ANNALES
POLONICI MATHEMATICI
LXVI (1997)

On the intertwinings of regular dilations

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Abstract. The aim of this paper is to find conditions that assure the existence of the commutant lifting theorem for commuting pairs of contractions (briefly, bicontractions) having ($*$-)regular dilations. It is known that in such generality, a commutant lifting theorem fails to be true. A positive answer is given for contractive intertwinings which doubly intertwine one of the components. We also show that it is possible to drop the doubly intertwining property for one of the components in some special cases, for instance for semi-subnormal bicontractions. As an application, a result regarding the existence of a unitary (isometric) dilation for three commuting contractions is obtained.

0. Introduction. It is well known that the theorem of B. Sz.-Nagy and C. Foiaş regarding the lifting of the commutant of a pair of contractions plays an important role in the applications of dilation theory in operator interpolation problems, optimization and control, in geology and geophysics. This is excellently illustrated in the book [5] of C. Foiaş and A. E. Frazho.

Lately, the dilation theory method was extended to the study of commuting multioperators by many authors (W. Mlak, M. Słociński, M. Kosiek, M. Ptak, E. Albrecht, V. Müller, R. E. Curto, F. H. Vasilescu, A. Octavio, B. Chevreau and others). In 1993, at the B. Sz.-Nagy Anniversary International Conference in Szeged, C. Foiaş raised the problem of obtaining a commutant lifting theorem for a pair of bicontractions having regular unitary dilations. In 1994, at the XV-th International Conference on Operator Theory in Timişoara, V. Müller proved that in such a generality, the commutant lifting theorem fails.

In the present work it is our aim to find conditions that assure the existence of such a lifting. In this frame a structure for regular (or $*$-regular) dilations is needed.

1991 *Mathematics Subject Classification*: Primary 47A20; Secondary 47A13.
Key words and phrases: commuting multioperator, $*$-regular dilation, contractive intertwining, (semi-)subnormal pair.

1. Preliminaries. For a complex separable Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ means the C^*-algebra of all bounded linear operators on $\mathcal{H}$ (with Hilbert adjoint as involution). The elements of the (closed) unit ball in $\mathcal{B}(\mathcal{H})$ are called *contractions* on $\mathcal{H}$. An n-tuple of operators will be called a *multioperator*. If the members of the n-tuple commute, then we have a *commuting multioperator*. A commuting multioperator consisting of contractions will be called a *multicontraction* (*bicontraction* if $n = 2$) on $\mathcal{H}$.

For a multicontraction $T := (T_1, \ldots, T_n)$ we define $T^* := (T_1^*, \ldots, T_n^*)$. We shall also use the multiindex notation

$$T^m := T_1^{m_1} \ldots T_n^{m_n}, \qquad m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n,$$

where $\mathbb{Z}$ (resp. $\mathbb{Z}_+$) is the set of all (resp. positive) integers. A multicontraction T on $\mathcal{H}$ will be briefly denoted by $[\mathcal{H}, T]$.

An *isometric* (*resp. unitary*) *dilation* of a multicontraction $[\mathcal{H}, T]$ is a multicontraction $[\mathcal{K}, U]$ consisting of isometric (resp. unitary) operators, such that $\mathcal{K}$ contains $\mathcal{H}$ as a closed subspace and

$$T^m = P_{\mathcal{H}} U^m | \mathcal{H} \qquad (m \in \mathbb{Z}_+^n), \tag{1}$$

where $P_{\mathcal{H}} = P_{\mathcal{K},\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ on $\mathcal{H}$.

It is known (see [1]) that each bicontraction has an isometric (and unitary) dilation, and generally speaking, an n-tuple consisting of more than three commuting contractions has no isometric dilation (see [15]). An isometric, respectively unitary, dilation $[\mathcal{K}, U]$ of $[\mathcal{H}, T]$ is called *minimal* if

$$\mathcal{K} = \bigvee_{m \in \mathbb{Z}_+^n} U^m \mathcal{H}, \tag{2}$$

or respectively,

$$\mathcal{K} = \bigvee_{m \in \mathbb{Z}^n} U^m \mathcal{H}. \tag{2'}$$

Let us first note that if $[\mathcal{K}, V]$ is an isometric minimal dilation of the multicontraction $[\mathcal{H}, T]$, then by (2), $\mathcal{H}$ is invariant with respect to V^* and $V_i^* | \mathcal{H} = T_i^*$ $(i = 1, \ldots, n)$. Let us also mention that if $[\mathcal{K}, V]$ is an isometric minimal dilation of $[\mathcal{H}, T]$, and $[\widetilde{\mathcal{K}}, U]$ is the minimal unitary extension (see [18]) of $[\mathcal{K}, V]$, then it is the unitary minimal dilation of $[\mathcal{H}, T]$.

On the other hand, it is known that in case of a single contraction, the minimality condition (2) or (2′) implies that the isometric (resp. unitary) dilation is uniquely determined up to a unitary equivalence which fixes $\mathcal{H}$. But for $n > 1$ this is not true ([1], [18]).

An isometric (resp. unitary) minimal dilation $[\mathcal{K}, U]$ of the multicontraction $[\mathcal{H}, T]$ is called *regular* (respectively $*$-*regular*) if it satisfies

$$T^{*m_-} T^{m_+} = P_{\mathcal{H}} U^{*m_-} U^{m_+} | \mathcal{H} \qquad (m \in \mathbb{Z}^n), \tag{3}$$

or respectively,

$$T^{m_+}T^{*m_-} = P_{\mathcal{H}}U^{*m_-}U^{m_+}|\mathcal{H} \quad (m \in \mathbb{Z}^n), \tag{3*}$$

where $m_+ := (m_1^+, \dots, m_n^+)$, $m_- = (m_1^-, \dots, m_n^-)$ and $m_i^+ := \max\{m_i, 0\}$, $m_i^- := \max\{-m_i, 0\}$.

Regular dilations were studied in [3], [10], [18] and recently in [4] and [8]. Their existence is not assured for any multicontraction, not even for $n = 2$. However, if such a dilation exists, then by the minimality condition (2) or (2′) it is uniquely determined up to unitary equivalence (see [18]). It is easy to see from (3) and (3*) that $[\mathcal{K}, U]$ is a regular (resp. $*$-regular) unitary dilation of $[\mathcal{H}, T]$ iff $[\mathcal{K}, U^*]$ is a $*$-regular (resp. regular) unitary dilation of $[\mathcal{H}, T^*]$. We also note that if $[\mathcal{K}, V]$ is a regular (resp. $*$-regular) isometric dilation of $[\mathcal{H}, T]$ then the minimal unitary extension $[\widetilde{\mathcal{K}}, U]$ of $[\mathcal{K}, V]$ is a regular (resp. $*$-regular) unitary dilation. On the other hand, if $[\mathcal{K}, U]$ is a regular unitary dilation, by putting $\mathcal{K}^+ := \bigvee_{m\in\mathbb{Z}_+^n} U^m\mathcal{H}$, $\mathcal{K}_*^+ := \bigvee_{m\in\mathbb{Z}_+^n} U^{*m}\mathcal{H}$, $V_{*i} := U_i^*|\mathcal{K}_*^+$ and $V_i := U_i|\mathcal{K}^+$ $(i = 1, \dots, n)$, then $[\mathcal{K}^+, V]$ is a regular isometric dilation of $[\mathcal{H}, T]$, whereas $[\mathcal{K}_*^+, V_*]$ is a $*$-regular isometric dilation of $[\mathcal{H}, T^*]$. Let us also recall that a multicontraction $[\mathcal{H}, T]$ has a regular isometric dilation iff

$$\Delta_T := \sum_{|m|\le n} (-1)^{|m|} T^{*m_-}T^{m_+} \ge 0,$$

where $|m| := m_1 + \ldots + m_n$ (see [18], [4]).

