Stabilization of solutions to a differential-delay equation in a Banach space

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Abstract. A parameter dependent nonlinear differential-delay equation in a Banach space is investigated. It is shown that if at the critical value of the parameter the problem satisfies a condition of linearized stability then the problem exhibits a stability which is uniform with respect to the whole range of the parameter values. The general theorem is applied to a diffusion system with applications in biology.

1. Introduction. In this work we investigate a class of parameter dependent differential-delay equations in a Banach space $X$ and apply the method of fixed points in the spaces of functions in $X$ tending to zero as $t \to \infty$ at an appropriate rate that was developed in [3]. In particular, we address the stability of the stationary solution of such an equation. The stability is shown to be uniform with respect to a small parameter on some finite interval.

The stability of solutions to differential-delay equations has been studied in a number of publications. Let us mention at least a few of them. The asymptotic stability for Problem (2.1) below with $\varepsilon = 1$ has been proved in [6, 7] under the assumption of the stability of the linearized problem. The results are applied to a parabolic equation with delay. In [10] stabilization of solutions to the fully nonlinear problem is established by means of monotonicity of the generator of the corresponding nonlinear semigroup. A similar approach is also used in [2], where a series of results on asymptotic behavior of solutions and their mean values is proved. Finally, in [8, 9] appropriate functionals and sufficiently strong \textit{a priori} bounds are used to show the (uniform) asymptotic stability of solutions under certain natural assumptions.

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In our approach the existence and stabilization of solution is shown by a fixed point argument. We obtain the rate of convergence and describe the global behavior of solution in connection with a singular parameter involved.

Our notation is consistent with that introduced in [3]; in particular, we adopt the usual notation $L^p(M; X)$ for the $L^p$-spaces of functions from a set $M \subset \mathbb{R}^N$ into a Banach space $X$, $W^{k,p}(M; X)$ for the Sobolev spaces of $k$th order, $C^k(M; X)$ for the spaces of functions with continuous derivatives up to order $k$, $L(X, Y)$ for the space of the continuous linear operators from $X$ into $Y$ with $L(X) = L(X, X)$, $L_s(X)$ being $L(X)$ equipped with the strong operator topology, and so on.

2. Formulation of the problem. Let us consider the following parameter dependent problem:

$$
\varepsilon u_\varepsilon'(t) + Au_\varepsilon(t) - Eu_\varepsilon(t - \tau) = Fu_\varepsilon(t) + Gu_\varepsilon(t - \tau), \quad t > 0,
$$

(2.1)

$$
u_\varepsilon(s) = x(s), \quad s \in (-\tau, 0], \quad \varepsilon \in [0, \varepsilon_0] \quad (\tau > 0, \quad \varepsilon_0 > 0),$$

where $A : X \supset D(A) \to X$ is linear, $E \in L(X)$, $F, G : X \to X$ are possibly nonlinear operators. The fixed number $\tau > 0$ is a given delay, $\varepsilon \in [0, \varepsilon_0]$ a parameter, and $x(\cdot) : (-\tau, 0) \to X$ a given initial datum. We are interested in the stabilization of $u_\varepsilon(t)$ as $t \to \infty$. This will be achieved by an appropriate splitting of the problem in a stable linear part and a nonlinear perturbation which is locally small. We shall work in the space

$$
L^\infty_w(0, \infty; X)
$$

= \{u \in L^\infty(0, \infty; X) : \|u\|_w := \text{ess sup}_{t \geq 0} w(t)|u(t)| < \infty\},

(2.2)

for some function $w \in L^\infty_{\text{loc}}(0, \infty)$ such that $w(t) \geq 1$ a.e. in $(0, \infty)$ and $\lim_{t \to \infty} w(t) = \infty$. It is a standard result that the space $L^\infty_w(0, \infty; X)$ is a Banach space under the norm $\| \cdot \|_w$.

