Convergence of orthogonal series of projections
in Banach spaces

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Abstract. For a sequence \((A_j)\) of mutually orthogonal projections in a Banach space, we discuss all possible limits of the sums \(S_n = \sum_{j=1}^{n} A_j\) in a "strong" sense. Those limits turn out to be some special idempotent operators (unbounded, in general). In the case of \(X = L_2(\Omega, \mu)\), an arbitrary unbounded closed and densely defined operator \(A\) in \(X\) may be the \(\mu\)-almost sure limit of \(S_n\) (i.e. \(S_n f \to Af \ \mu\)-a.e. for all \(f \in \mathcal{D}(A)\)).

Introduction. Monotone families of projections are important objects in both classical and functional analysis. Let us mention here the huge classical theory of Fourier series with respect to general or special orthonormal systems of functions, the theory of martingales, the spectral theory of normal operators in a Hilbert space or, more generally, the theory of spectral or well-bounded operators in a Banach space ([3], [5]). In the case of non-selfadjoint projections, the assumption that the systems of idempotent operators considered are uniformly bounded is important and, as a rule, necessary if we want to reach results similar to the classical well-known ones for selfadjoint projections in a Hilbert space. The most typical results of this kind are the integral spectral representations for well-bounded or power-bounded operators ([1], [11], [12]).

The main goal of this paper is to consider some "unbounded situations". We discuss the convergence problems concerning series of mutually orthogonal projections in Banach spaces. We do not assume that the partial sums of those series are bounded in the operator norm. This implies, in particular, that every unbounded closed and densely defined operator \(A\) in \(L_2(\mu)\)

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is the sum, in the sense of the almost sure convergence, of a certain series of mutually orthogonal idempotent operators. It should be stressed here that the possibility of almost sure approximation of $A$ by the multiples of orthogonal projections in $L_2(\mu)$ depends heavily on the properties of the spectral measure of $|A|$ (cf. [7], [8]).

0. Generalities. Let $X$ be a Banach space. For any fixed pair $(F, K)$ of linear subspaces of $X$ (not necessarily closed) satisfying the condition $F \cap K = (\Theta)$, we can define an operator $A$ by putting

$$D(A) = F \oplus K \quad \text{and} \quad A(f + k) = f \quad \text{for} \quad f \in F, \ k \in K.$$  

Obviously, $Ax = A^2x$ for $x \in D(A)$. Such an $A$ will be called an idempotent operator. $A$ may not be closed or densely defined. Such a general definition is motivated by the situations which will be described later and which appear in natural circumstances.

In the sequel, we shall often write $A = (F, K)$ or $A = (F_A, K_A)$.

A bounded idempotent defined on the whole space $X$ is called a projection in $X$. Then, obviously, $F$ and $K$ are closed subspaces of $X$ and $X = F \oplus K$. Let us remark that an idempotent $A = (F, K)$ is closable iff $F \cap K = (\Theta)$. Indeed, let $A$ be the closure of $A$; then $\overline{Ax} = x$, $\overline{Ay} = \Theta$ for any $x \in F$, $y \in K$. Thus $x = \overline{Ax} = \Theta$ for any $x \in \overline{F} \cap \overline{K}$. On the other hand, if $\overline{F} \cap \overline{K} = (\Theta)$, then for any $f_n \in F$, $k_n \in K$ with $f_n + k_n \to x$ and $A(f_n + k_n) \to f$, we have $f_n \to f \in \overline{F}$ and $k_n \to k \in \overline{K}$. This means that $(\overline{F}, \overline{K})$ represents the closure $(F, K)^-$ of $(F, K)$.

Clearly, the idempotent $(F, K)$ is closed iff the subspaces $F$ and $K$ are closed.

1. Quasi-strong convergence of orthogonal series of projections in a Banach space

1.1. Projections $A_i$, $i \in I$, are said to be mutually orthogonal iff $A_iA_j = 0$ for $i \neq j$, which is equivalent to $F_i \subseteq K_j$ for $i \neq j$. For two projections $A$ and $B$, we write $A \leq B$ iff $AB = BA = A$. Evidently, if the projections $A_1, A_2, \ldots$ are mutually orthogonal, then $A_1 + \ldots + A_n \leq A_1 + \ldots + A_{n+1}$. Conversely, if $S_1 \leq S_2 \leq \ldots$ are projections, then $S_2 - S_1, S_3 - S_2, \ldots$ are mutually orthogonal projections. It is easy to check that if $A_1, A_2, \ldots$ are mutually orthogonal projections and $A_j = (F_j, K_j)$, then

$$A_1 + \ldots + A_n = (F_1 \oplus \ldots \oplus F_n, K_1 \cap \ldots \cap K_n).$$

Obviously, $A \leq B$ iff $F_A \subseteq F_B$ and $K_A \supseteq K_B$ (for more details, see e.g. [4]).

1.2. Let $A_j = (F_j, K_j)$ ($j = 1, 2, \ldots$) be mutually orthogonal projections.
in a Banach space $X$. Put $S_n = \sum_{j=1}^{n} A_j$. Let

$$D(S) = \{ x \in X : \text{s-\lim}_{n \to \infty} S_n x \text{ exists} \}$$

and let

$$Sx = \text{s-\lim} S_n x \quad \text{for } x \in D(S).$$

Let us put

$$K = \bigcap_{s=1}^{\infty} K_s \quad \text{and} \quad F^0 = \{ f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent, } f_j \in F_j \}.$$ 

Then we have $S = (F^0, K)$, that is, $D(S) = F^0 \oplus K$ and $S(f^0 + k) = f^0$ for $f^0 + k \in F^0 \oplus K$. Indeed, for $x \in D(S)$, there exist a sequence $(f_1, f_2, \ldots)$, $f_j \in F_j$, and $k_n \in \bigcap_{j=1}^{n} K_j$ ($n = 1, 2, \ldots$), uniquely determined by $x$ and such that

$$x = f_1 + f_2 + \ldots + f_n + k_n \quad \text{for } n = 1, 2, \ldots$$

In particular, we have $f_j = A_j x$. Thus $x \in D(A)$ iff $f_1 + \ldots + f_n$ strongly converges to $Ax$ as $n \to \infty$ and $k_n \to k \in K$. Consequently, (4) leads us to the equality

$$x = (f_1 + f_2 + \ldots) + k$$

exactly for $x \in D(A)$.