A multicontraction is called a *polydisc isometry* ([4]) when $\Delta_T = 0$. It is easily seen that if T_i $(i = 1, \dots, n)$ are isometries (i.e. T is an n-toral isometry [2]), then T is a polydisc isometry. Now if $I - \sum_{i=1}^n T_i^*T_i \ge 0$, then $\Delta_T \ge 0$. If $\sum_{i=1}^n T_i^*T_i = I$, then $[\mathcal{H}, T]$ is called a *spherical isometry* ([2]). When $[\mathcal{H}, T^*]$ is a polydisc or a spherical isometry, we say that $[\mathcal{H}, T]$ is a *polydisc* or a *spherical coisometry*, respectively. If the multicontraction is doubly commuting, then obviously $\Delta_T = (I - T_1^*T_1)\ldots(I - T_n^*T_n) \ge 0$.

Furthermore, it is easy to verify

Proposition 0. *For a multicontraction* $[\mathcal{H}, T]$ *the following statements are equivalent*:

(i) $[\mathcal{H}, T]$ *is doubly commuting*;

(ii) $[\mathcal{H}, T]$ *has a regular isometric dilation which is doubly commuting*;

(iii) $[\mathcal{H}, T]$ *has a regular unitary dilation* $[\mathcal{K}, U]$ *such that* $[\mathcal{K}, U^*]$ *is a regular unitary dilation for* $[\mathcal{H}, T^*]$.

In particular, a multicontraction consisting of coisometries has a regular isometric (*or unitary*) *dilation iff it is doubly commuting.*

Let us observe that Proposition 0(iii) means more than that T has a regular and a $*$-regular dilation. For example, for a bicontraction $T=(T_0,T_1)$ with $\|T_0\|^2+\|T_1\|^2\le 1$, we have $\Delta_T\ge 0$ and $\Delta_{T^*}\ge 0$ but it is possible that $T_0T_1^*\ne T_1^*T_0$.

Finally, also recall that an isometric pair $[\mathcal{H},V]$ is called a *shift n-tuple* (see [7], [8]) or a *multishift* (see [4]) if there exists a wandering (closed) subspace $\mathcal{E}$ in $\mathcal{H}$ (i.e. $V^m\mathcal{E}\perp V^p\mathcal{E}$, $m\ne p$, $m,p\in\mathbb{Z}_+^n$) such that $\mathcal{H}=\bigoplus_{m\in\mathbb{Z}_+^n}V^m\mathcal{E}$. For the sake of simplicity we shall work in the case $n=2$.

2. $*$-Regular isometric dilations. The isometric dilations consisting of doubly commuting isometries are in some sense connected with regular dilations. Precisely this is given in

THEOREM 1. *For a bicontraction $[\mathcal{H},T]$ with $T=(T_0,T_1)$ the following assertions are equivalent*:

(i) *T has a doubly commuting minimal isometric dilation*;

(ii) *T has a minimal isometric dilation of the form $[\mathcal{M}\oplus\mathcal{G},W\oplus V]$, where W is a bishift on $\mathcal{M}$ and V is a bidisc coisometry on $\mathcal{G}$*;

(iii) *If $[\mathcal{K}_0,S_0]$ is the minimal isometric dilation of T_0, then there exists a contraction S_1 on $\mathcal{K}_0$ which doubly commutes with S_0, such that $P_{\mathcal{H}}S_1=T_1P_{\mathcal{H}}$*;

(iv) *T has a $*$-regular isometric (unitary) dilation.*

Proof. (i)$\Rightarrow$(ii). Let $[\mathcal{K},U]$ be a minimal isometric dilation of T with U_0,U_1 doubly commuting isometries on $\mathcal{K}$. By the Wold decomposition ([17], [7]) we have $\mathcal{K}=\mathcal{K}_u\oplus\mathcal{K}_s\oplus\mathcal{K}_{s0}\oplus\mathcal{K}_{s1}$, so that U_0 and U_1 reduce each subspace and U_0,U_1 are unitary on $\mathcal{K}_u$, U is a shift pair on $\mathcal{K}_s$ and U_i is unitary (resp. a shift) on $\mathcal{K}_{s1-i}$ (resp. $\mathcal{K}_{si}$), $i=0,1$. Put $\mathcal{G}=\mathcal{K}_u\oplus\mathcal{K}_{s0}\oplus\mathcal{K}_{s1}$, $V_i=V_i^i\oplus V_i^{1-i}$ with $V_i^i=U_i|\mathcal{K}_u\oplus\mathcal{K}_{s0}$, $V_i^{1-i}=U_i|\mathcal{K}_{s1}$ $(i=0,1)$ and $V=(V_0,V_1)$, $W'=(V_0^0,V_1^1)$, $W''=(V_0^1,V_1^0)$. Because V_1^1 is unitary, W'^* is a bidisc isometry on $\mathcal{K}_u\oplus\mathcal{K}_{s0}$, and since V_0^1 is unitary, W''^* is a bidisc isometry on $\mathcal{K}_{s1}$. Then

$$\Delta_{V^*}=\Delta_{W'^*}+\Delta_{W''^*}=0,$$

so V^* is a bidisc isometry on $\mathcal{G}$. Therefore since $W:=(U_0|\mathcal{K}_s,\ U_1|\mathcal{K}_s)$ is a bishift on $\mathcal{M}=\mathcal{K}_s$ and $\mathcal{K}=\mathcal{M}\oplus\mathcal{G}$, $W\oplus V=U$, we see that the dilation $[\mathcal{K},U]$ of T has the form described in (ii).

(ii)$\Rightarrow$(iii). Let $[\mathcal{M}\oplus\mathcal{G},W\oplus V]$ be as in (ii). Since W is a bishift on $\mathcal{M}$, the isometries W_0 and W_1 doubly commute on $\mathcal{M}$ ([16]). Also the isometries V_0 and V_1 doubly commute on $\mathcal{G}$, because V has a $*$-regular dilation. Therefore the isometries $U_0=W_0\oplus V_0$ and $U_1=W_1\oplus V_1$ doubly commute on $\mathcal{M}\oplus\mathcal{G}$.

Put

$$\mathcal{K}_0 = \bigvee_{m \in \mathbb{Z}_+} W_0^m \mathcal{H}, \qquad S_0 = U_0 | \mathcal{K}_0, \qquad S_1 = P_{\mathcal{K}_0} U_1 | \mathcal{K}_0.$$

Then $[\mathcal{K}_0, S_0]$ is the minimal isometric dilation of T_0, $S_0 S_1 = S_1 S_0$ and $P_{\mathcal{H}} S_1 = T_1 P_{\mathcal{H}}$. Since $U_0^* | \mathcal{H} = T_0^* = S_0^* | \mathcal{H}$, we also have $U_0^* | \mathcal{K}_0 = S_0^*$. Furthermore, for $k = \sum_{p \in \mathbb{Z}_+} S_0^p h_p \in \mathcal{K}$ with the sequence $\{h_p\} \subset \mathcal{H}$ with finite support, we obtain

$$\begin{aligned} S_1 S_0^* k &= S_1 S_0^* h_0 + \sum_{p \geq 1} S_1 S_0^{p-1} h_p \\ &= P_{\mathcal{K}_0} U_1 U_0^* h_0 + \sum_{p \geq 1} S_0^{p-1} S_1 h_p \\ &= P_{\mathcal{K}_0} U_0^* U_1 h_0 + \sum_{p \geq 1} S_0^* S_0^p S_1 h_p \\ &= S_0^* S_1 h_0 + S_0^* \sum_{p \geq 1} S_1 S_0^p h_p = S_0^* S_1 k, \end{aligned}$$

where we have used the fact that $P_{\mathcal{K}_0} U_0^* U_1 | \mathcal{K}_0 = S_0^* S_1$. Consequently, S_0 and S_1 doubly commute on $\mathcal{K}_0$.

