infty}(\tau_{k+1}-\tau_k)\tau_{k+1}\sigma_{\omega_k}<\infty. \tag{2.10}$$

If there exists $N_0 \geq 0$ *such that* $\gamma_N > 0$ *for* $N \geq N_0$ *and*

$$\sum_{N=N_0}^{\infty}\sigma_N/\gamma_N=\infty, \tag{2.11}$$

then $\mathbf{A}=\overline{\mathbf{A}}_0$ *and* $\mathbf{A}_0$ *is essentially selfadjoint.*

Our main result can be immediately applied to band matrices.

COROLLARY 2.5. *Assume that* $[A_{i,j}]$ *is a symmetric band matrix. If there exists* $N_0 \geq 0$ *such that* $\gamma_N > 0$ *for* $N \geq N_0$ *and*

$$\sum_{N=N_0}^{\infty}1/\gamma_N=\infty, \tag{2.12}$$

then $\mathbf{A}=\overline{\mathbf{A}}_0$ *and* $\mathbf{A}_0$ *is essentially selfadjoint.*

Proof. Apply Theorem 2.3 with $\sigma(n)=1$ for $n\in\mathbb{N}$, $\omega(n)=0$ for $1\leq n\leq s$ and $\omega(n)=\infty$ for $n>s$, s being the width of the band matrix $[A_{i,j}]$. ∎

The Carleman criterion for selfadjointness of Jacobi matrices [3] as well as the Berezanskiĭ one for selfadjointness of operator Jacobi matrices [2] are special cases of Corollary 2.5. All these criteria for essential selfadjointness of operators induced by matrices are independent of their main diagonals.

3. Symmetric operators with "almost invariant" subspaces. In this section we infer from our main result some criteria for essential selfadjointness of symmetric operators possessing a chain of "almost invariant" subspaces. This part of the paper is closely related to some results of Jørgensen (cf. [9, 10]).

We begin with an observation that Corollary 2.5 is equivalent to a result of Jørgensen (cf. [9, Th. 2]), which can be formulated for symmetric operators as follows.

PROPOSITION 3.1. *Assume that* L *is a symmetric operator in* $\mathcal{H}$ *with domain* $\bigcup_{j=0}^{\infty}\mathcal{K}_j$, *where* $\{\mathcal{K}_j\}_{j=0}^{\infty}$ *is an increasing sequence of closed subspaces of* $\mathcal{H}$ *such that for some* $m\geq 1$, $L\mathcal{K}_j\subseteq\mathcal{K}_{j+m}$ *for every* $j\geq 0$. *If* (2)

$$\sum_{n=0}^{\infty}\frac{1}{\|P_n^{\perp}LP_n\|}=\infty, \tag{3.1}$$

(2) The condition (3.1) means that either $\|P_n^{\perp}LP_n\|=0$ for infinitely many n's or $\|P_n^{\perp}LP_n\|>0$ for $n\geq n_0$ and $\sum_{n=n_0}^{\infty}1/\|P_n^{\perp}LP_n\|=\infty$.

then L is essentially selfadjoint (P_n being the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}_n$).

To explain the equivalence of Corollary 2.5 and Proposition 3.1 notice that if L is as in Proposition 3.1, then $L = \mathbf{A}_0$, where $\mathbf{A}_0$ is the operator associated with the band matrix $[A_{i,j}]$ of width m defined by $A_{i,j} = Q_i L|_{\mathcal{H}_j}$; here Q_i is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i := \mathcal{K}_i \ominus \mathcal{K}_{i-1}$ ($A_{i,j}$ is bounded by the closed graph theorem). Conversely, if $[A_{i,j}]$ is a band matrix of width $m \geq 1$, then $L := \mathbf{A}_0$ satisfies the condition $L\mathcal{K}_j \subseteq \mathcal{K}_{j+m}$ with $\mathcal{K}_j = \sum_{i=0}^{j} \oplus \mathcal{H}_i$.

By the above one-to-one correspondence the operator $P_n^{\perp} L|_{\mathcal{K}_n}$ has the operator $m \times m$-matrix representation $[A_{n+i,n-m+j}]_{i,j=1}^{m}$. Consequently, there exists a constant C_m (depending only on m) such that

$$\gamma_n(A) \leq \|P_n^{\perp} L P_n\| \leq C_m \gamma_n(A), \qquad n \in \mathbb{N}.$$

Therefore, the conditions (2.12) and (3.1) are equivalent.

Notice that also Corollary 2.4 has its counterpart in Jørgensen's language.

THEOREM 3.2. *Let L be a symmetric operator in $\mathcal{H}$ with domain $\bigcup_{j=0}^{\infty} \mathcal{K}_j$, where $\{\mathcal{K}_j\}_{j=0}^{\infty}$ is an increasing sequence of closed subspaces of $\mathcal{H}$ such that*

$$L\mathcal{K}_j \subseteq \mathcal{K}_{j+\tau_k}, \qquad \omega_k \leq j < \omega_{k+1},\ k \in \mathbb{N}, \tag{3.2}$$

$$\sum_{k=0}^{\infty} (\tau_{k+1} - \tau_k)\tau_{k+1}\sigma_{\omega_k} < \infty, \tag{3.3}$$

for some decreasing sequence $\{\sigma_k\}_{k=0}^{\infty} \subseteq \mathbb{R}_+$ and for some strictly increasing sequences $\{\tau_k\}_{k=0}^{\infty} \subseteq \mathbb{N}_1$ and $\{\omega_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ with $\omega_0 = 0$. If (3)

$$\sum_{n=0}^{\infty} \frac{\sigma_n}{\|P_n^{\perp} L P_n\|} = \infty, \tag{3.4}$$

then L is essentially selfadjoint (P_n being the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}_n$).