The idempotent $S = (F^0, K)$ satisfying (2), (3) (not necessarily bounded or densely defined) will be called a quasi-strong limit of $S_n = \sum_{j=1}^{n} A_j$. We shall also write q.s.-\lim_{n \to \infty} S_n = S$ or q.s.-\sum_{j=1}^{\infty} A_j = S = (F^0, K)$.

Obviously, in the case when the projections $A_j$ act in a Hilbert space and are selfadjoint, then $S$ is an orthogonal projection (onto $F^0 = F^0_0$). In general (i.e. when $A_j$ are not necessarily selfadjoint), the operator $S$ can be unbounded. It may be closed and densely defined but also it may happen that it is not closed or densely defined. It may also not be closable, even if $X$ is a Hilbert space.

In the sequel, we shall use the following notation. For vectors $x, y, \ldots$ in a Banach space $X$, the symbol $[x, y, \ldots]$ denotes the closed subspace of $X$ spanned by $x, y, \ldots$. We shall also write $[F_i; i \in I] = \bigcup_{i \in I} F_i$ for a family $(F_i)_{i \in I}$ of subspaces. If $X$ is a Hilbert space, $[x, y, \ldots]^{\perp}$ will stand for the orthogonal projection onto $[x, y, \ldots]$, and, for $e \in X$, we will write

$$\hat{e} = [e]^{\perp} = \langle \cdot, e \rangle / \| e \|^2.$$ 

1.3. Example ($S$ is unbounded, densely defined and closed). Let $H$ be a separable Hilbert space and let $(e_1, e_2, \ldots, f_1, f_2, \ldots)$ be an orthonormal
basis in $H$. We put
\[ g_k = \cos \frac{1}{k} e_k + \sin \frac{1}{k} f_k \quad \text{for } k = 1, 2, \ldots, \]
\[ \hat{F}_n = \sum_{k=1}^{n} \hat{g}_k, \quad \hat{K}_n = \sum_{k=1}^{\infty} \hat{c}_k + \sum_{k=n+1}^{\infty} \hat{f}_k \]
($\hat{F}_n$ is an orthogonal projection onto $F_n$). Then $\hat{F}_n \uparrow \hat{F} = \sum_{k=1}^{\infty} \hat{g}_k$ and $\hat{K}_n \downarrow \hat{K} = \sum_{k=1}^{\infty} \hat{c}_k$. For the sequence $S_n = (F_n, K_n)$, $S_0 = 0$, we have $S_n \leq S_{n+1}$, which means that $S_n = \sum_{i=1}^{n} A_i$ with mutually orthogonal projections
\[ A_i = S_i - S_{i-1}. \]
It can easily be checked that $F \cap K = (\Theta)$, $\overline{F \oplus K} = H$ and $F \oplus K \neq H$. To show that $F \oplus K \neq H$, it is enough to take $\varphi = \sum_{i=1}^{\infty} i^{-1} f_i \in H$. Then the assumption that $\varphi = f + k$, with $f \in F$, $k \in K$, leads directly to a contradiction. Thus the idempotent $(F, K)$ is unbounded, densely defined and closed. It remains to prove that $S_n \rightarrow S = (F, K)$ quasi-strongly, i.e.
\[ D(S) = \{ x \in H : S_n x \text{ converges strongly} \} = F \oplus K, \quad \text{and} \quad S(f + k) = f \quad \text{for } f \in F, k \in K. \]
To do this, let us remark that, for $x \in H$, we have the unique representation
\[ x = \sum_{i=1}^{\infty} \alpha_i e_i + \sum_{i=1}^{\infty} \beta_i f_i \quad \text{with } \sum_{i=1}^{\infty} (|\alpha_i|^2 + |\beta_i|^2) \leq \infty. \]
We shall first show that, for such an $x$ (of the form (5)), $x \in D(S)$ iff $\sum_{i=1}^{\infty} i^2 |\beta_i|^2 < \infty$.
Indeed, for $x \in H$ and $n = 1, 2, \ldots$,
\[ x = \sum_{i=1}^{n} c_i^{(n)} g_i + \sum_{i=1}^{n} \gamma_i^{(n)} e_i + \sum_{i=n+1}^{\infty} \delta_i^{(n)} f_i \]
and, for $x$ of the form (5),
\[ c_i^{(n)} = \frac{\beta_i}{\sin(1/i)} \quad \text{for } i = 1, \ldots, n. \]
Thus
\[ S_n x = \sum_{i=1}^{n} c_i^{(n)} g_i = \sum_{i=1}^{n} \frac{\beta_i}{\sin(1/i)} g_i \]
is convergent iff
\[ \sum_{i=1}^{\infty} \frac{|\beta_i|^2}{\sin^2(1/i)} < \infty. \]
On the other hand, \( x \in F \oplus K \) iff \( x = \sum_{i=1}^{\infty} (\gamma_i e_i + c_i g_i) \) with \( \sum_{i=1}^{\infty} (|\gamma_i|^2 + |c_i|^2) < \infty \). By the uniqueness of coefficients for \( x \) of the form (5), we have \( \beta_i = c_i \sin(1/i) \). This means that \( x \in F \oplus K \) iff (6) holds.

**Remark.** For any idempotent \((\hat{F}H, \hat{K}H)\) in a Hilbert space \( H \) with \( \hat{F} \), \( \hat{K} \) orthogonal projections, and for any finite-dimensional orthogonal projections \( \hat{P}_1 \leq \hat{P}_2 \leq \ldots \) tending to identity and commuting with \( \hat{F} \) and \( \hat{K} \), the projections \((\hat{P}_n \hat{F}H, (\hat{P}_n \hat{K} + 1 - \hat{P}_n)H)\) tend quasi-strongly to \((\hat{F}H, \hat{K}H)\).

