ANNALES
POLONICI MATHEMATICI
LXVI (1997)

Convergence of orthogonal series of projections in Banach spaces

by RYSZARD JAJTE and ADAM PASZKIEWICZ (Łódź)

To the memory of Włodzimierz Mlak

Abstract. For a sequence (A_j) of mutually orthogonal projections in a Banach space, we discuss all possible limits of the sums $S_n = \sum_{j=1}^{n} A_j$ in a "strong" sense. Those limits turn out to be some special idempotent operators (unbounded, in general). In the case of $X = L_2(\Omega, \mu)$, an arbitrary unbounded closed and densely defined operator A in X may be the μ-almost sure limit of S_n (i.e. $S_n f \to Af$ μ-a.e. for all $f \in \mathcal{D}(A)$).

Introduction. Monotone families of projections are important objects in both classical and functional analysis. Let us mention here the huge classical theory of Fourier series with respect to general or special orthonormal systems of functions, the theory of martingales, the spectral theory of normal operators in a Hilbert space or, more generally, the theory of spectral or well-bounded operators in a Banach space ([3], [5]). In the case of non-selfadjoint projections, the assumption that the systems of idempotent operators considered are uniformly bounded is important and, as a rule, necessary if we want to reach results similar to the classical well-known ones for selfadjoint projections in a Hilbert space. The most typical results of this kind are the integral spectral representations for well-bounded or power-bounded operators ([1], [11], [12]).

The main goal of this paper is to consider some "unbounded situations". We discuss the convergence problems concerning series of mutually orthogonal projections in Banach spaces. We do not assume that the partial sums of those series are bounded in the operator norm. This implies, in particular, that every unbounded closed and densely defined operator A in $L_2(\mu)$

1991 *Mathematics Subject Classification*: Primary 47A58, 46E30.
Key words and phrases: idempotent, mutually orthogonal projections, L_2-space, convergence almost everywhere.
Research supported by the KBN grant 2 P03A 048 08.

is the sum, in the sense of the almost sure convergence, of a certain series of mutually orthogonal idempotent operators. It should be stressed here that the possibility of almost sure approximation of A by the multiples of *orthogonal* projections in $L_2(\mu)$ depends heavily on the properties of the spectral measure of $|A|$ (cf. [7], [8]).

0. Generalities. Let X be a Banach space. For any fixed pair (F, K) of linear subspaces of X (not necessarily closed) satisfying the condition $F \cap K = (\Theta)$, we can define an operator A by putting

(1) $$\mathcal{D}(A) = F \oplus K \quad \text{and} \quad A(f+k) = f \quad \text{for } f \in F,\ k \in K.$$

Obviously, $Ax = A^2x$ for $x \in \mathcal{D}(A)$. Such an A will be called an *idempotent operator*. A may not be closed or densely defined. Such a general definition is motivated by the situations which will be described later and which appear in natural circumstances.

In the sequel, we shall often write $A = (F, K)$ or $A = (F_A, K_A)$.

A bounded idempotent defined on the whole space X is called a *projection* in X. Then, obviously, F and K are closed subspaces of X and $X = F \oplus K$. Let us remark that an idempotent $A = (F, K)$ is closable iff $\overline{F} \cap \overline{K} = (\Theta)$. Indeed, let $\overline{A}$ be the closure of A; then $\overline{A}x = x$, $\overline{A}y = \Theta$ for any $x \in \overline{F}$, $y \in \overline{K}$. Thus $x = \overline{A}x = \Theta$ for any $x \in \overline{F} \cap \overline{K}$. On the other hand, if $\overline{F} \cap \overline{K} = (\Theta)$, then for any $f_n \in F$, $k_n \in K$ with $f_n + k_n \to x$ and $A(f_n + k_n) \to f$, we have $f_n \to f \in \overline{F}$ and $k_n \to k \in \overline{K}$. This means that $(\overline{F}, \overline{K})$ represents the closure $(F, K)^-$ of (F, K).

Clearly, the idempotent (F, K) is closed iff the subspaces F and K are closed.

1. Quasi-strong convergence of orthogonal series of projections in a Banach space

1.1. Projections A_i, $i \in I$, are said to be *mutually orthogonal* iff $A_iA_j = 0$ for $i \neq j$, which is equivalent to $F_i \subseteq K_j$ for $i \neq j$. For two projections A and B, we write $A \le B$ iff $AB = BA = A$. Evidently, if the projections $A_1, A_2, \ldots$ are mutually orthogonal, then $A_1 + \ldots + A_n \le A_1 + \ldots + A_{n+1}$. Conversely, if $S_1 \le S_2 \le \ldots$ are projections, then $S_2 - S_1, S_3 - S_2, \ldots$ are mutually orthogonal projections. It is easy to check that if $A_1, A_2, \ldots$ are mutually orthogonal projections and $A_j = (F_j, K_j)$, then

$$A_1 + \ldots + A_n = (F_1 \oplus \ldots \oplus F_n, K_1 \cap \ldots \cap K_n).$$

Obviously, $A \le B$ iff $F_A \subseteq F_B$ and $K_A \supseteq K_B$ (for more details, see e.g. [4]).

1.2. Let $A_j = (F_j, K_j)$ $(j = 1, 2, \ldots)$ be mutually orthogonal projections in a Banach space X. Put $S_n = \sum_{j=1}^n A_j$. Let

$$\mathcal{D}(S) = \{x \in X : s\text{-}\lim_{n\to\infty} S_n x \text{ exists}\} \tag{2}$$

and let

$$Sx = s\text{-}\lim S_n x \quad \text{for } x \in \mathcal{D}(S). \tag{3}$$

Let us put

$$K = \bigcap_{s=1}^{\infty} K_s \quad \text{and}$$

$$F^0 = \{f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent, } f_j \in F_j\}.$$

Then we have $S = (F^0, K)$, that is, $\mathcal{D}(S) = F^0 \oplus K$ and $S(f^0 + k) = f^0$ for $f^0 + k \in F^0 \oplus K$. Indeed, for $x \in \mathcal{D}(S)$, there exist a sequence $(f_1, f_2, \ldots)$, $f_j \in F_j$, and $k_n \in \bigcap_{j=1}^n K_j$ $(n = 1, 2, \ldots)$, uniquely determined by x and such that

$$x = f_1 + f_2 + \ldots + f_n + k_n \quad \text{for } n = 1, 2, \ldots \tag{4}$$

In particular, we have $f_j = A_j x$. Thus $x \in \mathcal{D}(A)$ iff $f_1 + \ldots + f_n$ strongly converges to Ax as $n \to \infty$ and $k_n \to k \in K$. Consequently, (4) leads us to the equality

$$x = (f_1 + f_2 + \ldots) + k$$

exactly for $x \in \mathcal{D}(A)$.