Let $\mathbf{A}_0$ be the operator associated with the locally band matrix $[A_{i,j}]$ defined as in the paragraph following Proposition 3.1. Then $(\{\nu_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ is the barrier of $[A_{i,j}]$ with $\nu_k = \tau_{k-1} + 1$ for $k \geq 1$ and $\nu_0 = 1$. One can easily check that the series $\sum_{k=0}^{\infty} (\nu_{k+1} - \nu_k)\nu_{k+1}\sigma_{\omega_k}$ is convergent. On the other hand, $\gamma_n(A) \leq \|P_n^{\perp} L P_n\|$, so the series $\sum_{n=0}^{\infty} \sigma_n/\gamma_n(A)$ is divergent. Since $L = \mathbf{A}_0$, Theorem 3.2 follows from Corollary 2.4.

We conclude this section with an example which shows that Theorem 3.2 does not imply Corollary 2.4.

(3) See footnote (2).

EXAMPLE 3.3. Define a symmetric locally band matrix $[A_{i,j}]$ as follows:

$$A_{k+1}(j) = \begin{cases} 0, & 0 \le j < k^3, \\ 1, & j \ge k^3, \end{cases} \qquad k \in \mathbb{N},$$

and $A_0(j) = 1$ for $j \in \mathbb{N}$. Its barrier is given by $\tau_k = k+1$ and $\omega_k = k^3$. Set $\sigma_n = (n+1)^{-1}$. Then the series $\sum_{k=0}^{\infty} \sigma_k/\gamma_k(A)$ is divergent while the series $\sum_{k=0}^{\infty}(\tau_{k+1} - \tau_k)\tau_{k+1}\sigma_{\omega_k}$ is convergent, so by Corollary 2.4, $\mathbf{A}$ is selfadjoint. We claim that there is no decreasing sequence $\{\widetilde{\sigma}_n\}_{n=0}^{\infty} \subseteq \mathbb{R}_+$ such that

$$\sum_{k=0}^{\infty}(\tau_{k+1} - \tau_k)\tau_{k+1}\widetilde{\sigma}_{\omega_k} < \infty, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{\widetilde{\sigma}_n}{\|P_n^{\perp} L P_n\|} = \infty. \tag{3.6}$$

For if not, we proceed as follows. Denote by M_k, $k \in \mathbb{N}_1$, the uppertriangular $k \times k$ matrix given by $(M_k)_{i,j} = 1$ for $i \le j$ and $(M_k)_{i,j} = 0$ otherwise. Analyzing the form of the matrix B_n associated with $P_n^{\perp} L P_n$ one can easily see that for k large enough, the matrix B_n always "contains" the right upper triangle $[(M_k)_{i,j}]_{i \le j}$ of M_k, provided $k^3 \le n < (k+1)^3$. This and the fact that B_n has only $0, 1$-entries imply that there is a constant $c > 0$ such that

$$\|P_n^{\perp} L P_n\| \ge \frac{\|\mathbf{M}_k(e)\|}{c\|e\|} \ge \frac{k+1}{3c}, \qquad k \in \mathbb{N}_1,\ k^3 \le n < (k+1)^3, \tag{3.7}$$

where $e = (1, \ldots, 1)$. Applying (3.7), the monotonicity of $\{\widetilde{\sigma}_n\}_{n=0}^{\infty}$ and (3.5), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\widetilde{\sigma}_n}{\|P_n^{\perp} L P_n\|} &= \sum_{k=1}^{\infty} \sum_{n=k^3}^{(k+1)^3-1} \frac{\widetilde{\sigma}_n}{\|P_n^{\perp} L P_n\|} \le 3c \sum_{k=1}^{\infty} \sum_{n=k^3}^{(k+1)^3-1} \frac{\widetilde{\sigma}_n}{k+1} \\ &\le 3c \sum_{k=1}^{\infty} \frac{(k+1)^3 - k^3}{k+1} \widetilde{\sigma}_{k^3} \le 9c \sum_{k=1}^{\infty} (k+1)\widetilde{\sigma}_{k^3} \\ &\le 9c \sum_{k=1}^{\infty} (\tau_{k+1} - \tau_k)\tau_{k+1}\widetilde{\sigma}_{\omega_k} < \infty, \end{aligned}$$

which contradicts (3.6).

Remark 3.4. Notice that the operator $\mathbf{A}_0$ associated with a band matrix $[A_{i,j}]$ is bounded if and only if $\sup_{i,j} \|A_{i,j}\| < \infty$; if this happens, then

$$\|\mathbf{A}\| \le (2s+1) \sup_{i,j} \|A_{i,j}\|,$$

where s is the width of $[A_{i,j}]$. This is no longer true for locally band matrices. Indeed, by (3.7), the matrix from Example 3.3 induces an unbounded selfadjoint operator $\mathbf{A}$, though its entries are uniformly bounded.

4. Locally band matrices with tempered growth. It is clear that the operator $\mathbf{A}_0$ associated with a symmetric locally band matrix $[A_{i,j}]$ is always symmetric and $\mathbf{A}_0(\mathcal{D}(\mathbf{A}_0)) \subseteq \mathcal{D}(\mathbf{A}_0)$. It turns out that every symmetric operator with invariant domain in a separable Hilbert space has a matrix representation which is locally band. Moreover, this matrix can be chosen in such a way that its barrier is as good as we wish. The following result has been inspired by [13] (see also [6, Problem 36]).