**1.4. Example** (\( S \) is defined only on an arbitrary closed infinite-dimensional subspace \( F \)). Let \((f_{is}, g_{s}; i, s = 1, 2, \ldots)\) be a basis in \( H \) with \( F = [f_{is}; i, s = 1, 2, \ldots] \). Put \( A_i = \sum_{s=1}^{\infty} \langle \cdot, f_{is} + g_s \rangle f_{is} \). Then, obviously, \( A_i \) are mutually orthogonal projections. Moreover, \( \sum_{i=1}^{n} A_i f_{js} = f_{js} \) for \( n > j \), and

\[
\sum_{i=1}^{n} A_i x = \left( \sum_{i=1}^{n} \sum_{s=1}^{\infty} \hat{f}_{is} \right) x \to x \quad \text{for } x \in F
\]

as well as

\[
\sum_{i=1}^{n} A_i g_s = \sum_{i=1}^{n} f_{is}
\]

and

\[
\left\| \sum_{i=1}^{n} A_i x \right\|^2 = n \| x \|^2 \to \infty \quad \text{for } x \in [g_s; s = 1, 2, \ldots].
\]

**1.5. Example** (\( S \) is densely defined but not closable). Let \((e_1, e_2, \ldots)\) be an orthonormal basis in a Hilbert space \( H \). Let us fix \((f_n)\) by putting

\[
f_0 = e_1,
\]

\[
f_1 = \left( \cos \frac{1}{2^n} \right) f_0 + \left( \sin \frac{1}{2^n} \right) e_2,
\]

\[
f_n = \left( \cos \frac{1}{2^n} \right) f_{n-1} + \left( \sin \frac{1}{2^n} \right) e_{n+1},
\]

and define \( F_n = [f_n], K_n = [f_i; i = 0, 1, \ldots, i \neq n] \). Then \( F_n \cap K_n = (\emptyset) \). Indeed, suppose that \( f_n \in K_n \), where, obviously,

\[
K_n = [e_1, \ldots, e_n, f_{n+1}, e_{n+3}, e_{n+4}, \ldots] \quad \text{for } n = 0, 1, \ldots
\]

Thus

\[
f_n = \alpha f_{n+1} + \sum_{s \neq n+1} \sum_{s \neq n+2} c_s e_s.
\]
Consequently,
\[ 0 = \langle f_n, e_{n+2} \rangle = \alpha \langle f_{n+1}, e_{n+2} \rangle = \alpha \sin \frac{1}{2n+1}, \quad \text{so} \quad \alpha = 0 \]
and
\[ f_n = \sum_{s \neq n+1, s \neq n+2} c_s e_s, \]
which is impossible.

Obviously, \( F_n \oplus K_n = H \), so \((F_n, K_n)\) is a projection.

We know that \( S = (F^0, K) \), where
\[ F^0 = \{ \sum_{i=0}^{\infty} \alpha_i f_i : \text{the series is strongly convergent} \} \]
and \( K = \bigcap_{n=0}^{\infty} K_n \). Obviously, \( F^0 = H \). Moreover, \( K = [k] \) for
\[ k = e_1 + \frac{\sin(1/2)}{\cos(1/2)} e_2 + \frac{\sin(1/2^2)}{\cos(1/2) \cos(1/2^2)} e_3 + \ldots \]
Indeed, by (7), we have \( k \in K_n \) iff
\[ (\hat{e}_{n+1} \hat{e}_{n+2})k = (\hat{e}_{n+1} \hat{e}_{n+2}) \lambda_n f_{n+1} \]
\[ = \lambda_n \left[ \left( \cos \frac{1}{2n+1} \sin \frac{1}{2n+1} \right) e_{n+1} + \left( \sin \frac{1}{2n+1} \right) e_{n+2} \right], \]
and \( k \in K_0 \) iff
\[ (\hat{e}_1 \hat{e}_2)k = (\hat{e}_1 \hat{e}_2) \lambda_0 f_1 = \lambda_0 \left[ \left( \cos \frac{1}{2} \right) e_1 + \left( \sin \frac{1}{2} \right) e_2 \right]. \]

1.6. Example (\( S \) is not closed but closable with the closure \( 1 \)). Let, as before, \((e_1, e_2, \ldots)\) be an orthonormal basis in \( H \) and let
\[ f_0 = e_1, \]
\[ f_1 = \left( \cos \frac{1}{\sqrt{n}} \right) f_0 + \left( \sin \frac{1}{\sqrt{n}} \right) e_2, \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ f_n = \left( \cos \frac{1}{\sqrt{n}} \right) f_{n-1} + \left( \sin \frac{1}{\sqrt{n}} \right) e_{n+1}, \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
and \( F_n = [f_n], K_n = [f_i; i = 0, 1, \ldots, i \neq n] \). Then, as in Example 1.5, \( K_n \) is of the form (7) and \((F_n, K_n)\) denotes a projection for any \( n = 0, 1, \ldots \)

It remains to prove that \( S = (F^0, (\Theta)) \), according to (8). Indeed, \( k \in K_n \) iff
\[ (\hat{e}_{n+1} \hat{e}_{n+2})k = \lambda_n \left[ \left( \cos \frac{1}{\sqrt{n+1}} \sin \frac{1}{\sqrt{n+1}} \right) e_{n+1} + \left( \sin \frac{1}{\sqrt{n+1}} \right) e_{n+2} \right], \]
and $k \in K_0$ iff
\[ (\hat{e}_1 + \hat{e}_2)k = \lambda_0 \left( \cos \frac{1}{\sqrt{1}} \right) e_1 + \left( \sin \frac{1}{\sqrt{1}} \right) e_2, \]
and $x \in \bigcap_{n=0}^{\infty} K_n$ iff
\[ x = \lambda \left[ e_1 + \sin \left( \frac{1}{\sqrt{1}} \right) e_2 + \frac{\sin(1/\sqrt{2})}{\cos(1/\sqrt{1}) \cos(1/\sqrt{2})} e_3 \right. \]
\[ \left. + \frac{\sin(1/\sqrt{3})}{\cos(1/\sqrt{1}) \cos(1/\sqrt{2}) \cos(1/\sqrt{3})} e_4 + \ldots \right]. \]
This is equivalent to $x = \Theta$.