The idempotent $S = (F^0, K)$ satisfying (2), (3) (not necessarily bounded or densely defined) will be called a *quasi-strong limit* of $S_n = \sum_{j=1}^n A_j$. We shall also write q.s.-$\lim_{n\to\infty} S_n = S$ or q.s.-$\sum_{j=1}^\infty A_j = S = (F^0, K)$.

Obviously, in the case when the projections A_j act in a Hilbert space and are selfadjoint, then S is an orthogonal projection (onto $F^0 = \overline{F^0}$). In general (i.e. when A_j are not necessarily selfadjoint), the operator S can be unbounded. It may be closed and densely defined but also it may happen that it is not closed or densely defined. It may also not be closable, even if X is a Hilbert space.

In the sequel, we shall use the following notation. For vectors $x, y, \ldots$ in a Banach space X, the symbol $[x, y, \ldots]$ denotes the closed subspace of X spanned by $x, y, \ldots$ We shall also write $[F_i; i \in I] = [\bigcup_{i\in I} F_i]$ for a family $(F_i)_{i\in I}$ of subspaces. If X is a Hilbert space, $[x, y, \ldots]^\wedge$ will stand for the orthogonal projection onto $[x, y, \ldots]$, and, for $e \in X$, we will write $\widehat{e} = [e]^\wedge = \langle \cdot, e\rangle e/\|e\|^2$.

1.3. Example (S is unbounded, densely defined and closed). Let H be a separable Hilbert space and let $(e_1, e_2, \ldots, f_1, f_2, \ldots)$ be an orthonormal basis in H. We put

$$g_k = \cos\frac{1}{k} e_k + \sin\frac{1}{k} f_k \quad \text{for } k = 1, 2, \ldots,$$

$$\widehat{F}_n = \sum_{k=1}^{n} \widehat{g}_k, \qquad \widehat{K}_n = \sum_{k=1}^{\infty} \widehat{e}_k + \sum_{k=n+1}^{\infty} \widehat{f}_k$$

($\widehat{F}_n$ is an orthogonal projection onto F_n). Then $\widehat{F}_n \nearrow \widehat{F} = \sum_{k=1}^{\infty} \widehat{g}_k$ and $\widehat{K}_n \searrow \widehat{K} = \sum_{k=1}^{\infty} \widehat{e}_k$. For the sequence $S_n = (F_n, K_n)$, $S_0 = 0$, we have $S_n \le S_{n+1}$, which means that $S_n = \sum_{i=1}^{n} A_i$ with mutually orthogonal projections

$$A_i = S_i - S_{i-1}.$$

It can easily be checked that $F \cap K = (\Theta)$, $\overline{F \oplus K} = H$ and $F \oplus K \ne H$. To show that $F \oplus K \ne H$, it is enough to take $\varphi = \sum_{i=1}^{\infty} i^{-1} f_i \in H$. Then the assumption that $\varphi = f + k$, with $f \in F$, $k \in K$, leads directly to a contradiction. Thus the idempotent (F, K) is unbounded, densely defined and closed. It remains to prove that $S_n \to S = (F, K)$ quasi-strongly, i.e.

$$\mathcal{D}(S) = \{x \in H : S_n x \text{ converges strongly}\} = F \oplus K, \quad \text{and}$$
$$S(f + k) = f \quad \text{for } f \in F,\ k \in K.$$

To do this, let us remark that, for $x \in H$, we have the unique representation

$$x = \sum_{i=1}^{\infty} \alpha_i e_i + \sum_{i=1}^{\infty} \beta_i f_i \quad \text{with } \sum_{i=1}^{\infty} (|\alpha_i|^2 + |\beta_i|^2) \le \infty. \tag{5}$$

We shall first show that, for such an x (of the form (5)), $x \in \mathcal{D}(S)$ iff $\sum_{i=1}^{\infty} i^2 |\beta_i|^2 < \infty$.

Indeed, for $x \in H$ and $n = 1, 2, \ldots,$

$$x = \sum_{i=1}^{n} c_i^{(n)} g_i + \sum_{i=1}^{\infty} \gamma_i^{(n)} e_i + \sum_{i=n+1}^{\infty} \delta_i^{(n)} f_i$$

and, for x of the form (5),

$$c_i^{(n)} = \frac{\beta_i}{\sin(1/i)} \quad \text{for } i = 1, \ldots, n.$$

Thus

$$S_n x = \sum_{i=1}^{n} c_i^{(n)} g_i = \sum_{i=1}^{n} \frac{\beta_i}{\sin(1/i)} g_i$$

is convergent iff

$$\sum_{i=1}^{\infty} \frac{|\beta_i|^2}{\sin^2(1/i)} < \infty. \tag{6}$$

On the other hand, $x \in F \oplus K$ iff $x = \sum_{i=1}^{\infty}(\gamma_i e_i + c_i g_i)$ with $\sum_{i=1}^{\infty}(|\gamma_i|^2 + |c_i|^2) < \infty$. By the uniqueness of coefficients for x of the form (5), we have $\beta_i = c_i \sin(1/i)$. This means that $x \in F \oplus K$ iff (6) holds.

Remark. For any idempotent $(\widehat{F}H, \widehat{K}H)$ in a Hilbert space H with $\widehat{F}$, $\widehat{K}$ orthogonal projections, and for any finite-dimensional orthogonal projections $\widehat{P}_1 \le \widehat{P}_2 \le \ldots$ tending to identity and commuting with $\widehat{F}$ and $\widehat{K}$, the projections

$$(\widehat{P}_n \widehat{F} H, (\widehat{P}_n \widehat{K} + \mathbf{1} - \widehat{P}_n)H)$$

tend quasi-strongly to $(\widehat{F}H, \widehat{K}H)$.