PROPOSITION 4.1. *Let S be a symmetric operator in a separable Hilbert space $\mathcal{H}$ such that $S(\mathcal{D}(S)) \subseteq \mathcal{D}(S)$. Take $\omega : \mathbb{N}_1 \to \mathbb{N}$ such that $\omega(1) = 0$. Then there is an orthonormal basis $\{e_n\}_{n=0}^\infty \subseteq \mathcal{D}(S)$ such that the matrix $a_{i,j} := (Se_j, e_i)$ is locally band, ω is a barrier of $[a_{i,j}]$ and $\overline{S} = \overline{\mathbf{A}}_0$.*

Proof. We can assume that $\mathcal{H}$ is infinite-dimensional. Since $\mathcal{H}\oplus\mathcal{H}$ is separable, so is the graph $\Gamma(S)$ of S. Hence, there is a sequence $\{h_n\}_{n=0}^\infty \subseteq \mathcal{D}(S)$ such that $\{(h_n, Sh_n)\}_{n=0}^\infty$ is dense in $\Gamma(S)$. Since the algebraic dimension of $\mathcal{F} := \mathrm{LIN}\,\{h_n\}_{n=0}^\infty$ is $\aleph_0$, we can apply the Schmidt ortonormalization procedure to get an orthonormal basis $\{f_n\}_{n=0}^\infty$ such that $\mathcal{F} = \mathrm{LIN}\,\{f_n\}_{n=0}^\infty$. However, $\{(h_n, Sh_n)\}_{n=0}^\infty \subseteq \Gamma(S|_\mathcal{F}) \subseteq \Gamma(S)$, so we have

$$\overline{S} = (S|_\mathcal{F})^-. \tag{4.1}$$

Modifying ω if necessary we may assume that ω is of the form (2.2), where $\{\omega_k\}_{k=0}^\infty \subseteq \mathbb{N}$ is strictly increasing, $\omega_0 = 0$ and $\tau_k = k+1$, $k \in \mathbb{N}$.

It is clear that if $\mathcal{K}$ is a finite-dimensional subspace of $\mathcal{D}(S)$ and $g \in \mathcal{D}(S)$, then there is $e \in \mathcal{D}(S)$ such that $\|e\| = 1$, e is orthogonal to $\mathcal{K}$ and $g \in \mathrm{LIN}\,(\mathcal{K}\cup\{e\})$. Using repeatedly this fact and the induction procedure one can construct (4) an orthonormal sequence $\{e_n\}_{n=0}^\infty \subseteq \mathcal{D}(S)$ such that $e_0 = f_0$, $f_k \in \mathrm{LIN}\,\{e_0, \ldots, e_{\omega_k + k}\}$ and $Se_m \in \mathrm{LIN}\,\{e_0, \ldots, e_{m+k}\}$ for $\omega_{k-1} \le m < \omega_k$ and $k \in \mathbb{N}_1$. Since $\{f_n\}_{n=0}^\infty \subseteq \mathcal{E} := \mathrm{LIN}\,\{e_n\}_{n=0}^\infty$, we conclude that $\{e_n\}_{n=0}^\infty$ is an orthonormal basis of $\mathcal{H}$ and $(S|_\mathcal{F})^- \subseteq (S|_\mathcal{E})^- \subseteq \overline{S}$, so, by (4.1), we have

$$\overline{S} = (S|_\mathcal{E})^-. \tag{4.2}$$

It follows from the construction of $\{e_n\}_{n=0}^\infty$ that the matrix $[a_{i,j}]$ is locally band and that ω is its barrier. It is easily seen that $\mathbf{A}_0 = S|_\mathcal{E}$, so by (4.2) we have $\overline{S} = \overline{\mathbf{A}}_0$. ∎

If S is a symmetric operator with invariant domain in a separable Hilbert space, $\{\tau_k\}_{k=0}^\infty \subset \mathbb{N}$ is a strictly increasing sequence with $\tau_0 = 1$ and $\{\sigma_k\}_{k=0}^\infty \subset \mathbb{R}_+$ is a decreasing sequence which is convergent to 0, then there is a subsequence $\{\sigma_{\omega_k}\}_{k=0}^\infty$ such that (2.10) holds and $\omega_0 = 0$. By Proposition 4.1, there is an orthonormal basis $\{e_n\}_{n=0}^\infty \subseteq \mathcal{D}(S)$ such that the matrix $a_{i,j} := (Se_j, e_i)$ is locally band, $(\{\tau_k\}_{k=0}^\infty, \{\omega_k\}_{k=0}^\infty)$ is a barrier of $[a_{i,j}]$ and

(4) See [6, Problem 36] for a special case of this construction.

$\overline{S} = \overline{\mathbf{A}}_0$. In other words, for every such sequence $\{\sigma_k\}_{k=0}^{\infty}$ we can always find a matrix representation of S with a barrier satisfying (2.10); therefore the question is whether (2.11) holds.

In general, it is not true that symmetric operators with invariant domains in a separable Hilbert space have band matrix representations. This is a consequence of the following

PROPOSITION 4.2. *If $[a_{i,j}]$ is a band matrix of width s with scalar entries, then the deficiency indices of $\overline{\mathbf{A}}_0$ are both less than or equal to s. If moreover $a_{i,j} \in \mathbb{R}$ for all $i,j \in \mathbb{N}$, then the deficiency indices of $\overline{\mathbf{A}}_0$ coincide.*

In the proof of Proposition 4.2 we use the following result of general nature.