(iii)$\Rightarrow$(iv). If $\mathcal{K}_0, S_0$ and S_1 are as in (iii), then $S_i^* | \mathcal{H} = T_i^*$ $(i = 0, 1)$ and since S_0 and S_1 doubly commute, we have

$$\begin{aligned} \varDelta_{T^*} &= I - T_0 T_0^* - T_1 T_1^* + T_0 T_1 T_0^* T_1^* \\ &= P_{\mathcal{H}} (I - S_0 S_0^* - S_1 S_1^* + S_0 S_1 S_0^* S_1^*) | \mathcal{H} \\ &= P_{\mathcal{H}} (I - S_0 S_0^*)(I - S_1 S_1^*) | \mathcal{H} \geq 0. \end{aligned}$$

Consequently, T has a $*$-regular isometric (or unitary) dilation.

(iv)$\Rightarrow$(i). Suppose $\varDelta_{T^*} \geq 0$. Denote by $\mathcal{M} = H^2(\mathbf{T}^2, \mathcal{H})$ and $Z = (Z_0, Z_1)$ the shift pair (that is, Z_0 and Z_1 are the operators of multiplication with the coordinate functions) on $\mathcal{M}$. Using Theorem 3.15 of [4], there are a Hilbert space $\mathcal{H}_1$, a bicontraction $N = (N_0, N_1)$ on $\mathcal{H}_1$ with N_0 and N_1 normal operators and with N^* a bidisc isometry, and an isometry A of $\mathcal{H}$ in $\mathcal{M} \oplus \mathcal{H}_1$ such that $A\mathcal{H}$ is invariant for $(Z_i \oplus N_i)^*$ and $(Z_i \oplus N_i)^* A = A T_i^*$, $i = 0, 1$. Then it results that

$$T_0^p T_1^q = A^* (Z_0 \oplus N_0)^p (Z_1 \oplus N_1)^q A \qquad (p, q \in \mathbb{Z}_+).$$

Let now $[\mathcal{K}_1, (M_0, M_1)]$ be a minimal isometric dilation of N with M_0 and M_1 doubly commuting isometries on $\mathcal{K}_1$. Put

$$\mathcal{K} = \mathcal{M} \oplus \mathcal{K}_1, \quad U_i = Z_i \oplus M_i \quad (i = 0, 1).$$

Then U_0 and U_1 are doubly commuting isometries on $\mathcal{K}$. Denoting by J the embedding of $\mathcal{M} \oplus \mathcal{H}_1$ in $\mathcal{K}$, we find that JA is an isometry of $\mathcal{H}$ into $\mathcal{K}$.

For $m=(p,q)\in\mathbb{Z}_+^2$ and $h\in H$ we obtain

$$\begin{aligned}(JA)^*U^mJAh&=(JA)^*U_0^pU_1^q(P_{\mathcal{H}_0}Ah\oplus P_{\mathcal{H}_1}Ah)\\&=(JA)^*(Z^mP_{\mathcal{H}_0}Ah\oplus M^mP_{\mathcal{H}_1}Ah)\\&=A^*(P_{\mathcal{H}_0}Z^mP_{\mathcal{H}_0}Ah\oplus P_{\mathcal{H}_1}M^mP_{\mathcal{H}_1}Ah)\\&=A^*(Z^mP_{\mathcal{H}_0}Ah\oplus N^mP_{\mathcal{H}_1}Ah)\\&=A^*(Z_0\oplus N_0)^p(Z_1\oplus N_1)^qAh=T^mh.\end{aligned}$$

Let us observe that the subspace $A\mathcal{H}$ is invariant for U_i^* $(i=0,1)$, because for $h\in\mathcal{H}$ we have

$$\begin{aligned}U_i^*Ah&=U_i^*(P_{\mathcal{H}_0}Ah\oplus P_{\mathcal{H}_1}Ah)=Z_i^*P_{\mathcal{H}_0}Ah\oplus M_i^*P_{\mathcal{H}_1}Ah\\&=Z_i^*P_{\mathcal{H}_0}Ah\oplus N_i^*P_{\mathcal{H}_1}Ah=(Z_i\oplus N_i)^*Ah=AT_i^*h.\end{aligned}$$

Now define

$$\mathcal{K}_+=\bigvee_{m\in\mathbb{Z}_+^2}U^mAh,\qquad V_i=U_i|\mathcal{K}_+\quad(i=0,1)$$

and $B=J_+A$, where J_+ is the embedding of $\mathcal{M}\oplus\mathcal{H}_1$ in $\mathcal{K}_+$. Then B is an isometry of $\mathcal{H}$ into $\mathcal{K}_+$ and we have

$$T^m=B^*V^mB\qquad(m\in\mathbb{Z}_+^2).$$

Identifying $\mathcal{H}$ with $B\mathcal{H}$ in $\mathcal{K}_+$, we deduce that $[\mathcal{K}_+,V]$ is a minimal isometric dilation of T. It remains to prove that $V_0V_1^*=V_1^*V_0$. First, since U_0 and U_1 doubly commute on $\mathcal{K}$, it results that $\mathcal{K}_+$ is invariant for U_i^*, $i=0,1$. Indeed, for $k=\sum_{m\in\mathbb{Z}_+^2}U^mAh_m$ with the sequence $\{h_m\}\subset\mathcal{H}$ with finite support, we obtain

$$\begin{aligned}U_0^*k&=\sum_{q\ge0}U_0^*U_1^qAh_{0q}+\sum_{\substack{p\ge1\\q\ge0}}U_0^*U_0^pU_1^qAh_{pq}\\&=\sum_{q\ge0}U_1^qU_0^*Ah_{0q}+\sum_{\substack{p\ge1\\q\ge0}}U_0^{p-1}U_1^qAh_{pq}\\&=\sum_{q\ge0}U_1^qAT_0^*h_{0q}+\sum_{\substack{p\ge1\\q\ge0}}U_0^{p-1}U_1^qAh_{pq}.\end{aligned}$$

Therefore $U_0^*\mathcal{K}_+\subset\mathcal{K}_+$ and analogously $U_1^*\mathcal{K}_+\subset\mathcal{K}_+$. Then $V_i^*=U_i^*|\mathcal{K}_+$ $(i=0,1)$, and consequently, $V_0V_1^*=V_1^*V_0$. Hence $[\mathcal{K}_+,(V_0,V_1)]$ is a doubly commuting minimal isometric dilation of T.

Corollary 2. *A bicontraction T on $\mathcal{H}$ has a regular isometric* (*unitary*) *dilation if and only if T has a doubly commuting minimal coisometric extension.*

P r o o f. If $\Delta_T \geq 0$, then T^* has a doubly commuting minimal isometric dilation $[\mathcal{K},\ (W_0, W_1)]$. Hence (W_0^*, W_1^*) is a minimal coisometric extension of T and W_0^*, W_1^* doubly commute on $\mathcal{K}$. The converse is obvious.

R e m a r k. If $\mathcal{K} = \mathcal{K}_u^i \oplus \mathcal{K}_s^i$ is the Wold decomposition of $\mathcal{K}$ relative to W_i (in the previous proof), then W_{1-i} reduces $\mathcal{K}_u^i$ and $\mathcal{K}_s^i$, $i = 0, 1$. Thus, the matrix of W_{1-i}^* relative to the decomposition $\mathcal{K} = \mathcal{K}_u^i \oplus \mathcal{K}_s^i$ has diagonal form for $i = 0, 1$, that is, T is diagonally extendable (see [11]).

Now we can give the following characterization of the double commutativity of an isometric dilation of $[\mathcal{H}, T]$.