We make the following assumptions:

$$
\begin{cases}
(i) \quad -A \text{ is the generator of a } C_0\text{-semigroup in } L(X); \\
(ii) \quad F : X \to X, F(0) = 0; \\
(iii) \quad \text{the semigroup } T(t) \text{ generated by } -A \text{ satisfies} \\
\quad |T(t)| \leq \rho(t), \quad t \geq 0, \text{ with some } \rho \in L^\infty(0, \infty); \\
(iv) \quad E \in L(X) \text{ and } G : X \to X, G(0) = 0.
\end{cases}
$$

(2.3)

To invert the linear part of (2.1) in the space $L^\infty_w(0, \infty; X)$ with an appropriate weight $w$, define the following auxiliary problems.
Stabilization of solutions

- **Fundamental solution:**
  \[ \varepsilon U'_\varepsilon(t) + AU_\varepsilon(t) - EU_\varepsilon(t - \tau) = 0, \; t > 0, \]
  \[ U_\varepsilon(s) = 0 \quad \text{for} \; s \in (-\tau, 0), \quad U_\varepsilon(0) = I, \quad \varepsilon \in (0, \varepsilon_0]. \]

- **Homogeneous problem:**
  \[ \varepsilon v'_\varepsilon(t) + Av_\varepsilon(t) = \begin{cases} g(t), & t \in (0, \tau), \\ E(v_\varepsilon(t - \tau)), & t > \tau, \end{cases} \]
  \[ v_\varepsilon(0) = y, \quad \varepsilon \in (0, \varepsilon_0), \]

where \( g : (-\tau, 0) \to X \) and \( y \in X \) are given.

- **Inhomogeneous problem:**
  \[ \varepsilon z'_\varepsilon(t) + Az_\varepsilon(t) - Ez_\varepsilon(t - \tau) = h(t), \quad t > 0, \]
  \[ z_\varepsilon(s) = 0, \quad s \in (-\tau, 0], \quad \varepsilon \in (0, \varepsilon_0], \]

where \( h : (0, \infty) \to X \). Note that \( U_\varepsilon \) is an operator valued function.

Since \(-A\) generates a \(C_0\)-semigroup in \(X\), it is clear that for any \( \varepsilon \in (0, \varepsilon_0] \) there exists a unique generalized solution \( U_\varepsilon \in C([0, \infty); L_\varepsilon(X)) \) of (2.4), that is, \( U_\varepsilon \) satisfies (2.4) and the integral equation

\[ U_\varepsilon(t) = T(t/\varepsilon) + \varepsilon^{-1} \int_0^t T((t - s)/\varepsilon) EU_\varepsilon(s - \tau) \, ds, \quad t \geq 0. \]

Also, there exists \( g_\varepsilon \in L^\infty([0, \infty)) \) such that

\[ |U_\varepsilon(t)| \leq g_\varepsilon(t) \quad \text{for} \; t \geq 0 \; \text{and} \; \varepsilon \in (0, \varepsilon_0]. \]

It is a standard result (see e.g. [1]) that the solutions \( v_\varepsilon \) and \( z_\varepsilon \) may be expressed in terms of \( U_\varepsilon \) and \((y, g)\), and of \( U_\varepsilon \) and \( h \), respectively. This is the content of the following two propositions.

2.1. **Proposition.** Let \( y \in X \) and \( g \in L^1((0, \tau); X) \). Then problem (2.5) has a family of generalized solutions \( v_\varepsilon \in C([0, \infty); X), \; \varepsilon \in (0, \varepsilon_0], \) in the sense that

\[ v_\varepsilon(t) = \begin{cases} U_\varepsilon(t)y + \varepsilon^{-1} \int_0^t U_\varepsilon(t - s)g(s) \, ds & \text{for} \; t \in (0, \tau), \\ U_\varepsilon(t)y + \varepsilon^{-1} \int_0^\tau U_\varepsilon(t - s)g(s) \, ds & \text{for} \; t > \tau. \end{cases} \]

If, in addition, \( y \in D(A), \; g \in L^1((0, \tau); D(A)) \) and \( ED(A) \subset D(A) \), then the equation in (2.4) is satisfied pointwise a.e. in \((0, \infty)\).

