ANNALES
POLONICI MATHEMATICI
LXV.3 (1997)

Stabilization of solutions to a differential-delay equation in a Banach space

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Abstract. A parameter dependent nonlinear differential-delay equation in a Banach space is investigated. It is shown that if at the critical value of the parameter the problem satisfies a condition of linearized stability then the problem exhibits a stability which is uniform with respect to the whole range of the parameter values. The general theorem is applied to a diffusion system with applications in biology.

1. Introduction. In this work we investigate a class of parameter dependent differential-delay equations in a Banach space X and apply the method of fixed points in the spaces of functions in X tending to zero as $t \to \infty$ at an appropriate rate that was developed in [3]. In particular, we address the stability of the stationary solution of such an equation. The stability is shown to be uniform with respect to a small parameter on some finite interval.

The stability of solutions to differential-delay equations has been studied in a number of publications. Let us mention at least a few of them. The asymptotic stability for Problem (2.1) below with $\varepsilon = 1$ has been proved in [6, 7] under the assumption of the stability of the linearized problem. The results are applied to a parabolic equation with delay. In [10] stabilization of solutions to the fully nonlinear problem is established by means of monotonicity of the generator of the corresponding nonlinear semigroup. A similar approach is also used in [2], where a series of results on asymptotic behavior of solutions and their mean values is proved. Finally, in [8, 9] appropriate functionals and sufficiently strong *a priori* bounds are used to show the (uniform) asymptotic stability of solutions under certain natural assumptions.

1991 *Mathematics Subject Classification*: Primary 34D15, 34G20; Secondary 47H15.

Key words and phrases: abstract differential-delay equation, dependence on parameter, uniform stability.

Supported by the Australian Research Council Grant S6969557. The second author was partially supported by the Czech Republic Grant Agency Grant 201/93/2177.

In our approach the existence and stabilization of solution is shown by a fixed point argument. We obtain the rate of convergence and describe the global behavior of solution in connection with a singular parameter involved.

Our notation is consistent with that introduced in [3]; in particular, we adopt the usual notation $L^p(M;X)$ for the L^p-spaces of functions from a set $M\subset\mathbb{R}^N$ into a Banach space X, $W^{k,p}(M;X)$ for the Sobolev spaces of kth order, $C^k(M;X)$ for the spaces of functions with continuous derivatives up to order k, $L(X,Y)$ for the space of the continuous linear operators from X into Y with $L(X)=L(X,X)$, $L_{\mathrm{s}}(X)$ being $L(X)$ equipped with the strong operator topology, and so on.

2. Formulation of the problem. Let us consider the following parameter dependent problem:

$$
\begin{aligned}
&\varepsilon u'_\varepsilon(t)+Au_\varepsilon(t)-Eu_\varepsilon(t-\tau)=Fu_\varepsilon(t)+Gu_\varepsilon(t-\tau),\qquad t>0,\\
&u_\varepsilon(s)=x(s),\qquad s\in(-\tau,0],\ \varepsilon\in[0,\varepsilon_0]\ (\tau>0,\ \varepsilon_0>0),
\end{aligned}
\tag{2.1}
$$

where $A:X\supset D(A)\to X$ is linear, $E\in L(X)$, $F,G:X\to X$ are possibly nonlinear operators. The fixed number $\tau>0$ is a given delay, $\varepsilon\in[0,\varepsilon_0]$ a parameter, and $x(\cdot):(-\tau,0]\to X$ a given initial datum. We are interested in the stabilization of $u_\varepsilon(t)$ as $t\to\infty$. This will be achieved by an appropriate splitting of the problem in a stable linear part and a nonlinear perturbation which is locally small. We shall work in the space

$$
\begin{aligned}
L^\infty_w&(0,\infty;X)\\
&=\{u\in L^\infty(0,\infty;X):\|u\|_w:=\operatorname*{ess\,sup}_{t\ge0}w(t)|u(t)|<\infty\},
\end{aligned}
\tag{2.2}
$$

for some function $w\in L^\infty_{\mathrm{loc}}(0,\infty)$ such that $w(t)\ge1$ a.e. in $(0,\infty)$ and $\lim_{t\to\infty}w(t)=\infty$. It is a standard result that the space $L^\infty_w(0,\infty;X)$ is a Banach space under the norm $\|\cdot\|_w$.