PROPOSITION 3. *Suppose $T = (T_0, T_1)$ is a bicontraction on $\mathcal{H}$ and $[\mathcal{K}, (U_0, U_1)]$ a minimal isometric dilation of T. Then the isometries U_0 and U_1 doubly commute on $\mathcal{K}$ iff $[\mathcal{K}, U]$ is a $*$-regular isometric dilation of $[\mathcal{H}, T]$. In particular, the doubly commuting minimal isometric dilation of T (if it exists) is unique up to unitary equivalence.*

P r o o f. It is not difficult to see that the condition (3^*) is equivalent to

$$T_i^p T_{1-i}^{*q} = P_{\mathcal{H}} U_{1-i}^{*q} U_i^p|\mathcal{H} \quad (p, q \in \mathbb{Z}_+;\ i = 0, 1). \tag{4}$$

Suppose that the dilation $U = (U_0, U_1)$ satisfies (4) and let $[\widetilde{\mathcal{K}}, (\widetilde{U_0}, \widetilde{U_1})]$ be the minimal unitary extension of U. Then for $p, q \in \mathbb{Z}_+$ and $i = 0, 1$ we have

$$T_i^p T_{1-i}^{*q} = P_{\mathcal{H}} \widetilde{U}_{1-i}^{*q} \widetilde{U}_i^p|\mathcal{H} = P_{\mathcal{H}} \widetilde{U}_i^p \widetilde{U}_{1-i}^{*q}|\mathcal{H}$$

and we deduce that $[\widetilde{\mathcal{K}}, (\widetilde{U}_0^*, \widetilde{U}_1^*)]$ is a regular minimal unitary dilation for T^*. By Theorem 1, T has a doubly commuting minimal isometric dilation $[\mathcal{M}, (V_0, V_1)]$. Obviously, V_i satisfies (4) (in place of U_i) and consequently

$$P_{\mathcal{H}} U_i^{*q} U_{1-i}^p|\mathcal{H} = P_{\mathcal{H}} V_i^{*q} V_{1-i}^p|\mathcal{H} \quad (p, q \in \mathbb{Z}_+;\ i = 0, 1). \tag{$*$}$$

Let us prove that the dilations $U = (U_0, U_1)$ and $V = (V_0, V_1)$ are unitarily equivalent. Let $\{h_n\}_{n \in \mathbb{Z}_+^2} \subset \mathcal{H}$ be a sequence with finite support. Since U and V are dilations of T and satisfy $(*)$, by defining $m := (i, j) \in \mathbb{Z}_+^2$ and $n := (p, q) \in \mathbb{Z}_+^2$ we obtain

$$\begin{aligned}
\Big\| \sum_{n \in \mathbb{Z}_+^2} U^n h_n \Big\|^2 &= \sum_{m, n \in \mathbb{Z}_+^2} (U^n h_n, U^m h_m) \\
&= \sum_{j<q} (U_0^{*i} U_0^p U_1^{q-j} h_n, h_m) + \sum_{j \geq q} (U_0^{*i} U_1^{*(j-q)} U_0^p h_n, h_m) \\
&= \sum_{\substack{i<p \\ j<q}} (U_0^{p-i} U_1^{q-j} h_n, h_m) + \sum_{\substack{i \geq p \\ j<q}} (U_0^{*(i-p)} U_1^{q-j} h_n, h_m)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{\substack{i<p\\ j\geq q}} (U_1^{*(j-q)} U_0^{p-i} h_n, h_m) + \sum_{\substack{i\geq p\\ j\geq q}} (U_0^{*(i-p)} U_1^{*(j-q)} h_n, h_m)\\
&= \sum_{\substack{i<p\\ j<q}} (V_0^{p-i} V_1^{q-j} h_n, h_m) + \sum_{\substack{i\geq p\\ j<q}} (V_0^{*(i-p)} V_1^{q-j} h_n, h_m)\\
&\quad + \sum_{\substack{i<p\\ j\geq q}} (V_1^{*(j-q)} V_0^{p-i} h_n, h_m) + \sum_{\substack{i\geq p\\ j\geq q}} (V_0^{*(i-p)} V_1^{*(j-q)} h_n, h_m)\\
&= \sum_{m,n\in\mathbb{Z}_+^2} (V^{*m} V^n h_n, h_m) = \Big\| \sum_{n\in\mathbb{Z}_+^2} V^n h_n \Big\|^2.
\end{aligned}
$$

Using the minimality conditions of the spaces $\mathcal{K}$ and $\mathcal{M}$ and the norm equalities above, we deduce that there exists a unitary operator W from $\mathcal{K}$ to $\mathcal{M}$ satisfying

$$
W \sum_{n\in\mathbb{Z}_+^2} U^n h_n = \sum_{n\in\mathbb{Z}_+^2} V^n h_n
$$

for $\{h_n\} \subset \mathcal{H}$ with finite support. Consequently, $W|\mathcal{H} = I$ and $WU_i = V_i W$, $i = 0, 1$, and in particular, it results that U_0 and U_1 doubly commute on $\mathcal{K}$. Since the other assertions were also implicitly proved, the proof is finished.

Now, having in mind the condition (iii) of Theorem 1, we obtain

Corollary 4. *Let $T = (T_0, T_1)$ be a bicontraction on $\mathcal{H}$ and $[\mathcal{K}_0, S_0]$ (respectively $[\mathcal{K}_{*0}, S_{*0}]$) the minimal isometric dilation of T_0 (resp. T_0^*). Then T has a $*$-regular (resp. regular) isometric dilation if and only if T_1^* (resp. T_1) has a contractive extension on $\mathcal{K}_0$ (resp. $\mathcal{K}_{*0}$) which doubly commutes with S_0 (resp. S_{*0}).*

It is obvious (by the proof of Proposition 3) that if $[\mathcal{K}, V]$ is a $*$-regular (resp. regular) isometric dilation of $[\mathcal{H}, T]$ and if $[\widetilde{\mathcal{K}}, U]$ is the minimal unitary extension of V then the regular (resp. $*$-regular) isometric dilation of T^* is $[\mathcal{K}_*, (V_{*0}, V_{*1})]$, where

$$
\mathcal{K}_* = \bigvee_{m,n\in\mathbb{Z}_+} U_0^{*m} U_1^{*n} \mathcal{H}, \qquad V_{*i} = U_i^* | \mathcal{K}_* \quad (i = 0, 1). \tag{5}
$$

Furthermore, with the notations of Theorem 1(iii), the $*$-regular isometric dilation $[\mathcal{K}, V]$ of T is the regular and $*$-regular isometric dilation of the doubly commuting bicontraction $S = (S_0, S_1)$ (see Proposition 0), and in fact, $[\mathcal{K}, V_1]$ is the minimal isometric dilation of S_1.

3. Intertwinings of regular dilations. Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces and $T = (T_0, T_1)$ and $T' = (T_0', T_1')$ two bicontractions on $\mathcal{H}$ and $\mathcal{H}'$ respectively. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}'$ *intertwines* T and

T' if $AT_i = T_i'A$, $i = 0, 1$. The operator A *doubly intertwines* T and T' if $AT_i = T_i'A$ and $AT_i^* = T_i'^*A$, $i = 0, 1$.

V. Müller has shown in [14] that if A intertwines two bicontractions which have regular dilations, then in general, A cannot be "lifted" in the sense of [5], [18] to an operator which intertwines these dilations. In order to give conditions under which this *is* possible, we will first prove

THEOREM 5. *Let $[\mathcal{H}, T]$ and $[\mathcal{H}', T']$ be two bicontractions having $*$-regular isometric dilations $[\mathcal{K}, U]$ and $[\mathcal{K}', U']$ respectively. Let A be a contraction from $\mathcal{H}$ in $\mathcal{H}'$ such that $AT_i = T_i'A$ $(i = 0, 1)$ and $AT_0^* = T_0'^*A$. Then there is a contraction B from $\mathcal{K}$ in $\mathcal{K}'$ with $BU_i = U_i'B$ $(i = 0, 1)$, $BU_0^* = U_0'^*$ and $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$.*

Proof. Let $A : \mathcal{H} \to \mathcal{H}'$ be a contraction which satisfies $AT_i = T_i'A$ $(i = 0, 1)$ and $AT_0^* = T_0'^*A$. Let $[\mathcal{K}_0, S_0]$ and $[\mathcal{K}_0', S_0']$ be the minimal isometric dilations of T_0 and T_0' respectively. By Theorem 1(iii) there are contractions S_1 on $\mathcal{K}_0$ and S_1' on $\mathcal{K}_0'$ such that S_1 doubly commutes with S_0 and $P_{\mathcal{H}}S_1 = T_1P_{\mathcal{H}}$, while S_1' doubly commutes with S_0' and $P_{\mathcal{H}'}S_1' = T_1'P_{\mathcal{H}'}$. Since

$$\mathcal{K}_0 = \mathcal{H} \oplus \bigoplus_{p \in \mathbb{Z}_+} S_0^p \overline{(S_0 - T_0)\mathcal{H}}, \qquad \mathcal{K}_0' = \mathcal{H}' \oplus \bigoplus_{p \in \mathbb{Z}_+} S_0'^p \overline{(S_0' - T_0')\mathcal{H}'}$$