2.2. **Proposition.** Let \( h \in L^1_{\text{loc}}([0, \infty); X) \). Then problem (2.6) has a family of generalized solutions \( z_\varepsilon \in C([0, \infty); X), \; \varepsilon \in (0, \varepsilon_0], \) in the sense
that

\[ z_\varepsilon(t) = \varepsilon^{-1} \int_0^t U_\varepsilon(t - s)h(s)\,ds, \quad t \geq 0. \tag{2.10} \]

If, in addition, \( h \in L^1_{\text{loc}}([0, \infty); D(A)) \) and \( ED(A) \subset D(A) \), then the equation in (2.6) is satisfied pointwise a.e. in \((0, \infty)\).

Define operators \( V_\varepsilon \) and \( Z_\varepsilon \) by

\[
V_\varepsilon(y, g)(t) = v_\varepsilon(t), \quad t \in [0, \infty), \ y \in X, \ g \in L^1([0, \tau); X), \quad v_\varepsilon \text{ satisfies (2.9)},
\]

\[
Z_\varepsilon(h)(t) = z_\varepsilon(t), \quad t \in [0, \infty), \ h \in L^1_{\text{loc}}([0, \infty); X), \quad z_\varepsilon \text{ satisfies (2.10)}.
\] (2.11)

In accordance with the definitions of generalized solutions to Problems (2.5), (2.6) it is consistent to define a generalized solution to (2.1) as follows:

2.3. Definition. A function \( u_\varepsilon \in L^1_{\text{loc}}([0, \infty); X) \) (\( \varepsilon \in (0, \varepsilon_0] \)) is called a generalized solution to problem (2.1) if \( u_\varepsilon(s) = x(s) \) for \( s \in (-\tau, 0] \) and the following integral equation is satisfied:

\[
U_\varepsilon(t)x(0) + \varepsilon^{-1} \int_0^t U_\varepsilon(t - s)(Ex(s - \tau) + Gx(s - \tau) + Fu_\varepsilon(s))\,ds, \quad t \in (0, \tau],
\]

\[
U_\varepsilon(t)x(0) + \varepsilon^{-1} \int_0^\infty U_\varepsilon(t - s)(Ex(s - \tau) + Gx(s - \tau))\,ds + \varepsilon^{-1} \int_0^t U_\varepsilon(t - s)(Fu_\varepsilon(s) + Eu_\varepsilon(s - \tau) + Gu_\varepsilon(s - \tau))\,ds, \quad t > \tau,
\]

where \( U_\varepsilon \) is given by (2.7). Taking into account definitions (2.11), setting \( g(t) = Ex(t - \tau) + Gx(t - \tau) \) for \( t \in (0, \tau) \) and \( g(t) = 0 \) for \( t > \tau \),

\[
(\bar{G}u)(t) = \begin{cases} 0 & \text{for } t \in (0, \tau), \\ Eu(t - \tau) + Gu(t - \tau) & \text{for } t > \tau \end{cases}
\]

(2.13)

with \( u : \mathbb{R}^+ \to X \), we can write (2.12) in the form

\[
u_\varepsilon(t) = V_\varepsilon(x(0), g)(t) + Z_\varepsilon(Fu_\varepsilon(\cdot) + \bar{G}u_\varepsilon(\cdot))(t), \quad t \geq 0.
\]

(2.14)

Again, it may be shown in a standard way that the following assertion holds true.

2.4. Proposition. If \( x(0) \in D(A),\ Ex(\cdot) + Gx(\cdot) \in L^1((-\tau, 0); D(A)), ED(A) \subset D(A) \) and (2.13) has a solution \( u_\varepsilon \in W^{1,1}(0, T; X) \cap L^1(0, T; D(A)) \)
for some $T > 0$ and $\varepsilon \in (0, \varepsilon_0]$ then the first equation in (2.1) is satisfied pointwise a.e. in $(0, T)$.