LEMMA 4.3. *Let $f_1,\dots,f_k$ be linearly independent vectors in a Hilbert space $\mathcal{H}$ and let $\{T_n\}_{n=0}^{\infty}$ be a sequence of bounded linear operators in $\mathcal{H}$ strongly convergent to the identity operator I. Then there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, the vectors $T_n f_1,\dots,T_n f_k$ are linearly independent.*

Proof. Suppose, contrary to our claim, that there is an infinite subset E of $\mathbb{N}$ such that for every $n \in E$, the vectors $T_n f_1,\dots,T_n f_k$ are linearly dependent. Thus for every $n \in E$, there exists $\vec{v}_n = (\lambda_{n,1},\dots,\lambda_{n,k}) \in \mathbb{S}_k$ such that

$$\sum_{j=1}^{k} \lambda_{n,j} T_n f_j = 0, \qquad n \in E, \tag{4.3}$$

where $\mathbb{S}_k$ is the unit sphere of $\mathbb{C}^k$. Since $\mathbb{S}_k$ is compact, we can assume without loss of generality (passing to a subsequence, if necessary), that the sequence $\{\vec{v}_n\}_{n\in E}$ is convergent to some $\vec{v} = (\lambda_1,\dots,\lambda_k) \in \mathbb{S}_k$. This and (4.3) imply that $\sum_{j=1}^{k} \lambda_j f_j = 0$, which contradicts the linear independence of $f_1,\dots,f_k$. ∎

Proof of Proposition 4.2. Without loss of generality we may assume that $\mathcal{H} = \ell^2$. Denote by P_n the orthogonal projection of ℓ^2 onto $\mathbb{C}^{n+1} \oplus \{0\}$. Then clearly $\{P_n\}_{n=0}^{\infty}$ is strongly convergent to the identity operator I.

Suppose, contrary to our claim, that $\dim \mathcal{N}(\mathbf{A} \pm \mathrm{i}\, I) = \dim \mathcal{N}(\mathbf{A}_0^* \pm \mathrm{i}\, I) \geq s+1$ (the equality follows from Theorem 1.1). By Lemma 4.3, there is $n \geq 0$ such that $\dim P_{n+s} \mathcal{N}(\mathbf{A} \pm \mathrm{i}\, I) \geq s+1$. However, $P_{n+s}\mathcal{N}(\mathbf{A} \pm \mathrm{i}\, I) \subseteq \mathcal{N}(\mathbf{M}_\pm) \oplus \{0\}$, where $M_\pm = [a_{i,j} \pm \mathrm{i}\,\delta_{i,j}]_{i=0}^{n}\,{}_{j=0}^{n+s}$ ($\delta_{i,j}$ is the Kronecker symbol). Consequently,

$$\dim \mathcal{N}(\mathbf{M}_\pm) \geq s+1. \tag{4.4}$$

Since the matrix $[a_{i,j}]_{i,j=0}^{n}$ is symmetric, we have $\operatorname{rank} [a_{i,j} \pm \mathrm{i}\,\delta_{i,j}]_{i,j=0}^{n} =$

$n+1$ and consequently $\operatorname{rank} M_{\pm} = n+1$. By the Kronecker–Capelli theorem we obtain $\dim \mathcal{N}(\mathbf{M}_{\pm}) = (n+s+1) - \operatorname{rank} M_{\pm} = s$, which contradicts (4.4).

If $a_{i,j} \in \mathbb{R}$ for all $i,j \in \mathbb{N}$, then the antilinear operator J on ℓ^2 defined by $J(\{\lambda_n\}_{n=0}^{\infty}) = \{\overline{\lambda}_n\}_{n=0}^{\infty}$ is a conjugation such that $J\mathbf{A}_0 \subseteq \mathbf{A}_0 J$. Consequently, $J\overline{\mathbf{A}}_0 \subseteq \overline{\mathbf{A}}_0 J$, so by the von Neumann theorem, the deficiency indices of $\overline{\mathbf{A}}_0$ coincide. ■

By the above discussion, we may concentrate on symmetric locally band matrices $[A_{i,j}]$ with operator entries and with a barrier $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ satisfying

$$\tau_{k+1} - \tau_k \le C, \qquad k \in \mathbb{N}, \tag{4.5}$$

for some positive constant C. We show that if $\gamma_n = \mathcal{O}(n^{1-\varepsilon})$ with some $\varepsilon \in (0,1]$ and $\{\omega_k\}_{k=0}^{\infty}$ is of polynomial growth or if $\gamma_n = \mathcal{O}(n)$ and $\{\omega_k\}_{k=0}^{\infty}$ is of exponential growth, then $\mathbf{A}$ is selfadjoint.

PROPOSITION 4.4. *Let $\varepsilon \in (0,1]$ and $s > 2/\varepsilon$. Suppose that* (4.5) *holds. If $[A_{i,j}]$ is a symmetric matrix which satisfies*

(i) $$A_j(n) = 0, \qquad \tau_k \le j < \tau_{k+1},\ n < k^s,$$

(ii) $$\gamma_n = \mathcal{O}(n^{1-\varepsilon}),$$

then $\mathbf{A} = \overline{\mathbf{A}}_0$ is selfadjoint.