We are going to characterize those densely defined idempotents which are quasi-strong sums of series of mutually orthogonal projections. To clarify the situation a little bit, let us first consider the case $(F^0, (\Theta))$ with $F^0 = X$.

**1.7. Proposition.** The idempotent $(F^0, (\Theta))$ with $F^0 = X$ is a quasi-strong sum of a series of mutually orthogonal projections if and only if there exists a sequence $(F_i)_{i=1}^{\infty}$ of closed subspaces such that
\[ (9) \quad F_j \cap \tilde{F}_j = (\Theta), \quad F_j \oplus \tilde{F}_j = X, \]
\[ (10) \quad F^0 = \{ f^0 = f_1 + f_2 + \ldots : \]
\[ \text{the series is strongly convergent and } f_j \in F_j \}, \]
\[ (11) \quad \bigcap_{j=1}^{\infty} \tilde{F}_j = (\Theta). \]

**Proof.** Let $A_i = (F_i, K_i)$ be a sequence of mutually orthogonal projections and let $(F^0, (\Theta)) = \text{q.s.-}\sum_{i=1}^{\infty} A_i$ (with $F^0 = X$). Then the sequence $(F_i)$ satisfies (9)–(11). Indeed, by the mutual orthogonality of $A_i$, we have $\tilde{F}_i \subseteq K_i$, which implies (11), and $F_i \cap \tilde{F}_i = (\Theta)$ by 1.2. Moreover, for a fixed $i$ and $x \in X$, we have
\[ x = \lim_{s \to \infty} \left( f_i^{(s)} + \tilde{f}_i^{(s)} \right) \text{ for some } f_i^{(s)} \in F_i \text{ and } \tilde{f}_i^{(s)} \in \tilde{F}_i \subseteq K_i. \]
Thus $x = \lim_{s \to \infty} (f_i^{(s)} + \tilde{f}_i^{(s)})$ with $f_i^{(s)} \in F_i$ and $\tilde{f}_i^{(s)} \in K_i$. But the projection $(F_i, K_i)$ is continuous, therefore $f_i^{(s)} \to f_i = A_i x \in F_i$ and, consequently, $\tilde{f}_i^{(s)} \to \tilde{f}_i \in \tilde{F}_i$; so $x = f_i + \tilde{f}_i \in F_i \oplus \tilde{F}_i$, which means that $X = F_i \oplus \tilde{F}_i$.

Clearly, (10) holds by 1.2.

Let $(F_j)_{j=1}^{\infty}$ be a sequence of subspaces satisfying (9)–(11). Then, putting $A_j = (F_j, \tilde{F}_j)$, we obtain a sequence of mutually orthogonal projections such
that
\[ \text{q.s.-} \sum_{j=1}^{\infty} A_j = (F^0, (\Theta)). \]

In the case of the kernel \( K \neq (\Theta) \), the situation is more complicated. Let us start with the following

**1.8. Definition.** Let \( Y \) be a linear subspace of \( X \) (not closed in general). A sequence \((F_1, F_2, \ldots)\) of closed subspaces is said to be **basic for \( Y \)** relative to a closed subspace \( F_0 \) iff
\[
Y = \{ y = f_1 + f_2 + \ldots : \text{the series is strongly convergent and } f_j \in F_j \}
\]
and
\[
F_i \cap \tilde{F}_i^{(F_0)} = (\Theta), \quad F_i \oplus \tilde{F}_i^{(F_0)} = X \quad \text{for } i = 1, 2, \ldots,
\]
where \( \tilde{F}_i^{(F_0)} = [F_j : j = 0, 1, \ldots, j \neq i] \).

A sequence \((F_1, F_2, \ldots)\) basic for \( Y \) relative to the zero subspace \((\Theta)\) will simply be called **basic for \( Y \)**.

**Remark.** A sequence \((F_j)_{j=1}^{\infty}\) is basic for \( F_0 \) iff conditions (9) and (10) of Proposition 1.7 are satisfied.

If \((x_j)\) is a Schauder basis in \( X \), then the one-dimensional subspaces \([x_j]\) form a basic sequence for \( X \). Conversely, if one-dimensional spaces \([x_j]\) form a basic sequence for \( X \), then \((x_j)\) is a Schauder basis in \( X \).

Now, we are in a position to formulate the following characterization of limit idempotents.

**1.9. Theorem.** Let \((F^0, K)\) be a densely defined idempotent in \( X \). Then the following conditions are equivalent:

(i) \((F^0, K) = \text{q.s.-} \sum_{i=1}^{\infty} A_i\) for some mutually orthogonal projections \( A_i \);

(ii) there exists a sequence \((F_j)\) of subspaces of \( X \), basic for \( F_0 \) relative to \( K \), where \( K \) is closed and maximal in the sense that, if \((F_j)\) is basic for \( F_0 \) relative to some closed subspace \( K' \supseteq K \), then \( K = K' \).

**Proof.** (i)⇒(ii). Let \( A_i = (F_i, K_i) \). Then, by 1.2, \( K = \bigcap_{i=1}^{\infty} K_i \), so \( K \) is closed and \( F^0 = \{ f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent and } f_i \in F_i \} \). Moreover, for \( F_0 = K \), condition (13) is satisfied (comp. the proof of Proposition 1.7), so \((F_j)\) is basic for \( F^0 \) relative to \( K \).