3. Fundamental solution. We start with an investigation of the fundamental solution $U_{\varepsilon}(t)$ of (2.4).

3.1. Lemma. Let assumption (2.3) be satisfied and let

\begin{equation}
\rho(t) = Me^{-\alpha t}, \quad t \geq 0, \text{ with some constants } M > 0, \alpha > |E|.
\end{equation}

Assume further that $E$ commutes with $(\lambda I + A)^{-1}$ for some $\lambda$ with $\text{Re}\lambda > -\alpha$. Then for any $\varepsilon > 0$ there exists a generalized solution $U_{\varepsilon} \in L^\infty((-\tau, \infty); L(X))$ of (2.4), and it satisfies

\begin{equation}
|U_{\varepsilon}(t)| \leq M \left(1 - \frac{|E|e^{\beta t}}{\alpha - \varepsilon \beta}\right)^{-1} e^{-\beta t},
\end{equation}

\begin{equation*}
\varepsilon^{-1} \int_0^\infty e^{\beta t}|U_{\varepsilon}(t)| dt \leq M \alpha(\alpha - \varepsilon \beta)^{-1}(\alpha - e^{\beta t}|E|)^{-1},
\end{equation*}

for all $t \geq 0$, $\varepsilon \in (0, \varepsilon_0]$, $\beta \in [0, \beta_0(\varepsilon)) \supset (0, \beta_0)$, where $\beta_0(\varepsilon) := \sup\{\beta \in (0, \infty) : e^{\beta t}|E| < \alpha - \varepsilon \beta\}$, $\beta_0(\varepsilon) > 0$, $\beta_0(0+) = \tau^{-1} \log(\alpha/|E|)$.

Proof. A formal application of the Fourier transform to the function $U_{\varepsilon}$ (extended by zero for $t \leq -\tau$) suggests that we consider a solution of (2.4) in the form

\begin{equation}
U_{\varepsilon}(t) = \begin{cases} 
0, & t < 0, \\
\sum_{n=0}^{[s/\tau]} \frac{(t - n\tau)^n}{\varepsilon^n n!} T \left(\frac{t - n\tau}{\varepsilon}\right) E^n, & t \geq 0, \varepsilon > 0,
\end{cases}
\end{equation}

where $[s]$ stands for the integral part of $s$. Let $R(\lambda) = (\lambda I + A)^{-1}$. Then $R(\lambda) \in L(X)$ and, for each $\mu$ with $\text{Re}\mu > -\alpha$, $R(\mu) = f_\mu(R(\lambda))$ with a suitable analytic function $f_\mu$. By the functional calculus for bounded linear operators, $ER(\mu) = Ef_\mu(R(\lambda)) = f_\mu(R(\lambda))E = R(\mu)E$. To show that $E$ commutes with $T(s)$ for each $s \geq 0$ we use the Yosida approximation

\[ A_n = n^2 R(\alpha + n) - (\alpha + n)I, \quad n = 1, 2, \ldots; \]

then $T(s)x = \lim_{n \to \infty} \exp(-sA_n)x$ for all $x \in X$ and all $s \geq 0$ (see [5, Section 1.3]) and the commutativity follows. It can then be routinely verified that the function $U_{\varepsilon}$ given by (3.3) is a generalized solution of (2.4). We are going to use formula (3.3) to derive the estimates (3.2). Let $\beta \in [0, \beta_0(\varepsilon))$.