1.4. Example (S is defined only on an arbitrary closed infinite-dimensional subspace F). Let $(f_{is}, g_s; i, s = 1, 2, \ldots)$ be a basis in H with $F = [f_{i,s}; i, s = 1, 2, \ldots]$. Put $A_i = \sum_{s=1}^{\infty}\langle \cdot, f_{is} + g_s\rangle f_{is}$. Then, obviously, A_i are mutually orthogonal projections. Moreover, $\sum_{i=1}^{n} A_i f_{js} = f_{js}$ for $n > j$, and

$$\sum_{i=1}^{n} A_i x = \Big(\sum_{i=1}^{n}\sum_{s=1}^{\infty} \widehat{f}_{is}\Big)x \to x \quad \text{ for } x \in F$$

as well as

$$\sum_{i=1}^{n} A_i g_s = \sum_{i=1}^{n} f_{is}$$

and

$$\Big\|\sum_{i=1}^{n} A_i x\Big\|^2 = n\|x\|^2 \to \infty \quad \text{ for } x \in [g_s; s = 1, 2, , \ldots].$$

1.5. Example (S is densely defined but not closable). Let $(e_1, e_2, \ldots)$ be an orthonormal basis in a Hilbert space H. Let us fix (f_n) by putting

$$\begin{aligned}
f_0 &= e_1, \\
f_1 &= \Big(\cos\frac{1}{2^1}\Big) f_0 + \Big(\sin\frac{1}{2^1}\Big) e_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f_n &= \Big(\cos\frac{1}{2^n}\Big) f_{n-1} + \Big(\sin\frac{1}{2^n}\Big) e_{n+1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and define $F_n = [f_n]$, $K_n = [f_i; i = 0, 1, \ldots,\ i \ne n]$. Then $F_n \cap K_n = (\Theta)$. Indeed, suppose that $f_n \in K_n$, where, obviously,

$$K_n = [e_1, \ldots, e_n, f_{n+1}, e_{n+3}, e_{n+4}, \ldots] \quad \text{ for } n = 0, 1, \ldots \tag{7}$$

Thus

$$f_n = \alpha f_{n+1} + \sum_{\substack{s \ne n+1 \\ s \ne n+2}} c_s e_s.$$

Consequently,

$$0 = \langle f_n, e_{n+2}\rangle = \alpha\langle f_{n+1}, e_{n+2}\rangle = \alpha \sin\frac{1}{2^{n+1}}, \quad \text{so} \quad \alpha = 0$$

and

$$f_n = \sum_{\substack{s\neq n+1\\ s\neq n+2}} c_s e_s,$$

which is impossible.

Obviously, $F_n \oplus K_n = H$, so (F_n, K_n) is a projection.

We know that $S = (F^0, K)$, where

$$F^0 = \Big\{\sum_{i=0}^{\infty} \alpha_i f_i : \text{the series is strongly convergent}\Big\} \tag{8}$$

and $K = \bigcap_{n=0}^{\infty} K_n$. Obviously, $\overline{F^0} = H$. Moreover, $K = [k]$ for

$$k = e_1 + \frac{\sin(1/2)}{\cos(1/2)}e_2 + \frac{\sin(1/2^2)}{\cos(1/2)\cos(1/2^2)}e_3 + \ldots$$

Indeed, by (7), we have $k \in K_n$ iff

$$\begin{aligned}(\widehat{e}_{n+1} + \widehat{e}_{n+2})k &= (\widehat{e}_{n+1} + \widehat{e}_{n+2})\lambda_n f_{n+1}\\ &= \lambda_n\Big[\Big(\cos\frac{1}{2^{n+1}}\sin\frac{1}{2^n}\Big)e_{n+1} + \Big(\sin\frac{1}{2^{n+1}}\Big)e_{n+2}\Big],\end{aligned}$$

and $k \in K_0$ iff

$$(\widehat{e}_1 + \widehat{e}_2)k = (\widehat{e}_1 + \widehat{e}_2)\lambda_0 f_1 = \lambda_0\Big[\Big(\cos\frac{1}{2}\Big)e_1 + \Big(\sin\frac{1}{2}\Big)e_2\Big].$$

1.6. Example (S is not closed but closable with the closure $\mathbf{1}$). Let, as before, $(e_1, e_2, \ldots)$ be an orthonormal basis in H and let

$$\begin{aligned}&f_0 = e_1,\\ &f_1 = \Big(\cos\frac{1}{\sqrt{1}}\Big)f_0 + \Big(\sin\frac{1}{\sqrt{1}}\Big)e_2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &f_n = \Big(\cos\frac{1}{\sqrt{n}}\Big)f_{n-1} + \Big(\sin\frac{1}{\sqrt{n}}\Big)e_{n+1},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

and $F_n = [f_n]$, $K_n = [f_i; i = 0, 1, \ldots,\ i \neq n]$. Then, as in Example 1.5, K_n is of the form (7) and (F_n, K_n) denotes a projection for any $n = 0, 1, \ldots$

It remains to prove that $S = (F^0, (\Theta))$, according to (8). Indeed, $k \in K_n$ iff

$$(\widehat{e}_{n+1} + \widehat{e}_{n+2})k = \lambda_n\Big[\Big(\cos\frac{1}{\sqrt{n+1}}\sin\frac{1}{\sqrt{n}}\Big)e_{n+1} + \Big(\sin\frac{1}{\sqrt{n+1}}\Big)e_{n+2}\Big],$$

and $k \in K_0$ iff

$$(\widehat{e}_1 + \widehat{e}_2)k = \lambda_0 \left[\left(\cos \frac{1}{\sqrt{1}} \right) e_1 + \left(\sin \frac{1}{\sqrt{1}} \right) e_2 \right],$$

and $x \in \bigcap_{n=0}^{\infty} K_n$ iff

$$x = \lambda \left[e_1 + \frac{\sin(1/\sqrt{1})}{\cos(1/\sqrt{1})} e_2 + \frac{\sin(1/\sqrt{2})}{\cos(1/\sqrt{1})\cos(1/\sqrt{2})} e_3 + \frac{\sin(1/\sqrt{3})}{\cos(1/\sqrt{1})\cos(1/\sqrt{2})\cos(1/\sqrt{3})} e_4 + \ldots \right].$$

This is equivalent to $x = \Theta$.

We are going to characterize those densely defined idempotents which are quasi-strong sums of series of mutually orthogonal projections. To clarify the situation a little bit, let us first consider the case $(F^0, (\Theta))$ with $\overline{F^0} = X$.