(see [5], [18]), and A doubly intertwines T_0 and T_0', we can define a contraction $A_0 : \mathcal{K}_0 \to \mathcal{K}_0'$ by setting

$$A_0 k_0 := Ah + \sum_{p \geq 0} S_0'^p (S_0' - T_0')Ah_p$$

for $k_0 = h + \sum_{p \in \mathbb{Z}_+} S_0^p (S_0 - T_0)h_p$, where $h, h_p \in \mathcal{H}$. We have $A_0|\mathcal{H} = A$, and for $k_0 \in \mathcal{K}_0$ as above,

$$\begin{aligned} A_0 S_0 k_0 &= A_0 \Big[T_0 h + (S_0 - T_0)h + \sum_{p \geq 0} S_0^{p+1}(S_0 - T_0)h_p \Big] \\ &= AT_0 h + (S_0' - T_0')Ah + \sum_{p \geq 0} S_0'^{p+1}(S_0' - T_0')Ah_p \\ &= S_0' \Big[Ah + \sum_{p \geq 0} S_0'^p (S_0' - T_0')Ah_p \Big] = S_0' A_0 k_0. \end{aligned}$$

Therefore $A_0 S_0 = S_0' A_0$. Also, $A_0 S_0^* = S_0'^* A_0$ and $A_0^*|\mathcal{H}' = A^*$, because for $k = \sum_{p \geq 0} S_0^p h_p$ (finite sum) with $h_p \in \mathcal{H}$, we have

$$\begin{aligned} A_0 S_0^* k &= AT_0^* h_0 + \sum_{p \geq 1} A_0 S_0^{p-1} h_p = T_0'^* Ah_0 + \sum_{p \geq 1} S_0'^{p-1} Ah_p \\ &= S_0'^* Ah_0 + S_0'^* \sum_{p \geq 1} S_0'^p Ah_p = S_0'^* \sum_{p \geq 0} S_0'^p Ah_p = S_0'^* A_0 k, \end{aligned}$$

and for $h' \in \mathcal{H}'$,

$$(A_0^* h', k) = \sum_p (h', S_0'^p A h_p) = \sum_p (T_0'^{*p} h', A h_p) = \sum_p (A^* T_0'^{*p} h', h_p)$$
$$= \sum_p (T_0^{*p} A^* h', h_p) = \sum_p (A^* h', S_0^p h_p) = (A^* h', k).$$

Next, we also have $A_0 S_1 = S_1' A_0$, because for $\{h_p'\}_{p\ge 0} \subset \mathcal{H}'$ with finite support,

$$A_0^* S_1'^* \sum_p S_0'^p h_p' = \sum_p A_0^* S_0'^p S_1'^* h_p' = \sum_p S_0^p A_0^* T_1'^* h_p' = \sum_p S_0^p A^* T_1'^* h_p'$$
$$= \sum_p S_0^p T_1^* A^* h_p' = \sum_p S_0^p S_1^* A_0^* h_p' = S_1^* A_0^* \sum_p S_0'^p h_p',$$

and consequently, $A_0^* S_1'^* = S_1^* A_0^*$, whence $A_0 S_1 = S_1' A_0$. We conclude that A intertwines S_1 and S_1' and doubly intertwines S_0 and S_0', and A_0 is an extension for A, while A_0^* is an extension for A^*. Hence $P_{\mathcal{H}'} A_0 = A P_{\mathcal{H}}$.

Now let $[\mathcal{K}, U_1]$, $[\mathcal{K}', U_1']$ be the minimal isometric dilations of S_1, S_1' respectively, and let U_0, U_0' be the $*$-extensions of S_0 (on $\mathcal{K}$) and of S_0' (on $\mathcal{K}'$), respectively, such that U_0 doubly commutes with U_1 and U_0' doubly commutes with U_1'. Using the sequences of n-step dilations for S_1 and S_1' and the corresponding n-step intertwining liftings of A, we can define a contraction $B : \mathcal{K} \to \mathcal{K}'$ by

$$Bk = \lim_n A_n P_{\mathcal{K}_n} k \qquad (k \in \mathcal{K}),$$

where $\{\mathcal{K}_n\}$ and $\{A_n\}$ are inductively defined with $\mathcal{K}_1 = \mathcal{K}_0 \oplus \mathcal{D}_{S_1}$ and $A_1 : \mathcal{K}_0 \oplus \mathcal{D}_{S_1} \to \mathcal{K}_0' \oplus \mathcal{D}_{S_1'}$ of the form

$$A_1 = \begin{pmatrix} A_0 & 0 \\ X_1 D_{A_0} & Y_1 \end{pmatrix},$$

$\mathcal{D}_C$ being the defect space of the operator C. Here the operator $(X_1, Y_1) : \mathcal{D}_{A_0} \oplus \mathcal{D}_{S_1} \to \mathcal{D}_{S_1'}$ is $(X_1, Y_1) = \varGamma_0 P_0$, where P_0 is the orthogonal projection of $\mathcal{D}_{A_0} \oplus \mathcal{D}_{S_1}$ on the subspace $\{D_{A_0} S_1 k \oplus D_{S_1} k : k \in \mathcal{K}_0\}^-$ and $\varGamma_0(D_{A_0} S_1 k \oplus D_{S_1} k) = D_{S_1'} A_0 k$, $k \in \mathcal{K}_0$. Then B satisfies $BU_1 = U_1' B$, $BU_0 = U_0' B$, $BU_0^* = U_0'^* B$ and $P_{\mathcal{K}_0'} B = A_0 P_{\mathcal{K}_0}$ (see [9] for details). Hence $P_{\mathcal{H}'} B = B P_{\mathcal{H}}$ and since $[\mathcal{K}, (U_0, U_1)]$ and $[\mathcal{K}', (U_0', U_1')]$ are the $*$-regular isometric dilations for T and T' respectively, B is the desired operator. The proof is finished.

Corollary 6. *Let $[\mathcal{H}, T]$ and $[\mathcal{H}', T']$ be two bicontractions which have $*$-regular (or regular) isometric dilations with the minimal unitary extensions $[\widetilde{\mathcal{K}}, \widetilde{U}]$ and $[\widetilde{\mathcal{K}}', \widetilde{U}']$ respectively. If A is a contraction from $\mathcal{H}$ to $\mathcal{H}'$ which satisfies $AT_i = T_i' A$ $(i = 0, 1)$ and $AT_0^* = T_0'^* A$, then there exists a contraction $\widetilde{A}$ from $\widetilde{\mathcal{K}}$ to $\widetilde{\mathcal{K}}'$ such that $\widetilde{A}\widetilde{U}_i = \widetilde{U}_i' \widetilde{A}$ $(i = 0, 1)$ and $P_{\mathcal{H}'} \widetilde{A} | \mathcal{H} = A$.*

P r o o f. Suppose that T and T' have the $*$-regular isometric dilations $[\mathcal{K}, U]$ and $[\mathcal{K}', U']$ and let $[\widetilde{\mathcal{K}}, \widetilde{U}]$ and $[\widetilde{\mathcal{K}}', \widetilde{U}']$ be the minimal unitary extensions of U and U' respectively. If A is an intertwining contraction of T and T' and B is an intertwining contraction of U and U' with $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$ given by Theorem 5, then there exists (see [12]) a contraction $\widetilde{B}$ from $\widetilde{\mathcal{K}}$ into $\widetilde{\mathcal{K}}'$ which intertwines $\widetilde{U}$ and $\widetilde{U}'$, such that $\widetilde{B}|\mathcal{K} = B$. It results that $P_{\mathcal{H}'}\widetilde{B}|\mathcal{H} = A$, whence $P_{\mathcal{H}}\widetilde{B}^*|\mathcal{H}' = A^*$ and $\widetilde{B}^*$ intertwines $\widetilde{U}'^*$ and $\widetilde{U}^*$. Obviously, $[\widetilde{\mathcal{K}}, \widetilde{U}^*]$ and $[\widetilde{\mathcal{K}}', \widetilde{U}'^*]$ are the regular unitary dilations of T^* and T'^* respectively.