Setting

\begin{equation}
v_{\varepsilon}(t) = e^{\beta t}U_{\varepsilon}(t), \quad t \geq 0, \varepsilon \in (0, \varepsilon_0],
\end{equation}
$U_\varepsilon$ is a generalized solution of (2.4) if and only if $v_\varepsilon$ satisfies
\begin{equation}
\varepsilon v_\varepsilon'(t) + (A - \varepsilon \beta I)v_\varepsilon(t) = e^{\beta \tau}Ev_\varepsilon(t - \tau), \quad t \geq 0, \tag{3.5}
\end{equation}
\begin{align*}
v_\varepsilon(0) &= I, \\
v_\varepsilon(s) &= 0, \quad s \in (-\tau, 0).
\end{align*}
A consideration analogous to that for $U_\varepsilon$ above leads to the formula
\begin{equation}
v_\varepsilon(t) = \begin{cases}
0, & t < 0, \\
\sum_{n=0}^{[t/\tau]} \frac{(t - n\tau)^n}{\varepsilon^n n!} e^{\beta(t-n\tau)} T \left( \frac{t - n\tau}{\varepsilon} \right) e^{n\beta \tau} E^n, & t \geq 0, \varepsilon > 0.
\end{cases} \tag{3.6}
\end{equation}
We estimate the $n$th term of the sum in (3.6):
\begin{equation}
a_n(t) := \left| \frac{(t - n\tau)^n}{\varepsilon^n n!} e^{\beta(t-n\tau)} T \left( \frac{t - n\tau}{\varepsilon} \right) e^{n\beta \tau} E^n \right| \tag{3.7}
\end{equation}
\begin{align*}
&\leq M \frac{(t - n\tau)^n}{\varepsilon^n n!} \exp \left[ \frac{\varepsilon \beta - \alpha}{\varepsilon} (t - n\tau) \right] e^{n\beta \tau} |E|^n.
\end{align*}
Taking logarithm of $a_n(t)$ and using the estimate
\begin{align*}
\log(n!) &= \sum_{k=2}^{n} \log k \geq \int_{1}^{n} \log \nu d\nu = n \log n - n,
\end{align*}
we obtain
\begin{align*}
\log a_n(t) &\leq \log M + n \log \left( \frac{|E|}{\alpha - \varepsilon \beta} \right) \\
&\quad + \log \left( \sup_{s \geq 0} \left\{ s^n e^{-s} \right\} \right) + n\beta \tau - (n \log n - n) \\
&= \log M + n \left[ \beta \tau + \log \left( \frac{|E|}{\alpha - \varepsilon \beta} \right) \right].
\end{align*}
Hence we get
\begin{equation}
a_n(t) \leq Me^{-\kappa n}, \tag{3.8}
\end{equation}
where $\kappa := -\beta \tau - \log(|E|/(\alpha - \varepsilon \beta))$, which is positive by assumption. Consequently, by (3.6)–(3.8) we have
\begin{align*}
|v_\varepsilon(t)| &\leq \sum_{n=0}^{[t/\tau]} a_n(t) \leq M \sum_{n=0}^{[t/\tau]} e^{-\kappa n} \\
&\leq M \sum_{n=0}^{\infty} e^{-\kappa n} = \frac{M}{1 - e^{-\kappa}} = M \left( 1 - \frac{|E| e^{\beta \tau}}{\alpha - \varepsilon \beta} \right)^{-1},
\end{align*}
and (3.4) yields the first inequality in (3.2).
Now we prove the second inequality in \((3.2)\). Setting

\[
(3.9) \quad s = t/\varepsilon, \quad \sigma = \tau/\varepsilon, \quad v(s) = e^{\beta t} U_\varepsilon(t),
\]

we obtain

\[
(3.10) \quad v'(s) + (A - \varepsilon \beta I) v(s) = e^{\beta \tau} E v(s - \sigma), \quad s \geq 0,
\]

\[
 v(0) = I, \quad v(s) = 0, \quad s \in (-\sigma, 0).
\]

A similar reasoning to the above leads to the formula

\[
(3.11) \quad v(s) = \begin{cases} 
0, & s < 0, \\
\sum_{n=0}^{[s/\sigma]} \frac{(s - n \sigma)^n}{n!} e^{\varepsilon \beta (s - n \sigma)} e^{n \beta \tau} T(s - n \sigma) E^n, & s \geq 0.
\end{cases}
\]

Then we have

\[
J(\varepsilon) := \varepsilon^{-1} \int_0^\infty e^{\beta t} |U_\varepsilon(t)| \, dt = \int_0^\infty |v(s)| \, ds \leq \sum_{m=0}^\infty \sum_{n=0}^{(m+1)\sigma} \left| \sum_{n=0}^m \frac{(s - n \sigma)^n}{n!} e^{\varepsilon \beta (s - n \sigma)} e^{n \beta \tau} T(s - n \sigma) E^n \right| ds
\]

\[
\leq M \sum_{m=0}^\infty \sum_{n=0}^m e^{n \beta \tau} |E|^n \sum_{n=0}^{(m+1)\sigma} (s - n \sigma)^n e^{-(\alpha - \varepsilon \beta)(s - n \sigma)} ds
\]

\[
= M \sum_{m=0}^\infty \sum_{n=0}^m e^{n \beta \tau} |E|^n \sum_{n=0}^{(m+1-n)\sigma} \int_{(m-n)\sigma}^{(m+1-n)\sigma} s^n e^{-(\alpha - \varepsilon \beta)s} \, ds.
\]