1.7. PROPOSITION. *The idempotent $(F^0, (\Theta))$ with $\overline{F^0} = X$ is a quasi-strong sum of a series of mutually orthogonal projections if and only if there exists a sequence $(F_i)_{i=1}^{\infty}$ of closed subspaces such that, by putting $\widetilde{F}_i = [F_j; j = 1, 2, \ldots,\ j \neq i]$, we have*

$$F_j \cap \widetilde{F}_j = (\Theta), \qquad F_j \oplus \widetilde{F}_j = X, \tag{9}$$

$$F^0 = \{ f^0 = f_1 + f_2 + \ldots : \textit{the series is strongly convergent and } f_j \in F_j \}, \tag{10}$$

$$\bigcap_{j=1}^{\infty} \widetilde{F}_j = (\Theta). \tag{11}$$

Proof. Let $A_i = (F_i, K_i)$ be a sequence of mutually orthogonal projections and let $(F^0, (\Theta)) = \text{q.s.-}\sum_{i=1}^{\infty} A_i$ (with $\overline{F^0} = X$). Then the sequence (F_i) satisfies (9)–(11). Indeed, by the mutual orthogonality of A_i, we have $\widetilde{F}_i \subseteq K_i$, which implies (11), and $F_i \cap \widetilde{F}_i = (\Theta)$ by 1.2. Moreover, for a fixed i and $x \in X$, we have

$$x = \lim_{s\to\infty} (f_i^{(s)} + \widetilde{f}_i^{(s)}) \quad \text{for some } f_i^{(s)} \in F_i \text{ and } \widetilde{f}_i^{(s)} \in \widetilde{F}_i \subseteq K_i.$$

Thus $x = \lim_{s\to\infty} (f_i^{(s)} + \widetilde{f}_i^{(s)})$ with $f_i^{(s)} \in F_i$ and $\widetilde{f}_i^{(s)} \in K_i$. But the projection (F_i, K_i) is continuous, therefore $f_i^{(s)} \to f_i = A_i x \in F_i$ and, consequently, $\widetilde{f}_i^{(s)} \to \widetilde{f}_i \in \widetilde{F}_i$; so $x = f_i + \widetilde{f}_i \in F_i \oplus \widetilde{F}_i$, which means that $X = F_i \oplus \widetilde{F}_i$.

Clearly, (10) holds by 1.2.

Let $(F_j)_{j=1}^{\infty}$ be a sequence of subspaces satisfying (9)–(11). Then, putting $A_j = (F_j, \widetilde{F}_j)$, we obtain a sequence of mutually orthogonal projections such that

$$\text{q.s.-}\sum_{j=1}^{\infty} A_j = (F^0, (\Theta)). \ \blacksquare$$

In the case of the kernel $K \neq (\Theta)$, the situation is more complicated. Let us start with the following

1.8. Definition. Let Y be a linear subspace of X (not closed in general). A sequence $(F_1, F_2, \ldots)$ of closed subspaces is said to be *basic for Y relative to a closed subspace F_0* iff

$$(12)\qquad Y = \{y = f_1 + f_2 + \ldots : \text{the series is strongly convergent and } f_j \in F_j\}$$

and

$$(13)\qquad F_i \cap \widetilde{F}_i^{(F_0)} = (\Theta), \quad F_i \oplus \widetilde{F}_i^{(F_0)} = X \quad \text{for } i = 1, 2, \ldots,$$

where $\widetilde{F}_i^{(F_0)} = [F_j; j = 0, 1, \ldots,\ j \neq i]$.

A sequence $(F_1, F_2, \ldots)$ basic for Y relative to the zero subspace (Θ) will simply be called *basic for Y*.

Remark. A sequence $(F_j)_{j=1}^{\infty}$ is basic for F^0 iff conditions (9) and (10) of Proposition 1.7 are satisfied.

If (x_j) is a Schauder basis in X, then the one-dimensional subspaces $[x_j]$ form a basic sequence for X. Conversely, if one-dimensional spaces $[x_j]$ form a basic sequence for X, then (x_j) is a Schauder basis in X.

Now, we are in a position to formulate the following characterization of limit idempotents.

1.9. Theorem. *Let (F^0, K) be a densely defined idempotent in X. Then the following conditions are equivalent*:

(i) (F^0, K)=*q.s.*-$\sum_{i=1}^{\infty} A_i$ *for some mutually orthogonal projections* A_i;

(ii) *there exists a sequence (F_j) of subspaces of X, basic for F^0 relative to K, where K is closed and maximal in the sense that, if (F_j) is basic for F^0 relative to some closed subspace $K' \supseteq K$, then $K = K'$.*

Proof. (i)⇒(ii). Let $A_i = (F_i, K_i)$. Then, by 1.2, $K = \bigcap_{s=1}^{\infty} K_s$, so K is closed and $F^0 = \{f^0 = f_1 + f_2 + \ldots :$ the series is strongly convergent and $f_i \in F_i\}$. Moreover, for $F_0 = K$, condition (13) is satisfied (comp. the proof of Proposition 1.7), so (F_j) is basic for F^0 relative to K.

Let $K' \supseteq K$ and let

$$F_i \cap \widetilde{F}_i^{(K')} = (\Theta), \qquad F_i \oplus \widetilde{F}_i^{(K')} = X.$$

Since $F_i \cap \widetilde{F}_i^{(K')} = (\Theta)$ and $F_i \oplus \widetilde{F}_i^{(K')} = X$ and $K' \supseteq K$, we have

$$\widetilde{F}_i^{(K')} = \widetilde{F}_i^{(K)} \quad \text{for } i = 1, 2, \ldots$$

Consequently,

$$K_i \supseteq \widetilde{F}_i^{(K)} = \widetilde{F}_i^{(K')} \supseteq K',$$

so $K = \bigcap_i K_i \supseteq K'$ and, finally, $K = K'$.

(ii)⇒(i). Let us put $A_i = (F_i, \widetilde{F}_i^{(K)})$. Obviously, A_i are mutually orthogonal. We shall show that

$$\text{q.s.-}\sum_{i=1}^{\infty} A_i = (F^0, K).$$

Indeed, by 1.2,

$$F^0 = \{f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent}, \ \ f_i \in F_i\}.$$

It remains to show that

$$\bigcap_{i=1}^{\infty} \widetilde{F}_i^{(K)} = K.$$

To this end, let us put

$$K' = \bigcap_{i=1}^{\infty} \widetilde{F}_i^{(K)}.$$

Then we have $K \subset K'$ and, as can easily be checked, the sequence $(F_1, F_2, \ldots)$ is basic for F^0 relative to K'. By (ii), this implies $K = K'$, which ends the proof. ■

1.10. Example. Let $H = L_2[0,1]$ and let F_0 denote the space of all polynomials (considered on the interval $[0,1]$). Evidently, there is no basic sequence of subspaces for F_0. Thus $(F_0, (\Theta))$ is the idempotent (closable with the closure $\mathbf{1}$) which is not a quasi-strong sum of any series of mutually orthogonal projections in H.