THEOREM 7. *Let $[\mathcal{H}, T]$ and $[\mathcal{H}', T']$ be two bicontractions having regular isometric dilations $[\mathcal{K}, U]$ and $[\mathcal{K}', U']$ respectively, such that T_1^* or T_1' is an isometry. If A is a contraction of $\mathcal{H}$ into $\mathcal{H}'$ such that $AT_i = T_i'A$ $(i = 0, 1)$ and $AT_0^* = T_0'^*A$, then there exists a contraction B from $\mathcal{K}$ to $\mathcal{K}'$ with $BU_i = U_i'B$ $(i = 0, 1)$, and $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$.*

P r o o f. Suppose that the bicontractions $T = (T_0, T_1)$ and $T' = (T_0', T_1')$ have regular isometric dilations. Then $T^* = (T_0^*, T_1^*)$ has a $*$-regular isometric dilation and therefore if $[\mathcal{K}_{0*}, S_{0*}]$ is the minimal isometric dilation of T_0^*, then there is a contraction S_{1*} on $\mathcal{K}_{0*}$ which doubly commutes with S_{0*}, such that $P_{\mathcal{H}}S_{1*} = T_1^*P_{\mathcal{H}}$. Let $[\widetilde{\mathcal{K}}_0, \widetilde{S}_0]$ be the minimal isometric dilation of the coisometry S_{0*}^* and let $\widetilde{S}_1$ be the $*$-extension of S_{1*}^* to $\widetilde{\mathcal{K}}_0$ which doubly commutes with $\widetilde{S}_0$. But $\widetilde{S}_0$ is a unitary operator on $\widetilde{\mathcal{K}}_0$ and $[\mathcal{K}_0, S_0]$ given by

$$\mathcal{K}_0 = \bigvee_{n \in \mathbb{Z}_+} \widetilde{S}_0^n \mathcal{H}, \qquad S_0 = \widetilde{S}_0|\mathcal{K}_0,$$

is the minimal isometric dilation of T_0. We have $S_{1*}^*|\mathcal{H} = T_1$ and therefore $\widetilde{S}_1|\mathcal{H} = T_1$. Hence $\mathcal{K}_0$ is an invariant subspace for $\widetilde{S}_1$ and $S_1 = \widetilde{S}_1|\mathcal{K}_0$ is a contraction on $\mathcal{K}_0$ which satisfies $S_0S_1 = S_1S_0$ and $S_1|\mathcal{H} = T_1$.

Analogously, if $[\mathcal{K}_0', S_0']$ is the minimal isometric dilation of T_0', then there is a contraction S_1' on $\mathcal{K}_0'$ which satisfies $S_0'S_1' = S_1'S_0'$ and $S_1'|\mathcal{H}' = T_1'$.

Now let $A : \mathcal{H} \to \mathcal{H}'$ be a contraction which intertwines T_1 and T_1' and doubly intertwines T_0 and T_0'. As in the proof of Theorem 5 there is a contraction $A_0 : \mathcal{K}_0 \to \mathcal{K}_0'$ which doubly intertwines S_0 and S_0', such that $A_0|\mathcal{H} = A$. Then for any sequence $\{h_n\} \in \mathcal{H}$ with finite support we have

$$\begin{aligned} A_0S_1\sum_n S_0^n h_n &= \sum_n S_0'^n A_0S_1h_n = \sum_n S_0'^n AT_1h_n \\ &= \sum_n S_0'^n T_1'Ah_n = \sum_n S_0'^n S_1'A_0h_n = S_1'A_0\sum_n S_0^n h_n, \end{aligned}$$

therefore $A_0S_1 = S_1'A_0$. Let us remark that if T_1^* or T_1' is an isometry, then so is S_1^* (respectively S_1'). In this case it is known (see [5]) that A_0 has a unique contractive intertwining lifting of the minimal isometric dilations of S_1 and S_1'. Now as in the proof of Theorem 5 (see [9]) we can obtain a contraction $B : \mathcal{K} \to \mathcal{K}'$, where $[\mathcal{K}, U_1]$ and $[\mathcal{K}', U_1']$ are the minimal isometric dilations of S_1 and S_1' respectively, such that $P_{\mathcal{K}_0'} B = A_0 P_{\mathcal{K}_0}$ and $BU_1 = U_1'B$, $BU_0 = U_0'B$, U_0 and U_0' being the isometric extensions of S_0 and S_0' to $\mathcal{K}$ and $\mathcal{K}'$ which commute with U_1 and U_1' respectively. Finally, it is easy to see that $[\mathcal{K}, (U_0, U_1)]$ and $[\mathcal{K}', (U_0', U_1')]$ are the regular isometric dilations of T and T' respectively. The proof is finished.

Now we can obtain the versions of Theorems 5 and 7 for double intertwinings which complete those obtained in [14].

Proposition 8. *Let $[\mathcal{H}, T]$ and $[\mathcal{H}', T']$ be two bicontractions having regular* (*or $*$-regular*) *isometric dilations $[\mathcal{K}, V]$ and $[\mathcal{K}', V']$ respectively. If A is a contraction from $\mathcal{H}$ into $\mathcal{H}'$ which doubly intertwines T and T', then there exists a* (*unique*) *$*$-extension of A from $\mathcal{K}$ to $\mathcal{K}'$ which preserves the norm of A and doubly intertwines V and V'.*

Proof. Suppose first that T and T' have $*$-regular isometric dilations. Let $[\mathcal{K}_0, (S_0, S_1)]$ and $[\mathcal{K}_0', (S_0', S_1')]$ be as in the proof the Theorem 5. Consider $A : \mathcal{H} \to \mathcal{H}'$ a contractive double intertwining of T and T' and $A_0 : \mathcal{K}_0 \to \mathcal{K}_0'$ with $A_0 S_i = S_i' A_0$ $(i = 0, 1)$, $A_0 S_0^* = S_0'^* A_0$, $A_0|\mathcal{H} = A$, $A_0^*|\mathcal{H}' = A^*$ and $\|A_0\| = \|A\|$. Then for $\{h_p\}_{p\geq 0} \subset \mathcal{H}$ with finite support, we have

$$\begin{aligned} A_0 S_1^* \sum_p S_0^p h_p &= \sum_p S_0'^p A_0 S_1^* h_p = \sum_p S_0'^p A T_1^* h_p = \sum_p S_0'^p T_1'^* A h_p \\ &= \sum_p S_0'^p S_1'^* A_0 h_p = S_1'^* A_0 \sum_p S_0^p h_p, \end{aligned}$$

and so $A_0 S_1^* = S_1'^* A_0$.

Now let $[\mathcal{K}, V_1]$ and $[\mathcal{K}', V_1']$ be the minimal isometric dilations of S_1 and S_1' and let V_0, V_0' be the extensions of S_0, S_0' to $\mathcal{K}, \mathcal{K}'$ which doubly commute with V_1, V_1' respectively. As above, there exists a contraction $B : \mathcal{K} \to \mathcal{K}'$ with $BV_i = V_i'B$, $BV_i^* = V_i'^* B$, $(i = 0, 1)$, $B|\mathcal{K}_0 = A_0$, $B^*|\mathcal{K}_0' = A_0^*$, whence $B|\mathcal{H} = A$, $B^*|\mathcal{H}' = A^*$ and $\|B\| = \|A_0\| = \|A\|$. So the conclusion holds for the $*$-regular isometric dilations $[\mathcal{K}, V]$ and $[\mathcal{K}', V']$ of T and T'.