Since

\[
\int s^n e^{-s} \, ds = -\frac{1}{\delta} e^{-\delta s} \sum_{l=0}^n \frac{n! s^{n-l}}{(n-l)! \delta^l},
\]

we find that

\[
J(\varepsilon) \leq -\frac{M}{\alpha - \varepsilon \beta} \sum_{m=0}^\infty \sum_{n=0}^m e^{n \beta \tau} |E|^n
\]

\[
\times \sum_{l=0}^n \frac{1}{(\alpha - \varepsilon \beta)^l (n-l)!} s^{n-l} e^{-(\alpha - \varepsilon \beta)s} [s^{(m+1-n)\sigma} - s^{(m-n)\sigma}]
\]

\[
\leq \frac{M}{\alpha - \varepsilon \beta} \sum_{m=0}^\infty \sum_{n=0}^m e^{n \beta \tau} |E|^n \sum_{l=0}^n \frac{s^{n-l}}{(\alpha - \varepsilon \beta)^l (n-l)!} e^{-(\alpha - \varepsilon \beta)(m-n)\sigma}
\]

\[
\times [(m-n)^n e^{-(\alpha - \varepsilon \beta)(m-n)\sigma} - (m+1-n)^n e^{-(\alpha - \varepsilon \beta)(m+1-n)\sigma}]
\]
\[
\begin{align*}
&= \frac{M}{\alpha - \varepsilon \beta} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{l=0}^{n} \frac{e^{n \beta \tau} |E|^n \sigma^n l}{(\alpha - \varepsilon \beta)^l (n - l)!} (m - n)^{n - l} e^{-(\alpha - \varepsilon \beta)(m-n)\sigma} \\
&\quad - \frac{M}{\alpha - \varepsilon \beta} \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \sum_{l=0}^{n} \frac{e^{n \beta \tau} |E|^n \sigma^n l}{(\alpha - \varepsilon \beta)^l (n - l)!} (m - n)^{n - l} e^{-(\alpha - \varepsilon \beta)(m-n)\sigma} \\
&= \frac{M}{\alpha - \varepsilon \beta} \sum_{m=1}^{\infty} \sum_{l=0}^{m-1} \frac{e^{n \beta \tau} |E|^n \sigma^n l}{\alpha^m (m - l)!} (m - m)^{m - l} e^{-(\alpha - \varepsilon \beta)(m-m)\sigma} \\
&= \frac{M}{\alpha - \varepsilon \beta} + \frac{M}{\alpha - \varepsilon \beta} \sum_{m=1}^{\infty} \frac{e^{n \beta \tau} |E|^m \sigma^m l}{\alpha^m} = M \alpha (\alpha - \varepsilon \beta)^{-1} (\alpha - \varepsilon \beta |E|)^{-1},
\end{align*}
\]
and the second inequality in (3.2) follows immediately. ■

4. Uniform stability. In this last section we present a uniform stability theorem for problem (2.1).

4.1. Theorem. Let the assumptions of Lemma 3.1 hold, together with the following additional condition:

(v) there exists \( r_0 > 0 \) and a continuous nondecreasing function \( \lambda : [0, r_0) \to \mathbb{R}^+ \) with \( \lambda(0) = 0 \) such that for any \( r \in (0, r_0) \) we have

\[
\max\{|F(u) - F(v)|, |G(u) - G(v)|\} \leq \lambda(r)|u - v| \quad \text{for } u, v \in B_r(0; X).
\]

Then there exists \( R > 0 \) such that if

\[
\|x\|_{L^\infty(-\tau, 0)} + |x(0)| \leq R, \tag{4.1}
\]

then the corresponding generalized solution \( u_\varepsilon(t) \) of (2.1) exists and satisfies

\[
|u_\varepsilon(t)| \leq C(\beta)(\|x\|_{L^\infty(-\tau, 0)} + |x(0)|) e^{-\beta t} \quad \text{for } t \geq 0 \text{ and } \varepsilon \in (0, \varepsilon_0],
\]

with a constant \( C(\beta) \) independent of the function \( x \), and \( \beta \) in the same range as in Lemma 3.1.