2. Almost sure convergence of orthogonal series of projections in L_2-spaces

2.1. In this section we are interested in the following kind of convergence for operators acting in $X = L_2(\Omega, \mathcal{A}, \mu)$. Let (A_n) be a sequence of bounded linear operators in X and let A be linear (bounded or not). We say that A_n *converge to* A *almost surely* ($A_n \to A$ a.s.) iff $A_n f \to Af$ μ-almost everywhere for all $f \in \mathcal{D}(A)$.

There are several important results concerning this type of convergence. Let us mention here theorems on martingales, on iterates of conditional expectations, individual ergodic theorems or the results on orthogonal series [6], [9].

Let us start with the following theorem, which is an improvement of our previous results [2], [8].

2.2. THEOREM. *Let X be a closed subspace of a separable Hilbert space $L_2(\Omega,\mathcal{A},\mu)$ (over an arbitrary measure space $(\Omega,\mathcal{A},\mu)$). Let (B_k) be a sequence of finite-dimensional operators in X with $B_k\to B$ strongly. Then there exists an increasing sequence $n(k)$ of indices such that $B_{n(k)}\to B$ a.s.*

Proof. By the separability of $L_2(\Omega,\mathcal{A},\mu)$, there exists a sequence $(\mathcal{A}_n)$ of finite subfields of $\mathcal{A}$ such that the conditional expectations $P_n=\mathbb{E}^{\mathcal{A}_n}$ converge strongly and almost surely in $L_2(\Omega,\mathcal{A},\mu)$ to the identity operator $\mathbf{1}=\mathbb{E}^{\mathcal{A}}$. Obviously, P_n are finite-dimensional orthogonal projections in $L_2(\Omega,\mathcal{A},\mu)$. Assume first that the operators B_k act in the whole space $L_2(\Omega,\mathcal{A},\mu)$. Then $C_n=B_n-P_nB\to 0$ strongly and C_n are finite-dimensional. One can define, by induction, sequences $n(k)\nearrow\infty$ and $t(k)\nearrow\infty$ satisfying $t(1)=1$ and

$$\|C_{n(k)}P_{t(k)}\|<2^{-k},\quad \|C_{n(k)}P^{\perp}_{t(k+1)}\|<2^{-k}\quad \text{for } k=1,2,\ldots$$

with $P^{\perp}_t=1-P_t$. Then

$$\begin{aligned}C_{n(k)}f&=C_{n(k)}P_{t(k)}f+C_{n(k)}(P_{t(k+1)}-P_{t(k)})f+C_{n(k)}P^{\perp}_{t(k+1)}f\\&=\pi^1_k+\pi^2_k+\pi^3_k.\end{aligned}$$

Clearly,

$$\sum_k\|\pi^1_k\|^2<\infty,\qquad \sum_k\|\pi^3_k\|^2<\infty$$

and

$$\sum_k\|\pi^2_k\|^2\le\max_n\|C_n\|^2\|f\|^2<\infty.$$

Thus $C_{n(k)}\to 0$ a.s. Consequently, $B_{n(k)}\to B$ a.s.

Passing to the general case, assume that B_k act in a closed subspace X of $L_2(\Omega,\mathcal{A},\mu)$. Let (φ_s) be an orthonormal basis in X. Then $Q_n=\sum_{s=1}^n\widehat{\varphi}_s$ form a sequence of finite-dimensional projections in $L_2(\Omega,\mathcal{A},\mu)$. Passing, if necessary, to a subsequence, we can assume by the first part of the proof that $Q_n\to\mathbf{1}_X$ strongly and almost surely. Now it is enough to repeat the previous argument for $P_k=Q_k$. ■

2.3. COROLLARY. *Let (A_k) be a sequence of finite-dimensional mutually orthogonal projections (not necessarily selfadjoint) in the separable Hilbert space $L_2(\Omega,\mathcal{A},\mu)$. Denote by $(S,\mathcal{D}(S))$ a quasi-strong sum of the series $\sum_{k=1}^\infty A_k$. Assume that S is a closed operator (but not necessarily densely defined). Then there exists $n(k)\nearrow\infty$ such that $\sum_{s=1}^{n(k)}A_s\to S$ a.s.*

Proof. The corollary is an easy consequence of the previous theorem. Indeed, let $A_j=(F_j,K_j)$. Then, by 1.2, $\mathcal{D}(S)=F^0\oplus K$ and $S(f^0+k)=f^0$, where $K=\bigcap_{j=1}^\infty K_j$ and

$$F^0=\{f^0=f_1+f_2+\ldots:\text{the series is strongly convergent and } f_j\in F_j\}.$$

Since S is closed, F^0 is a closed subspace of $L_2(\Omega,\mathcal{A},\mu)$. Putting $S_n=\sum_{j=1}^n A_j$, we find that $S_n f^0\to f^0$ strongly for each $f^0\in F^0$, and S_n are finite-dimensional. By Theorem 2.2, there exists $n(k)\nearrow\infty$ such that $S_{n(k)}f^0\to f^0$ μ-almost everywhere for all $f^0\in F^0$. But this implies $S_n\to S$ a.s. ■

Remark. Theorem 2.2 and Corollary 2.3 can (and should) be treated as generalizations of the classical Marcinkiewicz result [10].

2.4. In contrast to the case of quasi-strong convergence, roughly speaking, each unbounded operator A in $L_2(\mu)$ is an almost sure sum of a series of mutually orthogonal projections. The construction of a suitable sequence of projections is based on the following facts:

(1) the generalization of Marcinkiewicz's theorem easily gives the existence of a sequence (A_j) of finite-dimensional operators such that $\sum_{j=1}^\infty A_j=A$ a.s.,

(2) mutually orthogonal projections (B_j) such that $\sum_{j=1}^\infty B_j=A$ a.s. can be obtained by a suitable perturbation of A_i's with the use of vectors with small supports and some vectors "asymptotically orthogonal" to the domain of A.

Passing to precise formulations, we have the following result.