Proof. Let ω_k be the largest integer less than or equal to k^s, $k \in \mathbb{N}$, and let $\sigma_n = 1/(n+2)^{\varepsilon}$. Then $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ is a barrier of $[A_{i,j}]$ and

$$\sum_{k=0}^{\infty} (\tau_{k+1} - \tau_k)\tau_{k+1}\sigma_{\omega_k} \le C^2 \sum_{k=0}^{\infty} \frac{k+2}{(k+1)^{s\varepsilon}} < \infty,$$

because $s\varepsilon > 2$. It is easily seen that the series $\sum_{n=0}^{\infty} \sigma_n/\gamma_n$ is divergent. Hence the conclusion follows from Corollary 2.4. ■

PROPOSITION 4.5. *Let $\varepsilon > 0$. Suppose that* (4.5) *holds. If $[A_{i,j}]$ is a symmetric matrix which satisfies*

(i) $$A_j(n) = 0, \qquad \tau_k \le j < \tau_{k+1},\ n < \exp(k^{2+\varepsilon}),$$

(ii) $$\gamma_n = \mathcal{O}(n),$$

then $\mathbf{A} = \overline{\mathbf{A}}_0$ is selfadjoint.

Proof. Let ω_k be the largest integer less than or equal to $\exp(k^{2+\varepsilon})$, $k \in \mathbb{N}$, and let $\sigma_n = 1/\ln(n+1)$ for $n \in \mathbb{N}_1$. Then $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ is a barrier of $[A_{i,j}]$ and

$$\sum_{k=1}^{\infty} (\tau_{k+1} - \tau_k)\tau_{k+1}\sigma_{\omega_k} \le C^2 \sum_{k=1}^{\infty} \frac{k+2}{k^{2+\varepsilon}} < \infty.$$

Since $\sum_{n=1}^{\infty} 1/(n\ln(n+1))$ is divergent, so is $\sum_{n=0}^{\infty} \sigma_n/\gamma_n$. Hence the conclusion follows from Corollary 2.4. ∎

Note that for every symmetric matrix $[A_{i,j}]$ satisfying the growth condition (2.12) and for all $s \in \mathbb{N}_1$, the operator $\mathbf{B}$ associated with the band matrix $[B_{i,j}]$ given by $B_{k,l} = A_{k,l}$ for $|k-l| \le s$ and $B_{k,l} = 0$ for $|k-l| > s$ is selfadjoint (this is obvious by Corollary 2.5). Even more, we can cut out from $[A_{i,j}]$ a symmetric locally band matrix of quite general form which also induces a selfadjoint operator.

PROPOSITION 4.6. *Let* $[A_{i,j}]$ *be a symmetric matrix satisfying the condition* (2.12) *and let* $\{\tau_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ *be a strictly increasing sequence with* $\tau_0 = 1$. *Then there is a strictly increasing sequence* ([5]) $\{\omega_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ *such that the operator* $\mathbf{B}$ *associated with the symmetric matrix* $[B_{i,j}]$ *defined by*

$$B_j(n) = \begin{cases} 0, & n < \omega_k,\ \tau_k \le j < \tau_{k+1},\ k \in \mathbb{N}, \\ A_j(n), & \textit{otherwise}, \end{cases}$$

is selfadjoint.

P r o o f. It follows from (2.12) that there is a decreasing sequence $\{\sigma_n\}_{n=0}^{\infty} \subseteq \mathbb{R}_+$ such that $\lim_{n\to\infty} \sigma_n = 0$ and $\sum_{n=N_0}^{\infty} \sigma_n/\gamma_n = \infty$. Since $\sigma_n \to 0$, there exists a subsequence $\{\sigma_{\omega_k}\}_{k=0}^{\infty}$ such that (2.10) holds. It is clear that $(\{\tau_k\}_{k=0}^{\infty}, \{\omega_k\}_{k=0}^{\infty})$ is a barrier of the symmetric matrix $[B_{i,j}]$. Therefore the conclusion follows from Corollary 2.4. ∎

5. Weighted shifts. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Denote by $\mathcal{E}$ the linear span of $\{e_n\}_{n=0}^{\infty}$. Given a sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}$, denote by D the (normal) diagonal operator with the diagonal elements λ_n ($De_n = \lambda_n e_n$), and by U the (isometric) unilateral shift ($Ue_n = e_{n+1}$). The operator $S = UD$ is called the (unilateral) *weighted shift* with weights $\{\lambda_n\}_{n=0}^{\infty}$. It is clear that S is injective if and only if $\lambda_n \ne 0$ for all n. Recall that a densely defined linear operator T in $\mathcal{H}$ is said to be *hyponormal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|T^* f\| \le \|Tf\|$ for $f \in \mathcal{D}(T)$. It is well known that a weighted shift is hyponormal if and only if the sequence $\{|\lambda_n|\}_{n=0}^{\infty}$ is increasing. Thus every hyponormal weighted shift S can be written as $S_0 \oplus S_1$, where S_0 is the zero operator and S_1 is a hyponormal injective weighted shift (more precisely, if $S \ne 0$ is not injective, then there is $n \in \mathbb{N}$ such that S_0 acts on $\mathrm{LIN}\{e_0, \ldots, e_n\}$). Hence there is no loss of generality in assuming that *all hyponormal weighted shifts under consideration are injective*.

Before stating the main result of this section, we prove a few selected facts concerning powers of hyponormal weighted shifts.

([5]) In fact, for each fixed $n \ge 0$, its entries $\omega_0, \ldots, \omega_n$ can be chosen arbitrarily.