Proof. Let \( \beta \in (0, \beta_0(\varepsilon)) \), where \( \beta_0(\varepsilon) \) is defined as in Lemma 3.1, \( \varepsilon \in (0, \varepsilon_0] \). Define \( w(t) = e^{\beta t} \) for \( t \geq 0 \), and let

\[
H_\varepsilon(u)(t) := U_\varepsilon(t) x(0) + \varepsilon^{-1} \int_0^t U_\varepsilon(t-s) g(s) \, ds \\
+ \varepsilon^{-1} \int_0^t U_\varepsilon(t-s) [F u(s) + \overline{G} u(s)] \, ds
\]
for \( u \in L^\infty(-\tau; X), t \geq 0 \) with \( x \in L^\infty(0, \tau), x(0) \in X \) given, and \( g \) and \( \overline{G} \) as in (2.13), (2.14). By Definition 2.3 and (2.14) it is sufficient to prove that if (4.1) is satisfied with \( R > 0 \) small enough, then for each \( \varepsilon \in (0, \varepsilon_0] \) the mapping \( H_\varepsilon \) has a fixed point in \( L^\infty_w(0, \infty; X) \). As in the proof of Theorem 3.3 of [3] we make use of the Banach contraction principle
in a sufficiently small ball $B_r(0; L^\infty_w(0, \infty; X))$, where $r > 0$. Then by (3.2) for $u \in B_r(0, L^\infty_w(0, \infty; X))$ we have

$$e^{\beta t}|H_\varepsilon(u)(t)| \leq e^{\beta t}|U_\varepsilon(t)| \cdot |x(0)| + \varepsilon^{-1} \int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)| \, ds \|g\|_w$$

$$+ 2\varepsilon^{-1} \int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)| \, ds \lambda(r)\|u\|_w$$

$$\leq M \left( 1 - \frac{|E|e^{\beta \tau}}{\alpha - \varepsilon \beta} \right)^{-1} R$$

$$+ M\alpha(\alpha - \varepsilon \beta)^{-1}(\alpha - e^{\beta \tau}|E|)^{-1}(\lambda(R) + |E|)\|x\|_{L^\infty(-\tau, 0)}e^{\beta \tau}$$

$$+ 2\lambda(r)M\alpha(\alpha - \varepsilon \beta)(\alpha - e^{\beta \tau}|E|)^{-1}r$$

$$\leq \text{const} \cdot (R + \lambda(r)r) \leq r,$$

the last inequality holding when $R$ and $r$ are sufficiently small.

Similarly we have

$$e^{\beta t}|H_\varepsilon(u)(t) - H_\varepsilon(v)(t)| \leq 2\varepsilon^{-1} \int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)| \, ds \lambda(r)\|u - v\|_w$$

$$\leq \text{const} \cdot \lambda(r)\|u - v\|_w,$$

and $r > 0$ can be chosen so that $\text{const} \lambda(r) < 1$. So we have proved that, for sufficiently small numbers $R > 0$ and $r > 0$, $H_\varepsilon$ maps the ball $B_r(0; L^\infty_w(0, \infty; X))$ into itself and is a contraction. The Banach contraction principle implies that, for any $\varepsilon > 0$ and $x$ satisfying (4.1), there exists a unique fixed point $u_\varepsilon$ of $H_\varepsilon$ in $B_r(0; L^\infty_w(0, \infty; X))$. This is clearly the generalized solution of (2.1) satisfying (4.2).}

### 4.2. Example

As an example of application let us consider the following problem:

$$\varepsilon \frac{\partial u_\varepsilon}{\partial t}(x,t) = \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left( a_{jk}(x) \frac{\partial u_\varepsilon}{\partial x_k}(x,t) \right) - bu_\varepsilon(x,t)$$

$$= f(u_\varepsilon(x,t)) + g(u_\varepsilon(x,t - \tau)),$$

(4.4)

$$x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \quad \varepsilon \in (0, \varepsilon_0] \quad (\varepsilon_0 > 0),$$

$$u_\varepsilon(x,t) = 0, \quad x \in \partial \Omega, \quad t > 0,$$

$$u_\varepsilon(x,s) = \varphi(x,s), \quad x \in \Omega, \quad s \in (-\tau, 0) \quad (\tau > 0).$$