2.5. THEOREM. *Let $H=L_2(\Omega,\mathcal{A},\mu)$ be a separable Hilbert space such that $0<\mu(Z_n)\to 0$ for some $(Z_n)\subset\mathcal{A}$. Let A be an unbounded closed and densely defined operator in H. Then there exists a sequence (B_j) of mutually orthogonal finite-dimensional projections in X such that the sums $S_n=\sum_{j=1}^n B_j$ converge almost surely to A.*

The following lemma is a key point in the proof of the theorem just formulated. In the sequel, for $Z\in\mathcal{A}$, the symbol $\mathbf{1}_Z$ denotes the multiplication operator in L_2 by the indicator χ_Z of Z.

2.6. LEMMA. *Let $H=L_2(\Omega,\mathcal{A},\mu)$ for an arbitrary measure space satisfying $0\ne\mu(Z_n)\to 0$ for some $(Z_n)\subset\mathcal{A}$. Let $D=\{f\in H:\int_0^\infty\lambda^2\|e(d\lambda)f\|^2<\infty\}$ for some spectral measure $e(\cdot)$ satisfying $e(\Lambda,\infty)\ne 0$ for all $\Lambda>0$. Then, for any sequence (A_n) of finite-dimensional operators in H, there exists a sequence (B_n) of mutually orthogonal finite-dimensional projections in H (not necessarily selfadjoint) satisfying, for some $Y_1\supseteq Y_2\supseteq\ldots$ $\ldots$, $(Y_n)\subset\mathcal{A}$, $\mu(Y_n)\to 0$, the condition*

$$\sum_{n=1}^\infty\Big\|\mathbf{1}_{\Omega\setminus Y_n}\Big(\sum_{i=1}^n A_i-\sum_{i=1}^n B_i\Big)f\Big\|^2<\infty\quad\textit{for all } f\in D.$$

Proof. Clearly, there exist sets $Z(1)\supseteq Z(2)\supseteq\ldots$ in $\mathcal{A}$ such that $\mu(Z(s))\to 0$ and the projections $\mathbf{1}_{Z(s)}$ are infinite-dimensional. Then some selfadjoint projections

$$L(1) \le L(2) \le \ldots, \quad L(s) \to 1,$$

can be fixed in such a way that the subspaces

$$(L(s+1) - L(s))H \cap \mathbf{1}_{Z(s)}H \quad \text{and} \quad (L(s+1) - L(s))H \cap e(s,\infty)H$$

are infinite-dimensional. Indeed, one can take, for example, an orthonormal system $(\xi_s, \zeta_t; s, t = 1, 2, \ldots)$ with $\xi_s \in \mathbf{1}_{Z(s)}H$, $\zeta_t \in e(t,\infty)H$ and put $L(\Lambda) = 1 - [\xi_s, \zeta_t; s, t \text{ are divisible by } 2^{\Lambda}]^{\wedge}$.

Now, fix an orthonormal system $(\varphi_s^{ij}, \psi^{ij}(\Lambda); i, j, s, \Lambda = 1, 2, \ldots)$ satisfying

$$\begin{gathered} \varphi_s^{ij} \in (L(s+1) - L(s))H \cap \mathbf{1}_{Z(s)}H, \\ \psi^{ij}(\Lambda) \in (L(\Lambda+1) - L(\Lambda))H \cap e(\Lambda,\infty)H. \end{gathered} \tag{14}$$

Define

$$\|f\|_e^2 = \int_0^\infty \lambda^2 \|e(d\lambda)f\|^2 \quad \text{for all } f \in D. \tag{15}$$

Let us begin the inductive construction of $B_1, B_2, \ldots$ Put $A_0 = B_0 = 0$ and assume that the mutually orthogonal projections $B_0, \ldots, B_n$ for some $n \ge 0$ have already been defined so that

$$\Big\|\mathbf{1}_{\Omega\setminus Y_k}\Big(\sum_{i=0}^{k} A_i - \sum_{i=0}^{k} B_i\Big)f\Big\| \le 2^{-k}(\|f\|_e + 2\|f\|), \quad \mu(Y_k) < \frac{1}{k} \tag{16}$$

for $f \in D$, $k = 1, \ldots, n$ and for some sets $Y_1 \supseteq \ldots \supseteq Y_n$ in $\mathcal{A}$. Assume additionally that the projections B_i for $i = 1, \ldots, n$ can be written in the form

$$B_i = \sum_{j=1}^{k(i)} \Big\langle \cdot, u_n^{ij} + \sum_{s=M(n)}^{\infty} \delta_s^{ij}\varphi_s^{ij}\Big\rangle \Big(v_n^{ij} + \sum_{\substack{t=1,2,\ldots \\ \Lambda_t^{ij} \ge M(n)}} \varepsilon_t^{ij}\psi^{ij}(\Lambda_t^{ij})\Big), \tag{17}$$

where $\delta_s^{ij} > 0$, $\varepsilon_t^{ij} > 0$, $M(n)$ is some positive integer, the sequences of positive integers $\Lambda_1^{ij} < \Lambda_2^{ij} < \ldots$ satisfy the inequalities

$$\Lambda_t^{ij} > \frac{t}{\varepsilon_t^{ij}}, \tag{18}$$

and

$$u_n^{ij}, v_n^{ij} \in L(M(n))H, \quad Z(M(n)) \subseteq Y_n. \tag{19}$$

We define an operator B_{n+1} satisfying (16) for $k = n+1$ and rewrite the operators $B_1, \ldots, B_n$ modifying (17) by taking $n+1$ instead of n. Let

$$\sum_{i=0}^{n+1} A_i - \sum_{i=0}^{n} B_i = \sum_{j=1}^{k(n+1)} \langle \cdot, u_j \rangle v_j.$$

There exists $M \geq 1$, $M > M(n)$ in the case $n > 0$, such that, for

$$x^{n+1,j} = L(M) u_j, \quad y^{n+1,j} = L(M) v_j,$$

we have

$$\begin{aligned} &\Big\| \sum_{j=0}^{k(n+1)} \langle \cdot, u_j \rangle v_j - \sum_{j=0}^{k(n+1)} \langle \cdot, x^{n+1,j} \rangle y^{n+1,j} \Big\| < 2^{-(n+1)}, \\ &\mu(Z(M)) < \frac{1}{n+1}. \end{aligned} \tag{20}$$