PROPOSITION 5.1. *If $m \geq 1$ and S is a hyponormal weighted shift with weights $\{\lambda_n\}_{n=0}^{\infty}$, then S^m is hyponormal and*

$$\text{(i)} \qquad \mathcal{D}(S^m) = \Big\{ f \in \mathcal{H} : \sum_{n=0}^{\infty} |(f, e_n)\lambda_n \ldots \lambda_{n+m-1}|^2 < \infty \Big\},$$

$$\text{(ii)} \qquad S^m f = \sum_{n=0}^{\infty} (f, e_n)\lambda_n \ldots \lambda_{n+m-1} e_{n+m}, \qquad f \in \mathcal{D}(S^m),$$

$$\text{(iii)} \qquad \mathcal{D}((S^m)^*) = \Big\{ f \in \mathcal{H} : \sum_{n=0}^{\infty} |(f, e_n)\lambda_{n-1} \ldots \lambda_{n-m}|^2 < \infty \Big\},$$

$$\text{(iv)} \qquad (S^m)^* f = \sum_{n=0}^{\infty} (f, e_n)\lambda_{n-1} \ldots \lambda_{n-m} e_{n-m}, \qquad f \in \mathcal{D}((S^m)^*),$$

$$\text{(v)} \qquad (S^m)^* = (S^*)^m,$$

$$\text{(vi)} \qquad \lim_{n\to\infty} S^m P_n f = S^m f, \qquad f \in \mathcal{D}(S^m),$$

$$\text{(vii)} \qquad \lim_{n\to\infty} (S^m)^* P_n f = (S^m)^* f, \qquad f \in \mathcal{D}((S^m)^*),$$

where P_n is the orthogonal projection of $\mathcal{H}$ onto $\mathrm{LIN}\{e_0, \ldots, e_n\}$.

Proof. Since S is hyponormal, the sequence $\{|\lambda_n|\}_{n=0}^{\infty}$ is increasing. The latter can be used to show that $S^m = U^m D_m$, where D_m is the diagonal operator given by $D_m e_k = \lambda_k \ldots \lambda_{k+m-1} e_k$, $k \in \mathbb{N}$. This directly implies conditions (i) through (iv) (because $(S^m)^* = D_m^* U^{*m}$). Exploiting the monotonicity of $\{|\lambda_n|\}_{n=0}^{\infty}$, we obtain $\mathcal{D}(S^m) \subseteq \mathcal{D}((S^m)^*)$. Similarly, by (ii) and (iv), we have $\|(S^m)^* f\|^2 \leq \|S^m f\|^2$ for $f \in \mathcal{D}(S^m)$, which proves the hyponormality of S^m. The conditions (vi) and (vii) follow easily from formulas (i) through (iv).

To prove (v) take $f \in \mathcal{D}((S^m)^*)$. Then, by the monotonicity of $\{|\lambda_n|\}_{n=0}^{\infty}$, there exists a positive constant C such that

$$\sum_{n=0}^{\infty} |(f, e_n)\lambda_{n-1} \ldots \lambda_{n-j}|^2 \leq C \sum_{n=0}^{\infty} |(f, e_n)\lambda_{n-1} \ldots \lambda_{n-m}|^2 < \infty,$$

for $j = 1, \ldots, m$. Applying (iii) we get $f \in \mathcal{D}((S^*)^m)$, which completes the proof. ∎

The next result concerns real parts of operators related to hyponormal weighted shifts. Given a densely defined linear operator T in $\mathcal{H}$ such that $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$, we write $\mathrm{Re}\, T := (T + T^*)/2$. It is clear that $\mathrm{Re}\, T$ is symmetric. Below $\mathbb{C}[Z]$ stands for the algebra of all complex polynomials in one variable Z.

PROPOSITION 5.2. *Let S be a hyponormal weighted shift. Then for every nonzero $p \in \mathbb{C}[Z]$, we have $\mathcal{D}(p(S)) \subset \mathcal{D}(p(S)^*)$, $\mathcal{D}(\mathrm{Re}\, p(S)) = \mathcal{D}(S^{\deg p})$ and*

(i) $$\overline{\operatorname{Re} p(S)|_{\mathcal{E}}} = \overline{\operatorname{Re} p(S)}.$$

Proof. Let $p = \sum_{j=0}^{m} \alpha_j Z^j$ with $m = \deg p$. By Proposition 5.1, S^m is hyponormal and $(S^m)^* = (S^*)^m$, so

$$\mathcal{D}(p(S)) = \mathcal{D}(S^m) \subseteq \mathcal{D}((S^*)^m) = \mathcal{D}\Big(\sum_{j=0}^{m} \overline{\alpha}_j (S^*)^j\Big) \subseteq \mathcal{D}(p(S)^*).$$

Consequently, $\mathcal{D}(\operatorname{Re} p(S)) = \mathcal{D}(S^m)$.

To prove (i), take $f \in \mathcal{D}(p(S))$. Then, by Proposition 5.1, $S^j P_n f$ and $S^{*j} P_n f$ tend to $S^j f$ and $S^{*j} f$, respectively, as $n \to \infty$ ($j = 0, \ldots, m$). Therefore $\operatorname{Re} p(S) P_n f$ tends to $\operatorname{Re} p(S) f$ as $n \to \infty$, which completes the proof. ∎

Here is an application of Corollary 2.5 to real parts of polynomials of hyponormal weighted shifts.