Let \( K' \supseteq K \) and let
\[
F_i \cap \tilde{F}_i^{(K')} = (\Theta), \quad F_i \oplus \tilde{F}_i^{(K')} = X.
\]
Since \( F_i \cap \tilde{F}_i^{(K')} = (\Theta) \) and \( F_i \oplus \tilde{F}_i^{(K')} = X \) and \( K' \supseteq K \), we have
\[
\tilde{F}_i^{(K')} = \tilde{F}_i^{(K)} \quad \text{for } i = 1, 2, \ldots
\]
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Consequently,

\[ K_i \supseteq \widehat{F}_i^{(K_i)} = \widehat{F}_i^{(K_i')} \supseteq K' \]

so \( K = \bigcap_i K_i \supseteq K' \) and, finally, \( K = K' \).

(ii)⇒(i). Let us put \( A_i = (F_i, \widehat{F}_i^{(K_i)}) \). Obviously, \( A_i \) are mutually orthogonal. We shall show that

\[ \text{q.s.-} \sum_{i=1}^{\infty} A_i = (F^0, K) \]

Indeed, by 1.2,

\[ F^0 = \{ f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent, } f_i \in F_i \} \]

It remains to show that

\[ \bigcap_{i=1}^{\infty} \widehat{F}_i^{(K_i)} = K \]

To this end, let us put

\[ K' = \bigcap_{i=1}^{\infty} \widehat{F}_i^{(K_i')} \]

Then we have \( K \subset K' \) and, as can easily be checked, the sequence \( (F_1, F_2, \ldots) \) is basic for \( F^0 \) relative to \( K' \). By (ii), this implies \( K = K' \), which ends the proof.

1.10. Example. Let \( H = L_2[0,1] \) and let \( F_0 \) denote the space of all polynomials (considered on the interval \([0,1]\)). Evidently, there is no basic sequence of subspaces for \( F_0 \). Thus \( (F_0, (\Theta)) \) is the idempotent (closable with the closure 1) which is not a quasi-strong sum of any series of mutually orthogonal projections in \( H \).

2. Almost sure convergence of orthogonal series of projections in \( L_2 \)-spaces

2.1. In this section we are interested in the following kind of convergence for operators acting in \( X = L_2(\Omega, A, \mu) \). Let \((A_n)\) be a sequence of bounded linear operators in \( X \) and let \( A \) be linear (bounded or not). We say that \( A_n \) converge to \( A \) almost surely (\( A_n \to A \) a.s.) iff \( A_n f \to A f \) \( \mu \)-almost everywhere for all \( f \in D(A) \).

There are several important results concerning this type of convergence. Let us mention here theorems on martingales, on iterates of conditional expectations, individual ergodic theorems or the results on orthogonal series \([6], [9]\).

Let us start with the following theorem, which is an improvement of our previous results \([2], [8]\).
2.2. Theorem. Let $X$ be a closed subspace of a separable Hilbert space $L_2(\Omega, \mathcal{A}, \mu)$ (over an arbitrary measure space $(\Omega, \mathcal{A}, \mu)$). Let $(B_k)$ be a sequence of finite-dimensional operators in $X$ with $B_k \to B$ strongly. Then there exists an increasing sequence $n(k)$ of indices such that $B_{n(k)} \to B$ a.s.

Proof. By the separability of $L_2(\Omega, \mathcal{A}, \mu)$, there exists a sequence $(A_n)$ of finite subfields of $\mathcal{A}$ such that the conditional expectations $P_n = \mathbb{E}^{A_n}$ converge strongly and almost surely in $L_2(\Omega, \mathcal{A}, \mu)$ to the identity operator $1 = \mathbb{E}^A$. Obviously, $P_n$ are finite-dimensional orthogonal projections in $L_2(\Omega, \mathcal{A}, \mu)$. Assume first that the operators $B_k$ act in the whole space $L_2(\Omega, \mathcal{A}, \mu)$. Then $C_n = B_n - P_nB \to 0$ strongly and $C_n$ are finite-dimensional. One can define, by induction, sequences $n(k) \nearrow \infty$ and $t(k) \nearrow \infty$ satisfying $t(1) = 1$ and

$$\|C_{n(k)}P_t(k)\| < 2^{-k}, \quad \|C_{n(k)}P_{t(k+1)}\| < 2^{-k} \quad \text{for } k = 1, 2, \ldots$$

with $P_t^k = 1 - P_t$. Then

$$C_{n(k)}f = C_{n(k)}P_{t(k)}f + C_{n(k)}(P_{t(k+1)} - P_{t(k)})f + C_{n(k)}P_{t(k+1)}f = \pi_k + \sum_{k+1}^{\infty} \pi_k + \pi_k^3.$$ 

Clearly,

$$\sum_k \|\pi_k^1\|^2 < \infty, \quad \sum_k \|\pi_k^3\|^2 < \infty$$

and

$$\sum_k \|\pi_k^2\|^2 \leq \max_n \|C_n\|^2 < \infty.$$ 

Thus $C_{n(k)} \to 0$ a.s. Consequently, $B_{n(k)} \to B$ a.s.

Passing to the general case, assume that $B_k$ act in a closed subspace $X$ of $L_2(\Omega, \mathcal{A}, \mu)$. Let $(\varphi_s)$ be an orthonormal basis in $X$. Then $Q_n = \sum_{s=1}^{n} \varphi_s$ form a sequence of finite-dimensional projections in $L_2(\Omega, \mathcal{A}, \mu)$. Passing, if necessary, to a subsequence, we can assume by the first part of the proof that $Q_n \to 1_X$ strongly and almost surely. Now it is enough to repeat the previous argument for $P_k = Q_k$. $\blacksquare$

2.3. Corollary. Let $(A_k)$ be a sequence of finite-dimensional mutually orthogonal projections (not necessarily selfadjoint) in the separable Hilbert space $L_2(\Omega, \mathcal{A}, \mu)$. Denote by $(S, \mathcal{D}(S))$ a quasi-strong sum of the series $\sum_{k=1}^{\infty} A_k$. Assume that $S$ is a closed operator (but not necessarily densely defined). Then there exists $n(k) \nearrow \infty$ such that $\sum_{s=1}^{n(k)} A_s \to S$ a.s.