Next let $[\widetilde{\mathcal{K}}, \widetilde{U}]$, $[\widetilde{\mathcal{K}}', \widetilde{U}']$ be the minimal unitary extensions of V, V' and $[\mathcal{K}_*, V_*]$, $[\mathcal{K}_*', V_*']$ be the regular isometric dilations of T^*, T'^* respectively (as in (5)). Then there exists ([12], [7]) a contraction $\widetilde{A} : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}'$ such that $\widetilde{A}\widetilde{U}_i = \widetilde{U}_i'\widetilde{A}$ $(i = 0, 1)$, $\widetilde{A}|\mathcal{K} = B$, $\widetilde{A}^*|\mathcal{K}' = B^*$ and $\|\widetilde{A}\| = \|B\|$. Because $\widetilde{A}|\mathcal{H} = A$ and $\widetilde{A}\widetilde{U}_i^* = \widetilde{U}_i'^* \widetilde{A}$ $(i = 0, 1)$, we have $\widetilde{A}\mathcal{K}_* \subset \mathcal{K}_*'$. But $\widetilde{A}^*|\mathcal{H}' = A^*$ and $\widetilde{A}^* \widetilde{U}_i'^* = \widetilde{U}_i^* A^*$ $(i = 0, 1)$ imply $\widetilde{A}^* \mathcal{K}_*' \subset \mathcal{K}_*$. So we can define the operator $C : \mathcal{K}_* \to \mathcal{K}_*'$ by $C = \widetilde{A}|\mathcal{K}_*$. Then $C^* = \widetilde{A}^*|\mathcal{K}_*'$ and $C|\mathcal{H} = A$, $C^*|\mathcal{H}' = A^*$ and since $V_{*i} = \widetilde{U}_i^*|\mathcal{K}_*$, $V_{*i}' = \widetilde{U}_i'^*|\mathcal{K}_*'$ $(i = 0, 1)$, it results that

$CV_{*i} = V'_{*i}C$ and $CV^*_{*i} = V'^*_{*i}C$ $(i = 0, 1)$. Finally, $\|A\| \le \|C\| \le \|\widetilde{A}\| = \|B\| = \|A\|$ and so $\|C\| = \|A\|$. Thus the conclusion holds for the regular isometric dilations of T^* and T'^*, and consequently, in the case when T and T' have regular isometric dilations.

Under certain conditions we can drop the doubly intertwining property on a component. The first fact in this context is contained in

PROPOSITION 9. *Let $[\mathcal{H}, T]$ be a bicontraction with the first component T_0 a coisometry, and $[\mathcal{H}', T']$ be another bicontraction which has a $*$-regular isometric dilation. Let $[\mathcal{K}, U]$ and $[\mathcal{K}', U']$ be the $*$-regular isometric dilations of T and T' respectively. If A is a contraction from $\mathcal{H}$ to $\mathcal{H}'$ with $AT_i = T'_iA$ $(i = 0, 1)$, then there exists a contraction B from $\mathcal{K}$ to $\mathcal{K}'$ such that $BU_i = U'_iB$ $(i = 0, 1)$, and $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$.*

Proof. Let $A : \mathcal{H} \to \mathcal{H}'$ be a contractive intertwining of T and T'. Preserving the notations of the proof of Theorem 5, we find (by the lifting theorem) that there exists a contraction A_0 from $\mathcal{K}_0$ to $\mathcal{K}'_0$ which satisfies $A_0S_0 = S'_0A_0$ and $P_{\mathcal{H}'}A_0 = AP_{\mathcal{H}}$. Because T_0 is a coisometry, its minimal isometric dilation S_0 is a unitary operator on $\mathcal{K}_0$. Then from Theorem B of [6] (which can be extended to operators acting on different spaces) it results that $A_0S^*_0 = S'^*_0A_0$. So A_0 doubly intertwines S_0 and S'_0. Furthermore, A_0 intertwines S_1 and S'_1, the doubly commuting commutants of S_0 and S'_0 which lift T_1 and T'_1 respectively, given by Theorem 1(iii). By Theorem 5, A_0 has a contractive lift, which intertwines the $*$-regular isometric dilations of $S = (S_0, S_1)$ (of T) and $S' = (S'_0, S'_1)$ (of T'), whence the conclusion follows.

The dual version of Proposition 9 is in fact an extension of Proposition 5.2 from [12] (for bicontractions) and of Proposition 10 from [2].

COROLLARY 10. *Let $[\mathcal{H}, T]$ be a bicontraction which has a regular isometric dilation, and $[\mathcal{H}', T']$ be a bicontraction with T'_0 an isometry. We also suppose that T^*_1 or T'_1 is an isometry. If A is a contraction of $\mathcal{H}$ into $\mathcal{H}'$ which intertwines T and T', then there exists a contraction B which intertwines the regular isometric dilations of T and T' and satisfies $P_{\mathcal{H}'}B = AP_{\mathcal{H}}$.*

Proof. If $[\mathcal{K}_0, S_0]$ is the minimal isometric dilation of T_0 and S_1 is a contraction on $\mathcal{K}_0$ with $S_0S_1 = S_1S_0$ and $S_1|\mathcal{H} = T_1$, then $A_0 = AP_{\mathcal{H}}$ is a contraction from $\mathcal{K}_0$ into $\mathcal{H}'$ and satisfies $A_0S_0 = T'_0A_0$, $A_0S_1 = T'_1A_0$, and A_0 is a lifting for A. Since S_0 and T'_0 and respectively S^*_1 or T'_1 are isometries, there is a lifting for A_0 which intertwines the regular isometric dilations for (S_0, S_1) and (T'_0, T'_1).

Recall ([13]) that a bounded linear operator S on $\mathcal{H}$ is *subnormal* if there exists a normal operator N on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant for N and $S = N|\mathcal{H}$. If, furthermore, $\mathcal{K} = \bigvee_{p\geq 0} N^{*p}\mathcal{H}$, then N is said to be the *minimal normal extension* of S. In this case, N is unique (up to unitary equivalence) and $\|N\| = \|S\|$.

A bicontraction $T = (T_0, T_1)$ will be called *semi-subnormal* if one of the contractions is subnormal and the other one has an extension which commutes (therefore doubly commutes) with the minimal normal extension of the subnormal one. Such a bicontraction T has a regular isometric dilation because, if T_0 is subnormal and N_1 is an extension of T_1 commuting with the minimal normal extension N_0 of T_0, then we have $\Delta_T = P_{\mathcal{H}}\Delta_{(N_0,N_1)}|\mathcal{H} \geq 0$.

It is easy to see that every subnormal bicontraction is semi-subnormal.

Now we have the following completion of Corollary 10.

PROPOSITION 11. *Let $T = (T_0, T_1)$ and $T' = (T_0', T_1')$ be two semi-subnormal bicontractions on $\mathcal{H}$ and $\mathcal{H}'$ respectively, such that T_0 and T_0' are subnormal and T_1' is an isometry. If A is a contraction from $\mathcal{H}$ to $\mathcal{H}'$ which intertwines T and T' and A has an extension which intertwines the minimal normal extensions of T_0 and T_0', then A has an extension which intertwines the regular isometric dilations of T and T'.*

Proof. Let T, T' and A be as in the hypothesis and let $[\widetilde{\mathcal{H}}, N]$ and $[\widetilde{\mathcal{H}}', N']$, where $N = (N_0, N_1)$ and $N' = (N_0', N_1')$, be such that N_0 (resp. N_0') is the minimal normal extension on $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{H}}'$) of T_0 (resp. T_0') and N_1 (resp. N_1') is a contraction on $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{H}}'$) which extends T_1 (resp. T_1') and doubly commutes with N_0 (resp. N_0'). From the hypothesis and the Fuglede–Putnam Theorem, there is an operator $\widetilde{A} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}'$ which doubly intertwines N_0 and N_0', and $\widetilde{A}|\mathcal{H} = A$. Then for $q \geq 0$ and $h \in \mathcal{H}$ we have

$$\begin{aligned}\widetilde{A}N_1N_0^{*q}h = {}* \widetilde{A}N_0^{*q}N_1h &= N_0'^{*q}\widetilde{A}T_1h = N_0'^{*q}AT_1h \\ &= N_0'^{*q}T_1'Ah = N_0'^{*q}N_1'\widetilde{A}h = N_1'\widetilde{A}N_0^{*q}h.\end{aligned}$$