THEOREM 5.3. *Let S be a hyponormal weighted shift with weights $\{\lambda_n\}_{n=0}^{\infty}$.*

(i) *If $\sum_{n=0}^{\infty} |\lambda_n|^{-m} = \infty$ for some $m \geq 1$, then $\overline{\operatorname{Re} p(S)}$ is selfadjoint for all $p \in \mathbb{C}[Z]$ with $\deg p \leq m$.*

(ii) *If $|\lambda_n| = \mathcal{O}(\ln n)$, then $\overline{\operatorname{Re} p(S)}$ is selfadjoint for all $p \in \mathbb{C}[Z]$.*

Proof. Since (ii) follows from (i), it is enough to prove (i). Considering, if necessary, the hyponormal weighted shift tS and the polynomial $p(t^{-1}Z)$ with suitable $t \in \mathbb{C} \setminus \{0\}$, we may assume that $\lambda_0 = 1$. Let $p = \sum_{j=0}^{m} \alpha_j Z^j$ be such that $\deg p \geq 1$. Adopting the convention that $\lambda_j = 0$ and $e_j = 0$ for $j < 0$, we get

$$p(S)e_n = \sum_{j=0}^{m} \alpha_j \lambda_n \ldots \lambda_{n+j-1} e_{n+j},$$
$$p(S)^* e_n = \sum_{j=0}^{m} \overline{\alpha}_j \overline{\lambda}_{n-1} \ldots \overline{\lambda}_{n-j} e_{n-j}$$

and consequently

$$2\operatorname{Re} p(S)e_n = \sum_{j=1}^{m} \alpha_j \lambda_n \ldots \lambda_{n+j-1} e_{n+j} + 2\operatorname{Re}\alpha_0 e_n + \sum_{j=1}^{m} \overline{\alpha}_j \overline{\lambda}_{n-1} \ldots \overline{\lambda}_{n-j} e_{n-j}.$$

It follows that $[a_{i,j}] = [2(\operatorname{Re} p(S)e_j, e_i)]$ is a symmetric band matrix of width m with $a_j(n) = \overline{\alpha}_j \overline{\lambda}_n \ldots \overline{\lambda}_{n+j-1}$, $j = 1, \ldots, m$. Since the sequence $\{|\lambda_n|\}_{n=0}^{\infty}$

is increasing and $\lambda_0 = 1$, we get

$$\begin{aligned}\gamma_N &\le \max\{|a_j(n)| : 1 \le j \le m,\ N-m+1 \le n \le N\}\\ &\le \alpha \max\{|\lambda_{n+j-1}|^j : 1 \le j \le m,\ N-m+1 \le n \le N\}\\ &\le \alpha|\lambda_{N+m-1}|^m, \qquad N \in \mathbb{N},\end{aligned}$$

where $\alpha = \max\{|\alpha_j| : 0 \le j \le m\}$. This implies

$$\sum_{N=0}^{\infty} \frac{1}{\gamma_N} \ge \frac{1}{\alpha} \sum_{N=m-1}^{\infty} \frac{1}{|\lambda_N|^m} = \infty.$$

Applying Proposition 5.2 and Corollary 2.5 we conclude that $\operatorname{Re} p(S)$ is essentially selfadjoint. ∎

The following result on Toeplitz operators in the Segal–Bargmann space $\mathcal{B}_1$ has been proved by the first-named author in [7, Th. 3.4] using a different method.

Corollary 5.4. *If $p \in \mathbb{C}[Z]$ and $\deg p \le 2$, then the Toeplitz operator $T_{\operatorname{Re} p}$ is essentially* (6) *selfadjoint in $\mathcal{B}_1$.*

P r o o f. Let $\{f_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis of $\mathcal{B}_1$ and let $\mathcal{P} := \operatorname{LIN}\{f_n\}_{n=0}^{\infty}$. Then (cf. [1]) the Toeplitz operator T_z defined by

$$T_z f(w) = w f(w), \qquad w \in \mathbb{C}, f \in \mathcal{D}(T_z),$$

is a hyponormal weighted shift with weights $\lambda_n = \sqrt{n+1}$, i.e. $T_z f_n = \lambda_n f_n$. Since $T_{\operatorname{Re} p}|_{\mathcal{P}} = \operatorname{Re} p(T_z)|_{\mathcal{P}}$ and $\sum_{n=0}^{\infty} |\lambda_n|^{-2} = \infty$, Proposition 5.2 and Theorem 5.3 imply that $T_{\operatorname{Re} p}|_{\mathcal{P}}$ is essentially selfadjoint. However, $T_{\operatorname{Re} p}$ is symmetric, so $T_{\operatorname{Re} p}$ is essentially selfadjoint and $(T_{\operatorname{Re} p})^- = (T_{\operatorname{Re} p}|_{\mathcal{P}})^-$. ∎

Note that the restriction on degrees of polynomials appearing in Theorem 5.3 cannot be removed. Indeed, consider the monomial $p = Z^3$ and the weighted shift S with weights $\lambda_n = \sqrt{n+1}$, $n \ge 0$. Then $\sum_{n=0}^{\infty} |\lambda_n|^{-2} = \infty$ and $\operatorname{Re} p(S)$ is not essentially selfadjoint. In fact, one can check (compare with Example 3.6 in [7]) that the deficiency indices of $\overline{\operatorname{Re} p(S)|_{\mathcal{E}}}$ are equal to $(3,3)$. However, by Proposition 5.2, $\overline{\operatorname{Re} p(S)|_{\mathcal{E}}} = \overline{\operatorname{Re} p(S)}$, so $\operatorname{Re} p(S)$ is not essentially selfadjoint.

Theorem 5.3 can be generalized as follows. Denote by $\mathfrak{P}$ the algebra of all complex polynomials in two noncommuting variables X and Y. Then there is a unique involution * in $\mathfrak{P}$ such that $X^* = Y$. Set $\mathfrak{S} = \{p \in \mathfrak{P} : p = p^*\}$.