We make the following assumptions:

$$
\begin{cases}
\text{(i) } -A \text{ is the generator of a } C_0\text{-semigroup in } L(X);\\
\text{(ii) } F:X\to X, F(0)=0;\\
\text{(iii) the semigroup } T(t) \text{ generated by } -A \text{ satisfies}\\
\qquad |T(t)|\le\varrho(t),\ t\ge0, \text{ with some } \varrho\in L^\infty(0,\infty);\\
\text{(iv) } E\in L(X) \text{ and } G:X\to X,\ G(0)=0.
\end{cases}
\tag{2.3}
$$

To invert the linear part of (2.1) in the space $L^\infty_w(0,\infty;X)$ with an appropriate weight w, define the following auxiliary problems.

• *Fundamental solution*:

$$\text{(2.4)}\qquad \begin{aligned} &\varepsilon U_\varepsilon'(t) + AU_\varepsilon(t) - EU_\varepsilon(t-\tau) = 0,\ t > 0,\\ &U_\varepsilon(s) = 0 \quad \text{for } s \in (-\tau, 0), \quad U_\varepsilon(0) = I, \quad \varepsilon \in (0, \varepsilon_0]. \end{aligned}$$

• *Homogeneous problem*:

$$\text{(2.5)}\qquad \begin{aligned} &\varepsilon v_\varepsilon'(t) + Av_\varepsilon(t) = \begin{cases} g(t), & t \in (0,\tau),\\ E(v_\varepsilon(t-\tau)), & t > \tau, \end{cases}\\ &v_\varepsilon(0) = y, \quad \varepsilon \in (0, \varepsilon_0], \end{aligned}$$

where $g : (-\tau, 0) \to X$ and $y \in X$ are given.

• *Inhomogeneous problem*:

$$\text{(2.6)}\qquad \begin{aligned} &\varepsilon z_\varepsilon'(t) + Az_\varepsilon(t) - Ez_\varepsilon(t-\tau) = h(t), \quad t > 0,\\ &z_\varepsilon(s) = 0, \quad s \in (-\tau, 0],\ \varepsilon \in (0, \varepsilon_0], \end{aligned}$$

where $h : (0,\infty) \to X$. Note that U_ε is an operator valued function.

Since $-A$ generates a C_0-semigroup in X, it is clear that for any $\varepsilon \in (0, \varepsilon_0]$ there exists a unique generalized solution $U_\varepsilon \in C([0,\infty); L_s(X))$ of (2.4), that is, U_ε satisfies $(2.4)_2$ and the integral equation

$$\text{(2.7)}\qquad U_\varepsilon(t) = T(t/\varepsilon) + \varepsilon^{-1}\int_0^t T((t-s)/\varepsilon)EU_\varepsilon(s-\tau)\,ds, \quad t \ge 0.$$

Also, there exists $\varrho_\varepsilon \in L^\infty_{\mathrm{loc}}([0,\infty))$ such that

$$\text{(2.8)}\qquad |U_\varepsilon(t)| \le \varrho_\varepsilon(t) \quad \text{for } t \ge 0 \text{ and } \varepsilon \in (0, \varepsilon_0].$$

It is a standard result (see e.g. [1]) that the solutions v_ε and z_ε may be expressed in terms of U_ε and (y, g), and of U_ε and h, respectively. This is the content of the following two propositions.