Here $\Omega$ is a bounded domain with $C^2$-boundary $\partial \Omega$; $a_{jk} \in C^2(\overline{\Omega})$, $a_{jk} = a_{kj}$ for $j, k = 1, \ldots, n$; $\sum_{j,k=1}^N a_{jk}\xi_j\xi_k \geq c_0|\xi|^2$ for $\xi \in \mathbb{R}^N$ with $c_0 > 0$; $b \in \mathbb{R}$.
\( f, g : \mathbb{R}^N \rightarrow \mathbb{R}, f(0) = g(0) = 0; \varphi : \Omega \times (-\tau, 0] \rightarrow \mathbb{R} \). Moreover, assume that

\[(v') \ f, f', g, g' \text{ are locally Lipschitz continuous and there exists } r_0 > 0 \text{ and a continuous function } \lambda = \lambda(r), \ r \in [0, r_0), \ \lambda(0) = 0 \text{ such that for any } r \in (0, r_0] \text{ we have } \max\{|f(u) - f(v)|, |f'(u) - f'(v)|, |g(u) - g(v)|, |g'(u) - g'(v)|\} \leq \lambda(r)|u - v| \text{ for } u, v \in \mathbb{R} \text{ satisfying } \max\{|u|, |v|\} \leq r. \]

Let \( p > N \) and \( X = \dot{W}^{1,p}(\Omega) \). It is a standard result [5] that the operator \(-A\) defined by \(Av = \sum_{j,k=1}^N \frac{\partial}{\partial x_j} (a_{jk}(x) \frac{\partial u}{\partial x_k})\) for \( v \in W^{2,p}(\Omega) \cap \dot{W}^{1,p}(\Omega) \) generates an exponentially decreasing semigroup on \( L^p(\Omega) \). This semigroup is invariant on \( X \) and is also exponentially decreasing (see e.g. [3], Proposition 6.1), which means that the assumptions (i) and (iii) of (2.3) are satisfied, and it can easily be shown (see [3], proof of Proposition 6.1) that \( \alpha \) in (3.1) can be chosen as

\[
(4.5) \quad \alpha := 4c_0 m^2 \frac{p-1}{p^2},
\]

where \( m = \inf\{\int_\Omega |\nabla v|^2 dx/\int_\Omega v^2 dx : v \in \dot{W}^{1,2}(\Omega), \ v \neq 0\} \). Assuming \( b < \alpha \) we meet the demands of (3.1). Finally, it is a routine matter to verify from (v') the assumption (ii) and (iv) of (2.3) and the assumption (v) of Theorem 4.1, since \( X \hookrightarrow L^\infty(\Omega) \). Then Theorem 4.1 has the following consequence:

**Corollary 4.3.** Under the above assumptions there exists \( R > 0 \) such that if \( \|\varphi(\cdot, \cdot)\|_{L^\infty(-\tau, 0; W^{1,p}(\Omega))} + \|\varphi(\cdot, 0)\|_{W^{1,p}(\Omega)} \leq R \) then the corresponding generalized solution of the problem (4.4) exists with values in \( \dot{W}^{1,p}(\Omega) \) and satisfies

\[
\|u_\varepsilon(\cdot, t)\|_{W^{1,p}(\Omega)} \leq C(\beta)(\|\varphi(\cdot, \cdot)\|_{L^\infty(-\tau, 0; W^{1,p}(\Omega))} + \|\varphi(\cdot, 0)\|_{W^{1,p}(\Omega)} e^{-\beta t} \text{ for } t \geq 0 \text{ and } \varepsilon \in (0, \varepsilon_0],
\]

with a constant \( C(\beta) \) independent of the function \( u_0 \) and \( \beta \) in the same range as in Lemma 3.1, \( \alpha \) being given by (4.5). \( \blacksquare \)

Let us note that the diffusive functional differential equations of the type (4.4) are important in biological models (cf. [4]).

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