Obviously, $B_1, \ldots, B_n$ can be written in the form

$$B_i = \sum_{j=1}^{k(i)} \Big\langle \cdot, x^{ij} + \sum_{s=M}^{\infty} \delta_s^{ij} \varphi_s^{ij} \Big\rangle \Big(y^{ij} + \sum_{\substack{t=1,2,\ldots \\ \Lambda_t^{ij} \geq M}} \varepsilon_t^{ij} \psi^{ij}(\Lambda_t^{ij}) \Big),$$

where

$$\begin{aligned} x^{ij} &= u_n^{ij} + \sum_{s=M(n)}^{M-1} \delta_s^{ij} \varphi_s^{ij} \in L(M)H, \\ y^{ij} &= v_n^{ij} + \sum_{\substack{t=1,2,\ldots \\ M(n) \leq \Lambda_t^{ij} < M}} \varepsilon_t^{ij} \psi^{ij}(\Lambda_t^{ij}) \in L(M)H. \end{aligned}$$

Let us fix

$$\begin{gathered} \delta_s^{n+1,j} > 0, \quad \varepsilon_t^{n+1,j} > 0, \\ \Lambda_1^{n+1,j} < \Lambda_2^{n+1,j} < \ldots, \quad \Lambda_t^{n+1,j} \geq \frac{t}{\varepsilon_t^{n+1,j}}, \quad \Lambda_t^{n+1,j} \geq M, \end{gathered}$$

such that, for

$$\begin{aligned} \widetilde{B}_{n+1} = &\sum_{j=1}^{k(n+1)} \Big\langle \cdot, x^{n+1,j} + \sum_{s=M}^{\infty} \delta_s^{n+1,j} \varphi_s^{n+1,j} \Big\rangle \\ &\times \Big(y^{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_t^{n+1,j} \psi^{n+1,j}(\Lambda_t^{n+1,j}) \Big), \end{aligned} \tag{21}$$

we have the estimate

$$\left\|\widetilde{B}_{n+1} - \sum_{j=1}^{k(n+1)} \langle \cdot, x^{n+1,j}\rangle y^{n+1,j}\right\| < 2^{-(n+1)}. \tag{22}$$

Now, one can find a perturbation B_{n+1} of $\widetilde{B}_{n+1}$ such that $B_0, \ldots, B_{n+1}$ are the required mutually orthogonal projections. Namely, there exist matrices

$$(a_j^{ij'})_{j=1,\ldots,k(n+1)}^{i=1,\ldots,n+1,\, j'=1,\ldots,k(i)}, \qquad (b_j^{ij''})_{j=1,\ldots,k(n+1)}^{i=1,\ldots,n,\, j''=1,\ldots,k(i)}$$

such that, for an arbitrary orthonormal system

$$(e^{ij}, f^{ij}; i = 1, \ldots, n,\ j = 1, \ldots, k(i)) \cup (f^{n+1,j}; j = 1, \ldots, k(n+1))$$

orthogonal to

$$(x^{ij}, y^{ij}; i = 1, \ldots, n+1,\ j = 1, \ldots, k(i)),$$

the operators

$$\begin{aligned} C_1 &= \sum_{j=1}^{k(1)} \langle \cdot, x^{1j} + e^{1j}\rangle (y^{1j} + f^{1j}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_n &= \sum_{j=1}^{k(n)} \langle \cdot, x^{nj} + e^{nj}\rangle (y^{nj} + f^{nj}), \\ C_{n+1} &= \sum_{j=1}^{k(n+1)} \Big\langle \cdot, x^{n+1,j} + \sum_{i=1}^{n+1}\sum_{j'=1}^{k(i)} a_j^{ij'} f^{ij'} \Big\rangle \\ &\quad \times \Big(y^{n+1,j} + f^{n+1,j} + \sum_{i=1}^{n}\sum_{j''=1}^{k(i)} b_j^{ij''} e^{ij''} \Big) \end{aligned}$$

satisfy the conditions

$$C_{n+1} \text{ is a projection,} \tag{23}$$

$$C_{n+1}C_i = 0, \tag{24}$$

$$C_iC_{n+1} = 0, \tag{25}$$

for $i = 1, \ldots, n$. In fact, (23) uniquely determines the entries $(a_j^{n+1,j'}; j, j' = 1, \ldots, k(n+1))$, and, for $n > 0$, condition (24) (respectively (25)) uniquely determines the entries $(a_j^{i,j'}; j = 1, \ldots, k(n+1),\ j' = 1, \ldots, k(i))$ (respectively $(b_j^{i,j''}; j = 1, \ldots, k(n+1),\ j'' = 1, \ldots, k(i))$).

Let us remark that, for $q > 0$, $\varepsilon > 0$, $\psi \in H$, $\|\psi\| = 1$, we see, by (15), that

$$\psi \in e\Big(\frac{q}{\varepsilon}, \infty\Big) \quad \text{implies} \quad \Big|\Big\langle f, \frac{\psi}{\varepsilon}\Big\rangle\Big| < \frac{1}{q}\|f\|_e \quad \text{for any } f \in \mathcal{D}.$$

This makes it possible to find q (large enough) such that

$$(26) \qquad \Big\| \sum_{j=1}^{k(n+1)} \Big\langle f, \sum_{i=1}^{n+1} \sum_{j'=1}^{k(i)} \frac{a_j^{ij'}}{\varepsilon_{j+q}^{ij'}} \psi^{ij'}(\varLambda_{j+q}^{ij'}) \Big\rangle \times \Big(y^{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_t^{n+1,j} \psi^{n+1,j}(\varLambda_t^{n+1,j}) \Big) \Big\| < 2^{-(n+1)} \|f\|_e$$

(by (14), (18)). Finally, we define B_{n+1} by putting

$$\begin{aligned}(27) \quad B_{n+1} = & \sum_{j=1}^{k(n+1)} \Big\langle \cdot, x^{n+1,j} + \sum_{s=M}^{\infty} \delta_s^{n+1,j} \varphi_s^{n+1,j} \\ & + \sum_{i=1}^{n+1} \sum_{j'=1}^{k(i)} \frac{a_j^{ij'}}{\varepsilon_{j+q}^{ij'}} \psi^{ij'}(\varLambda_{j+q}^{ij'}) \Big\rangle \\ & \times \Big(y^{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_t^{n+1,j} \psi^{n+1,j}(\varLambda_t^{ij}) + \sum_{i=1}^{n} \sum_{j''=1}^{k(i)} \frac{b_j^{ij''}}{\delta_{j+M}^{ij''}} \varphi_{j+M}^{ij''} \Big) \\ = & \sum_{j=1}^{k(n+1)} \langle \cdot, \widetilde{x}^{n+1,j} \rangle \widetilde{y}^{n+1,j}.\end{aligned}$$