Proof. The corollary is an easy consequence of the previous theorem. Indeed, let $A_j = (F_j, K_j)$. Then, by 1.2, $\mathcal{D}(S) = F^0 \oplus K$ and $S(f^0 + k) = f^0$,

where $K = \bigcap_{j=1}^{\infty} K_j$ and

$$F^0 = \{f^0 = f_1 + f_2 + \ldots : \text{the series is strongly convergent and } f_j \in F_j\}.$$
Since $S$ is closed, $F^0$ is a closed subspace of $L_2(Ω, A, μ)$. Putting $S_n = \sum_{j=1}^n A_j$, we find that $S_n f^0 \to f^0$ strongly for each $f^0 \in F^0$, and $S_n$ are finite-dimensional. By Theorem 2.2, there exists $n(k) \not\to \infty$ such that $S_{n(k)} f^0 \to f^0$ $μ$-almost everywhere for all $f^0 \in F^0$. But this implies $S_n \to S$ a.s.

Remark. Theorem 2.2 and Corollary 2.3 can (and should) be treated as generalizations of the classical Marcinkiewicz result [10].

2.4. In contrast to the case of quasi-strong convergence, roughly speaking, each unbounded operator $A$ in $L_2(μ)$ is an almost sure sum of a series of mutually orthogonal projections. The construction of a suitable sequence of projections is based on the following facts:

1) the generalization of Marcinkiewicz’s theorem easily gives the existence of a sequence $(A_j)$ of finite-dimensional operators such that $\sum_{j=1}^{∞} A_j = A$ a.s.,

2) mutually orthogonal projections $(B_j)$ such that $\sum_{j=1}^{∞} B_j = A$ a.s. can be obtained by a suitable perturbation of $A_i$’s with the use of vectors with small supports and some vectors “asymptotically orthogonal” to the domain of $A$.

Passing to precise formulations, we have the following result.

2.5. Theorem. Let $H = L_2(Ω, A, μ)$ be a separable Hilbert space such that $0 \neq μ(Z_n) \to 0$ for some $(Z_n) \subset A$. Let $A$ be an unbounded closed and densely defined operator in $H$. Then there exists a sequence $(B_j)$ of mutually orthogonal finite-dimensional projections in $X$ such that the sums $S_n = \sum_{j=1}^{n} B_j$ converge almost surely to $A$.

The following lemma is a key point in the proof of the theorem just formulated. In the sequel, for $Z \in A$, the symbol $1_Z$ denotes the multiplication operator in $L_2$ by the indicator $χ_Z$ of $Z$.

2.6. Lemma. Let $H = L_2(Ω, A, μ)$ for an arbitrary measure space satisfying $0 \neq μ(Z_n) \to 0$ for some $(Z_n) \subset A$. Let $D = \{f \in H : \int_0^∞ \lambda^2 ||e(\lambda)f||^2 < ∞\}$ for some spectral measure $e(\cdot)$ satisfying $e(A, ∞) \neq 0$ for all $A > 0$. Then, for any sequence $(A_n)$ of finite-dimensional operators in $H$, there exists a sequence $(B_n)$ of mutually orthogonal finite-dimensional projections in $H$ (not necessarily selfadjoint) satisfying, for some $Y_1 \supseteq Y_2 \supseteq \ldots \ldots$, $(Y_n) \subset A$, $μ(Y_n) \to 0$, the condition

$$\sum_{n=1}^{∞} ||1_{Ω \setminus Y_n} \left( \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i \right) f ||^2 < ∞$$

for all $f \in D$.

Proof. Clearly, there exist sets $Z(1) \supseteq Z(2) \supseteq \ldots$ in $A$ such that $μ(Z(s)) \to 0$ and the projections $1_{Z(s)}$ are infinite-dimensional. Then some
selfadjoint projections

\[ L(1) \leq L(2) \leq \ldots, \quad L(s) \to 1, \]

can be fixed in such a way that the subspaces

\[ (L(s + 1) - L(s)) H \cap 1_{Z(s)} H \quad \text{and} \quad (L(s + 1) - L(s)) H \cap e(s, \infty) H \]

are infinite-dimensional. Indeed, one can take, for example, an orthonormal system \((\xi_s, \zeta_t; s, t = 1, 2, \ldots)\) with \(\xi_s \in 1_{Z(s)} H\), \(\zeta_t \in e(t, \infty) H\) and put \(L(\Lambda) = 1 - [\xi_s, \zeta_t; s, t \text{ are divisible by } 2^\lambda] \wedge\).

Now, fix an orthonormal system \((\varphi_{ij}^s, \psi_{ij}^s(\Lambda); i, j, s, \Lambda = 1, 2, \ldots)\) satisfying

\[ \begin{align*}
\varphi_{ij}^s &\in (L(s + 1) - L(s)) H \cap 1_{Z(s)} H, \\
\psi_{ij}^s(\Lambda) &\in (L(\Lambda + 1) - L(\Lambda)) H \cap e(\Lambda, \infty) H.
\end{align*} \]

Define

\[ \|f\|_e^2 = \int_0^\infty \lambda^2 \|e(d\lambda)f\|^2 \quad \text{for all } f \in D. \]