Theorem 5.5. *Let S be a hyponormal weighted shift with weights $\{\lambda_n\}_{n=0}^{\infty}$.*

(i) *If $\sum_{n=0}^{\infty} |\lambda_n|^{-m} = \infty$ for some $m \ge 1$, then $\overline{p(S,S^*)|_{\mathcal{E}}}$ is selfadjoint for all $p \in \mathfrak{S}$ with $\deg p \le m$.*

(ii) *If $|\lambda_n| = \mathcal{O}(\ln n)$, then $\overline{p(S,S^*)|_{\mathcal{E}}}$ is selfadjoint for all $p \in \mathfrak{S}$.*

(6) If $\deg p = 2$, then the operator $T_{\operatorname{Re} p}$ may not be closed (cf. [8, Ex. 6.2]).

Proof. Once again we can concentrate on the proof of (i). Since $p = p^*$, the operator $p(S,S^*)|_{\mathcal{E}}$ is symmetric. If $S_1,\dots,S_j \in \{S,S^*\}$, $1 \le j \le m$, then

$$S_1 \dots S_j e_n = \mu_1 \dots \mu_j \, e_{n+k}$$

for some $\mu_1,\dots,\mu_j \in \{\lambda_n,\dots,\lambda_{n+m-1},\overline{\lambda}_{n-1},\dots,\overline{\lambda}_{n-m}\}$ and $k = |\{i : S_i = S\}| - |\{i : S_i = S^*\}|$. It is clear that $-j \le k \le j$. It follows that

$$p(S,S^*)e_n = \sum_{k=-m}^{m} \alpha_{n,k} e_{n+k},$$

where $\alpha_{n,k}$ is some finite linear combination of products of the form $\mu_1 \dots \mu_j$ ($j \le m$) with coefficients that do not depend on n and k. We can now repeat estimates from the proof of Theorem 5.3 and then apply Corollary 2.5. ■

A careful inspection of proofs shows that Propositions 5.1 and 5.2, and Theorems 5.3 and 5.5 hold for those injective weighted shifts whose weights are increasing for indices large enough.

We conclude this paper with the following observation. If $[a_{i,j}]$ is a symmetric locally band matrix with scalar entries, then $a_{i,j} = (2\operatorname{Re}\varphi(U)e_j, e_i)$ (or equivalently $\mathbf{A}_0 = 2\operatorname{Re}\varphi(U)$), where φ is the formal power series $\sum_{j=0}^{\infty} Z^j D_j$, D_j is the diagonal operator with diagonal $\{d_{j,n}\}_{n=0}^{\infty}$ given by

$$d_{j,n} = \begin{cases} \overline{a}_j(n), & j \in \mathbb{N}_1, \\ \frac{1}{2}a_0(n), & j = 0, \end{cases} \qquad n \in \mathbb{N},$$

(i.e. $D_j e_n = d_{j,n} e_n$) and $\varphi(U)$ is defined on $\mathcal{D}(\varphi(U)) =: \mathcal{E}$ by $\varphi(U)e_n = \sum_{j=0}^{\infty} d_{j,n} e_{n+j}$. It is clear that D_j, $j \in \mathbb{N}$, are commuting normal operators. Moreover, $\varphi^*(U^*) \subseteq \varphi(U)^*$, where φ^* is the formal power series $\sum_{j=0}^{\infty} D_j^* Z^j$ and $\varphi^*(U^*)$ is defined on $\mathcal{D}(\varphi^*(U^*)) := \mathcal{E}$ by $\varphi^*(U^*)e_n = \sum_{j=0}^{\infty} \overline{d}_{j,n-j} e_{n-j}$. The matrix $[a_{i,j}]$ is a band matrix of width s if and only if φ is a polynomial of degree less than or equal to s, i.e. $\varphi = \sum_{j=0}^{s} Z^j D_j$.

References

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, *Part I*, Comm. Pure Appl. Math. 14 (1961), 187–214.

[2] Yu. M. Berezanskiĭ, *Eigenfunction Expansions for Selfadjoint Operators*, Naukova Dumka, Kiev, 1965 (in Russian).

[3] T. Carleman, *Sur la théorie mathématique de l'équation de Schrödinger*, Ark. Mat. Astr. Fys. 24 (1934), 1–7.

[4] W. Dahmen and C. A. Micchelli, *Banded matrices with banded inverses*, *II*: *locally finite decomposition of spline spaces*, Constr. Approx 9 (1993), 263–281.

[5] S. Demko, *Inverses of band matrices and local convergence of spline projectors*, SIAM J. Numer. Anal. 14 (1977), 616–619.

[6] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N.J., 1967.

[7] J. Janas, *Unbounded Toeplitz operators in the Bargmann–Segal space*, Studia Math. 99 (1991), 87–99.

[8] J. Janas and J. Stochel, *Unbounded Toeplitz operators in the Segal–Bargmann space. II*, J. Funct. Anal. 126 (1994), 418–447.

[9] P. E. T. Jørgensen, *Approximately reducing subspaces for unbounded linear operators*, ibid. 23 (1976), 392–414.

[10] —, *Essential self-adjointness of semibounded operators*, Math. Ann. 237 (1978), 187–192.

[11] W. Mlak, *The Schrödinger type couples related to weighted shifts*, Univ. Iagel. Acta Math. 27 (1988), 297–301.

[12] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, Amer. Math. Soc. Colloq. Publ. 15, Amer. Math. Soc., Providence, R.I., 1932.

[13] O. Toeplitz, *Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen*, Göttingen Nachr. 1910, 489–506.

[14] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

Jan Janas
Institute of Mathematics
Polish Academy of Sciences
Św. Tomasza 30
31-027 Kraków, Poland

Jan Stochel
Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland

Reçu par la Rédaction le 2.10.1995