Using the structure of the space $\widetilde{\mathcal{H}}$, it results that $\widetilde{A}N_1 = N_1'\widetilde{A}$. Let us remark that because T_1' is an isometry on $\mathcal{H}', N_1'$ is also an isometry on $\widetilde{\mathcal{H}}'$, hence the minimal unitary extension $[\mathcal{K}', V_1']$ of N_1' is just the minimal coisometry extension of N_1'. Let $[\mathcal{K}, V_1]$ be the minimal coisometry extension of N_1 and let V_0, V_0' be the $*$-extensions of N_0, N_0' which doubly commute with V_1, V_1' respectively. Since $\widetilde{A}$ intertwines N_1 and N_1', there exists a contraction $\widetilde{A}_1$ from $\mathcal{K}$ into $\mathcal{K}'$ which satisfies $\widetilde{A}_1V_1 = V_1'\widetilde{A}_1$ and $\widetilde{A}_1|\widetilde{\mathcal{H}} = \widetilde{A}$. In fact, $\widetilde{A}_1$ doubly intertwines V_1 and V_1' (see [6]). Moreover, $\widetilde{A}_1$ doubly intertwines the normal operators V_0 and V_0', because for $\{h_n\}_{n\geq 0} \subset \widetilde{\mathcal{H}}$ with finite support we have

$$\widetilde{A}_1 V_0 \sum_n V_1^{*n} h_n = \sum_n V_1'^{*n} \widetilde{A}_1 V_0 h_n = \sum_n V_1'^{*n} \widetilde{A} N_0 h_n$$
$$= \sum_n V_1'^{*n} N_0' \widetilde{A} h_n = \sum_n V_1'^{*n} V_0' \widetilde{A}_1 h_n = V_0' \widetilde{A}_1 \sum_n V_1^{*n} h_n.$$

Hence $\widetilde{A}_1$ doubly intertwines the bicontractions $V = (V_0, V_1)$ and $V' = (V_0', V_1')$. Then by Proposition 8, $\widetilde{A}_1$ has an $*$-extension $\widetilde{B}$ which intertwines the regular isometric dilations $M = (M_0, M_1)$ on $\mathcal{M}$ of V and $M' = (M_0', M_1')$ on $\mathcal{M}'$ of V' respectively. Setting

$$\mathcal{K} = \bigvee_{n \in \mathbb{Z}_+^2} M^n \mathcal{H}, \quad \mathcal{K}' = \bigvee_{n \in \mathbb{Z}_+^2} M'^n \mathcal{H}'$$

and $U_i = M_i|\mathcal{K}$, $U_i' = M_i'|\mathcal{K}'$ $(i = 0, 1)$, we deduce that $[\mathcal{K}, (U_0, U_1)]$ and $[\mathcal{K}', (U_0', U_1')]$ are the regular isometric dilations of T and T' respectively. Since $\widetilde{B}\mathcal{H} = \widetilde{A}\mathcal{H} \subset \mathcal{H}'$ and $\widetilde{B}$ intertwines M_i and M_i' $(i = 0, 1)$, it results that $\widetilde{B}\mathcal{K} \subset \mathcal{K}'$. Consequently, $B = \widetilde{B}|\mathcal{K}$ is a contraction of $\mathcal{K}$ in $\mathcal{K}'$ with $BU_i = U_i'B$ $(i = 0, 1)$ and $B|\mathcal{H} = A$.

Proposition 11 can be applied, in particular, to the case when T is a subnormal bicontraction, which includes the spherical isometries ([2]) and the bidisc isometries ([4]).

Now we can give sufficient conditions for three commuting contractions in order to have unitary dilations.

THEOREM 12. *Let T_0, T_1, T_2 be three pairwise commuting contractions on $\mathcal{H}$. If the bicontractions (T_0, T_1) and (T_0, T_2) have regular (or $*$-regular) isometric dilations, then (T_0, T_1, T_2) has a isometric (unitary) dilation.*

Proof. Suppose first that (T_0, T_1) and (T_0, T_2) have $*$-regular isometric dilations. Let $[\mathcal{K}_0, S_0]$ be the minimal isometric dilation of T_0 and S_1, S_2 be contractions on $\mathcal{K}_0$ which doubly commute with S_0, such that $P_{\mathcal{H}} S_i = T_i P_{\mathcal{H}}$, $i = 1, 2$. Then it results that $S_1^* S_2^* = S_2^* S_1^*$, and consequently, $S_1 S_2 = S_2 S_1$ on $\mathcal{K}_0$. Let $[\mathcal{K}_1, (V_0, V_1)]$ be the $*$-regular isometric dilation of (S_0, S_1), therefore with V_0 and V_1 doubly commuting isometries on $\mathcal{K}_1$. By Theorem 5 there exists a contraction $V_2 : \mathcal{K}_1 \to \mathcal{K}_1$ with $P_{\mathcal{K}_0} V_2 = S_2 P_{\mathcal{K}_0}$ and $V_2 V_i = V_i V_2$ $(i = 0, 1)$, $V_2 V_0^* = V_0^* V_2$. Therefore V_0 doubly commutes with (V_1, V_2) and by Propositions 8 or 11, if $[\mathcal{K}, (U_1, U_2)]$ is the regular isometric dilation of (V_1, V_2), then there is an isometry U_0 on $\mathcal{K}$ such that $U_0|\mathcal{K}_1 = V_0$ and U_0 commutes with U_1 and respectively with U_2. So for $m, n, j \in \mathbb{Z}_+$ and $h \in \mathcal{H}$ we have

$$P_{\mathcal{H}} U_0^m U_1^n U_2^j h = P_{\mathcal{K}_1, \mathcal{H}} P_{\mathcal{K}_1} U_1^n U_2^j V_0^m h$$
$$= P_{\mathcal{K}_1, \mathcal{H}} V_1^n V_2^j V_0^m h = P_{\mathcal{K}_0, \mathcal{H}} P_{\mathcal{K}_0} V_2^j V_0^m V_1^n h$$

$$= P_{\mathcal{K}_0,\mathcal{H}} S_2^j P_{\mathcal{K}_0} V_0^m V_1^n h = P_{\mathcal{H}} S_0^m S_1^n S_2^j h$$
$$= T_0^m P_{\mathcal{H}} S_1^n S_2^j h = T_0^m T_1^n P_{\mathcal{H}} S_2^j h = T_0^m T_1^n T_2^j h.$$

Therefore $[\mathcal{K}, (U_0, U_1, U_2)]$ is an isometric dilation of (T_0, T_1, T_2).

Now if (T_0, T_1) and (T_0, T_2) have regular isometric dilations, then (T_0^*, T_1^*) and (T_0^*, T_2^*) have $*$-regular isometric dilations, so (T_0^*, T_1^*, T_2^*) and consequently (T_0, T_1, T_2) have isometric dilations.

COROLLARY 13. *Let T_0, T_1 and T_2 be three pairwise commuting contractions on $\mathcal{H}$, such that T_0 is subnormal and (T_0, T_1) and (T_0, T_2) are semi-subnormal bicontractions. Then (T_0, T_1, T_2) has a unitary dilation.*

Proof. The bicontractions (T_0, T_1) and (T_0, T_2) have regular isometric dilations and we apply Theorem 12.

Acknowledgements. The work of the first named author at this research is partially supported by the Alexander von Humboldt-Stiftung. The final form of the article was presented during his visit to the Universität des Saarlandes in Saarbrücken. The work of the second named author was partially completed during his visit to the University "La Sapienza" in Rome. We would like to thank Professors Ernst Albrecht and Stefano Marchiafava for their hospitality.

We are also very grateful to the reviewer for pointing out some misprints and language errors in the first variant of the paper.

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Reçu par la Rédaction le 12.7.1995
Révisé le 8.1.1996