Let us begin the inductive construction of \(B_1, B_2, \ldots\). Put \(A_0 = B_0 = 0\) and assume that the mutually orthogonal projections \(B_0, \ldots, B_n\) for some \(n \geq 0\) have already been defined so that

\[ \left\| 1_{\Omega_k \setminus Y_k} \left( \sum_{i=0}^k A_i - \sum_{i=0}^k B_i \right) f \right\| \leq 2^{-k} (\|f\|_e + 2\|f\|), \quad \mu(Y_k) < \frac{1}{k} \]

for \(f \in D, k = 1, \ldots, n\) and for some sets \(Y_1 \supseteq \ldots \supseteq Y_n\) in \(A\). Assume additionally that the projections \(B_i\) for \(i = 1, \ldots, n\) can be written in the form

\[ B_i = \sum_{j=1}^{k(i)} \left\langle \cdot, u_{ij}^n \right\rangle + \sum_{s=M(n)}^\infty \delta_{ij}^s \varphi_{ij}^s \right\rangle \left( v_{ij}^n + \sum_{t=1,2,\ldots}^{\varepsilon_{ij}^t} \psi_{ij}^t(\Lambda_{ij}^t)^n \right), \]

where \(\delta_{ij}^s > 0, \varepsilon_{ij}^t > 0, M(n)\) is some positive integer, the sequences of positive integers \(\Lambda_{ij}^1 < \Lambda_{ij}^2 < \ldots\) satisfy the inequalities

\[ \Lambda_{ij}^t \geq \frac{t}{\varepsilon_{ij}^t}, \]

and

\[ \begin{align*}
u_{ij}^n, v_{ij}^n &\in L(M(n)) H, \\
Z(M(n)) &\subseteq Y_n.
\end{align*} \]
We define an operator $B_{n+1}$ satisfying (16) for $k = n + 1$ and rewrite the operators $B_1, \ldots, B_n$ modifying (17) by taking $n + 1$ instead of $n$. Let
\[
\sum_{i=0}^{n+1} A_i - \sum_{i=0}^{n} B_i = \sum_{j=1}^{k(n+1)} \langle \cdot, u_j \rangle v_j.
\]

There exists $M \geq 1$, $M > M(n)$ in the case $n > 0$, such that, for $x^{n+1,j} = L(M)u_j$, $y^{n+1,j} = L(M)v_j$, we have
\[
\left\| \sum_{j=0}^{k(n+1)} \langle \cdot, u_j \rangle v_j - \sum_{j=0}^{k(n+1)} \langle \cdot, x^{n+1,j} \rangle y^{n+1,j} \right\| < 2^{-(n+1)},
\]
\[
\mu(Z(M)) < \frac{1}{n + 1}.
\]

Obviously, $B_1, \ldots, B_n$ can be written in the form
\[
B_i = \sum_{j=1}^{k(i)} \left\langle \cdot, x^{ij} + \sum_{s=M}^{\infty} \delta^{ij}_s \varphi^s \right\rangle \left( y^{ij} + \sum_{t=1,2,\ldots}^{\Lambda(t)} \varepsilon_t^{ij} \psi^t (A_t^{ij}) \right),
\]
where
\[
x^{ij} = u^{ij} + \sum_{s=M(n)}^{M-1} \delta^{ij}_s \varphi^s \in L(M)H,
\]
\[
y^{ij} = v^{ij} + \sum_{t=1,2,\ldots}^{\Lambda(t)} \varepsilon_t^{ij} \psi^t (A_t^{ij}) \in L(M)H.
\]

Let us fix
\[
\delta^{n+1,j}_s > 0, \quad \varepsilon^{n+1,j}_t > 0,
\]
\[
A_1^{n+1,j} < A_2^{n+1,j} < \ldots, \quad A_t^{n+1,j} \geq \frac{t}{\varepsilon_t^{n+1,j}}, \quad A_t^{n+1,j} \geq M,
\]
such that, for
\[
\tilde{B}_{n+1} = \sum_{j=1}^{k(n+1)} \left\langle \cdot, x^{n+1,j} + \sum_{s=M}^{\infty} \delta^{n+1,j}_s \varphi^s \right\rangle \left( y^{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_t^{n+1,j} \psi^{n+1,j} (A_t^{n+1,j}) \right).
\]
we have the estimate
\[
\| \tilde{B}_{n+1} - \sum_{j=1}^{k(n+1)} \langle \cdot, x^{n+1,j} \rangle y^{n+1,j} \| < 2^{-(n+1)}.
\] (22)

Now, one can find a perturbation $B_{n+1}$ of $\tilde{B}_{n+1}$ such that $B_0, \ldots, B_{n+1}$ are the required mutually orthogonal projections. Namely, there exist matrices
\[
\left( a_{ij}' \right)_{i=1, \ldots, n+1, j'=1, \ldots, k(i)}, \quad \left( b_{ij}''' \right)_{i=1, \ldots, n, j''=1, \ldots, k(i)}
\]
such that, for an arbitrary orthonormal system
\[
(e^{ij}, f^{ij}; i = 1, \ldots, n, j = 1, \ldots, k(i)) \cup (f^{n+1,j}; j = 1, \ldots, k(n+1))
\]
orthogonal to
\[
(x^{ij}, y^{ij}; i = 1, \ldots, n+1, j = 1, \ldots, k(i)),
\]
the operators
\[
C_1 = \sum_{j=1}^{k(1)} \langle \cdot, x^{1j} + e^{1j} \rangle (y^{1j} + f^{1j}),
\]
\[
\vdots
\]
\[
C_n = \sum_{j=1}^{k(n)} \langle \cdot, x^{nj} + e^{nj} \rangle (y^{nj} + f^{nj}),
\]
\[
C_{n+1} = \sum_{j=1}^{k(n+1)} \left( \langle \cdot, x^{n+1,j} \rangle + \sum_{i=1}^{n+1} \sum_{j'=1}^{k(i)} a_{ij}' f^{ij}' \right)
\times \left( y^{n+1,j} + f^{n+1,j} + \sum_{i=1}^{n} \sum_{j''=1}^{k(i)} b_{ij}''' e^{ij''} \right)
\]
satisfy the conditions
\[
\text{(23)} \quad C_{n+1} \text{ is a projection,}
\]
\[
\text{(24)} \quad C_{n+1} C_i = 0,
\]
\[
\text{(25)} \quad C_i C_{n+1} = 0,
\]
for $i = 1, \ldots, n$. In fact, (23) uniquely determines the entries $(a^{n+1,j'}_{ij}; j, j' = 1, \ldots, k(n+1))$, and, for $n > 0$, condition (24) (respectively (25)) uniquely determines the entries $(a_{ij}'; j = 1, \ldots, k(n+1), j' = 1, \ldots, k(i))$ (respectively $(b_{ij}''''; j = 1, \ldots, k(n+1), j'' = 1, \ldots, k(i))$).
Let us remark that, for $q > 0$, $\varepsilon > 0$, $\psi \in H$, $\|\psi\| = 1$, we see, by (15), that
\[ \psi \in e\left(\frac{q}{\varepsilon}, \infty\right) \text{ implies } \left| \left\langle f, \frac{\psi}{\varepsilon} \right\rangle \right| < \frac{1}{q} \|f\|_e \text{ for any } f \in \mathcal{D}. \]