One can easily observe, for $Y_{n+1} = Z(M)$, that (14) implies

$$\mathbf{1}_{\Omega \setminus Y_{n+1}} \varphi_{j+M}^{ij''} = 0 \quad \text{for } j'' = 1, \ldots, k(i),\ j = 1, \ldots, k(n+1)$$

and, in the case $n > 0$, $Y_n \supseteq Y_{n+1}$. Thus, by (26) and by (20)–(22),

$$\begin{aligned}\Big\| \mathbf{1}_{\Omega \setminus Y_{n+1}} \Big(\sum_{i=1}^{n+1} A_i - \sum_{i=1}^{n+1} B_i \Big) f \Big\| & \\ \le 2^{-n} \|f\|_e + & \Big\| \mathbf{1}_{\Omega \setminus Y_{n+1}} \Big(\sum_{i=1}^{n+1} A_i - \sum_{i=1}^{n} B_i - \widetilde{B}_{n+1} \Big) f \Big\| \\ \le 2^{-(n+1)} & (\|f\|_e + 2\|f\|), \\ \mu(Y_{n+1}) < \frac{1}{n+1}. &\end{aligned}$$

To conclude the proof, it remains to show that the operators $B_1, \ldots, B_{n+1}$ can be rewritten in form (17) with $n+1$ instead of n. To this end, it is enough to fix $M(n+1)$ such that

$$\begin{aligned}
&M(n+1) \geq M,\\
&M(n+1) \geq \max(M+k(i); i=1,\ldots,n+1),\\
&M(n+1) \geq \max(\Lambda_{j+q}^{ij'}; i=1,\ldots,n+1,\\
&\qquad\qquad\qquad j=1,\ldots,k(n+1),\ j''=1,\ldots,k(i)),
\end{aligned}$$

and to put

$$\begin{aligned}
u_{n+1}^{ij} &= L(M(n+1))\Big(u_n^{ij} + \sum_{s=M(n)}^{\infty} \varepsilon_s^{ij}\varphi_s^{ij}\Big) \quad \text{for } i=1,\ldots,n,\\
u_{n+1}^{n+1,j} &= L(M(n+1))\widetilde{x}^{\,n+1,j},\\
v_{n+1}^{ij} &= L(M(n+1))\Big(v_n^{ij} + \sum_{\substack{t=1,2,\ldots\\ \Lambda_t^{ij}\geq M(n)}} \varepsilon_t^{ij}\psi^{ij}(\Lambda_t^{ij})\Big) \quad \text{for } i=1,\ldots,n,\\
v_{n+1}^{n+1,j} &= L(M(n+1))\widetilde{y}^{\,n+1,j}
\end{aligned}$$

(compare (27)). ∎

Proof of Theorem 2.5. For a closed and densely defined A, one can easily construct a sequence (C_n) of finite-dimensional operators such that $\|C_n f - Af\| \to 0$ for all $f \in \mathcal{D}(A)$. Let $e(\cdot)$ be the spectral measure of $|A|$. Putting $B = \int_0^\infty \min(1,\lambda^{-1})\, e(d\lambda)$, we can apply Theorem 2.2 to a sequence $C_n B$, so there exists $n(k) \nearrow \infty$ such that $C_{n(k)}Bg \to ABg$ for all $g \in H$. Moreover, $f \in \mathcal{D}(A)$ iff $f = Bg$ for some g. Thus we have $S_k = C_{n(k)}B \to A$ a.s.

Putting $A_k = S_k - S_{k-1}$, $S_0 = 0$, we have $\sum_{i=1}^\infty A_i = A$ a.s. Now, it is enough to apply Lemma 2.6. ∎

Remark. In the theorem just proved, the assumption that $H = L_2(\Omega, \mathcal{A}, \mu)$ with $0 < \mu(Z_n) \to 0$ for some $(Z_n) \subset \mathcal{A}$ can be replaced by a weaker condition. Namely, it is enough to assume that H is a closed subspace of $L_2(\Omega, \mathcal{A}, \mu)$ such that there exists $(f_n) \subset H$ with $\|f_n\| = 1$, $f_n \to 0$ in measure μ.

References

[1] E. Berkson and T. A. Gillespie, *Stečkin's theorem, transference and spectral decompositions*, J. Funct. Anal. 70 (1987), 140–170.

[2] J. L. Ciach, R. Jajte and A. Paszkiewicz, *On the almost sure approximation and convergence of linear operators in L_2-spaces*, Probab. Math. Statist. 15 (1995), 215–225.

[3] H. R. Downson, *Spectral Theory of Linear Operators*, London Math. Soc. Monographs 12, Academic Press, New York, 1978.

[4] N. Dunford and J. T. Schwartz, *Linear Operators Part I: General Theory*, Interscience, New York, 1958.

[5] —, —, *Linear Operators Part III: Spectral Operators*, Interscience, New York, 1971.

[6] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Lectures in Adv. Math. 4, Markham, Chicago, 1970.

[7] R. Jajte and A. Paszkiewicz, *Almost sure approximation of unbounded operators in* $L_2(X, \mathcal{A}, \mu)$, to appear.

[8] —, —, *Topics in almost sure approximation of unbounded operators in* L_2*-spaces*, in: Interaction between Functional Analysis, Harmonic Analysis, and Probability, Columbia, 1994, Lecture Notes in Pure and Appl. Math. 175, Dekker, New York, 1996, 219–228.

[9] S. Kaczmarz, *Sur la convergence et la sommabilité des développements orthogonaux*, Studia Math. 1 (1929), 87–121.

[10] J. Marcinkiewicz, *Sur la convergence des séries orthogonales*, ibid. 6 (1933), 39–45.

[11] I. R. Ringrose, *On well-bounded operators*, J. Austral. Math. Soc. 1 (1960), 334–343.

[12] —, *On well-bounded operators, II*, Proc. London Math. Soc. 13 (1963), 613–630.

Institute of Mathematics
Łódź University
S. Banacha 22
90-238 Łódź, Poland
E-mail: rjajte@plunlo51.bitnet

Reçu par la Rédaction le 12.7.1995