This makes it possible to find $q$ (large enough) such that
\[ \sum_{j=1}^{k(n+1)} \left\langle f, \sum_{i=1}^{n+1} \sum_{j'=1}^{k(i)} \frac{a_{ij'}}{\varepsilon_{j'+q}} \psi^{j'}(A_{j+q}^{ij'}) \right\rangle \times \left( y_{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_{t}^{n+1,j} \psi^{n+1,j}(A_{t}^{n+1,j}) \right) \left\| f \right\|_e < 2^{-(n+1)} \|f\|_e \]
(by (14), (18)). Finally, we define $B_{n+1}$ by putting
\[ B_{n+1} = \sum_{j=1}^{k(n+1)} \left\langle \cdot, \tilde{x}_{n+1,j} + \sum_{s=M}^{\infty} \delta_{s}^{n+1,j} \phi_{s}^{n+1,j} \right\rangle \times \left( y_{n+1,j} + \sum_{t=1}^{\infty} \varepsilon_{t}^{n+1,j} \psi^{n+1,j}(A_{t}^{n+1,j}) + \sum_{i=1}^{n} \sum_{j'=1}^{k(i)} \frac{b_{ij'}}{\delta_{j'+M}} \phi_{j'+M}^{ij''} \right) \]
\[ = \sum_{j=1}^{k(n+1)} \left\langle \cdot, \tilde{x}_{n+1,j} \right\rangle \tilde{y}_{n+1,j}. \]

One can easily observe, for $Y_{n+1} = Z(M)$, that (14) implies
\[ 1_{\Omega \setminus Y_{n+1}} \phi_{j''}^{j''} = 0 \text{ for } j'' = 1, \ldots, k(i), j = 1, \ldots, k(n+1) \]
and, in the case $n > 0$, $Y_n \supseteq Y_{n+1}$. Thus, by (26) and by (20)–(22),
\[ \left\| 1_{\Omega \setminus Y_{n+1}} \left( \sum_{i=1}^{n+1} A_i - \sum_{i=1}^{n} B_i \right) f \right\| \leq 2^{-n} \|f\|_e + \left\| 1_{\Omega \setminus Y_{n+1}} \left( \sum_{i=1}^{n+1} A_i - \sum_{i=1}^{n} B_i - \tilde{B}_{n+1} \right) f \right\| \leq 2^{-(n+1)} (\|f\|_e + 2\|f\|), \]
\[ \mu(Y_{n+1}) < \frac{1}{n+1}. \]

To conclude the proof, it remains to show that the operators $B_1, \ldots, B_{n+1}$ can be rewritten in form (17) with $n+1$ instead of $n$. To this end, it
is enough to fix $M(n + 1)$ such that
\[
M(n + 1) \geq M,
\]
\[
M(n + 1) \geq \max(M + k(i); i = 1, \ldots, n + 1),
\]
\[
M(n + 1) \geq \max(A_{j+q'}^{ij}; i = 1, \ldots, n + 1,
\]
\[
j = 1, \ldots, k(n + 1), \ j'' = 1, \ldots, k(i)),
\]
and to put
\[
u_{n+1}^{ij} = L(M(n + 1)) \left( \sum_{s=M(n)}^{\infty} \varepsilon_s^{ij} v_s^{ij} \right) \quad \text{for} \quad i = 1, \ldots, n,
\]
\[
u_{n+1}^{ij} = L(M(n + 1)) \zeta_{n+1,j},
\]
\[
u_{n+1}^{ij} = L(M(n + 1)) \left( \sum_{t=1,2,\ldots} \varepsilon_t^{ij} \psi_t^{ij}(A_t^{ij}) \right) \quad \text{for} \quad i = 1, \ldots, n,
\]
\[
u_{n+1}^{ij} = L(M(n + 1)) \tilde{y}_{n+1,j}
\]
(compare (27)).

**Proof of Theorem 2.5.** For a closed and densely defined $A$, one can easily construct a sequence $(C_n)$ of finite-dimensional operators such that $\|C_n f - Af\| \to 0$ for all $f \in D(A)$. Let $\lambda(\cdot)$ be the spectral measure of $|A|$. Putting $B = \int_0^\infty \min(1, \lambda^{-1}) e(d\lambda)$, we can apply Theorem 2.2 to a sequence $C_n B$, so there exists $n(k) \to \infty$ such that $C_n(k) B g \to A B g$ for all $g \in H$. Moreover, $f \in D(A)$ iff $f = B g$ for some $g$. Thus we have $S_k = C_n(k) B \to A$ a.s.

Putting $A_k = S_k - S_{k-1}$, $S_0 = 0$, we have $\sum_{i=1}^{\infty} A_i = A$ a.s. Now, it is enough to apply Lemma 2.6. 

**Remark.** In the theorem just proved, the assumption that $H = L_2(\Omega, \mathcal{A}, \mu)$ with $0 < \mu(Z_n) \to 0$ for some $(Z_n) \subset \mathcal{A}$ can be replaced by a weaker condition. Namely, it is enough to assume that $H$ is a closed subspace of $L_2(\Omega, \mathcal{A}, \mu)$ such that there exists $(f_n) \subset H$ with $\|f_n\| = 1$, $f_n \to 0$ in measure $\mu$.

**References**


Convergence of orthogonal series of projections


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