2.1. PROPOSITION. *Let $y \in X$ and $g \in L^1((0,\tau); X)$. Then problem* (2.5) *has a family of generalized solutions $v_\varepsilon \in C([0,\infty); X)$, $\varepsilon \in (0, \varepsilon_0]$, in the sense that*

$$\text{(2.9)}\qquad v_\varepsilon(t) = \begin{cases} U_\varepsilon(t)y + \varepsilon^{-1}\displaystyle\int_0^t U_\varepsilon(t-s)g(s)\,ds & \text{for } t \in (0,\tau),\\ U_\varepsilon(t)y + \varepsilon^{-1}\displaystyle\int_0^\tau U_\varepsilon(t-s)g(s)\,ds & \text{for } t > \tau. \end{cases}$$

If, in addition, $y \in D(A)$, $g \in L^1((0,\tau); D(A))$ and $ED(A) \subset D(A)$, then the equation in (2.4) *is satisfied pointwise a.e. in* $(0,\infty)$.

2.2. PROPOSITION. *Let $h \in L^1_{\mathrm{loc}}([0,\infty); X)$. Then problem* (2.6) *has a family of generalized solutions $z_\varepsilon \in C([0,\infty); X)$, $\varepsilon \in (0, \varepsilon_0]$, in the sense* *that*

$$z_\varepsilon(t) = \varepsilon^{-1}\int_0^t U_\varepsilon(t-s)h(s)\,ds, \qquad t \ge 0. \tag{2.10}$$

If, in addition, $h \in L^1_{\rm loc}([0,\infty); D(A))$ *and* $ED(A) \subset D(A)$, *then the equation in* (2.6) *is satisfied pointwise a.e. in* $(0,\infty)$.

Define operators V_ε and Z_ε by

$$\begin{aligned} &V_\varepsilon(y,g)(t) = v_\varepsilon(t), \quad t \in [0,\infty),\ y \in X,\ g \in L^1((0,\tau);X),\\ &\qquad\qquad\qquad\qquad\qquad\qquad v_\varepsilon \text{ satisfies (2.9)},\\ &Z_\varepsilon(h)(t) = z_\varepsilon(t), \quad t \in [0,\infty),\ h \in L^1_{\rm loc}([0,\infty);X),\\ &\qquad\qquad\qquad\qquad\qquad\qquad z_\varepsilon \text{ satisfies (2.10)}. \end{aligned} \tag{2.11}$$

In accordance with the definitions of generalized solutions to Problems (2.5), (2.6) it is consistent to define a generalized solution to (2.1) as follows:

2.3. Definition. A function $u_\varepsilon \in L^\infty_{\rm loc}([0,\infty);X)$ $(\varepsilon \in (0,\varepsilon_0])$ is called a *generalized solution* to problem (2.1) if $u_\varepsilon(s) = x(s)$ for $s \in (-\tau, 0]$ and the following integral equation is satisfied:

$$u_\varepsilon(t) = \begin{cases} U_\varepsilon(t)x(0) \\ \quad + \varepsilon^{-1}\int_0^t U_\varepsilon(t-s)\big(Ex(s-\tau) + Gx(s-\tau) + Fu_\varepsilon(s)\big)\,ds, & t \in (0,\tau],\\ U_\varepsilon(t)x(0) + \varepsilon^{-1}\int_0^\tau U_\varepsilon(t-s)(Ex(s-\tau) + Gx(s-\tau))\,ds \\ \quad + \varepsilon^{-1}\int_0^t U_\varepsilon(t-s)(Fu_\varepsilon(s) + Eu_\varepsilon(s-\tau) + Gu_\varepsilon(s-\tau))\,ds, & t > \tau, \end{cases} \tag{2.12}$$

where U_ε is given by (2.7). Taking into account definitions (2.11), setting $g(t) = Ex(t-\tau) + Gx(t-\tau)$ for $t \in (0,\tau)$ and $g(t) = 0$ for $t > \tau$,

$$(\overline{G}u)(t) = \begin{cases} 0 & \text{for } t \in (0,\tau),\\ Eu(t-\tau) + Gu(t-\tau) & \text{for } t > \tau \end{cases} \tag{2.13}$$

with $u : \mathbb{R}^+ \to X$, we can write (2.12) in the form

$$u_\varepsilon(t) = V_\varepsilon(x(0), g)(t) + Z_\varepsilon(Fu_\varepsilon(\cdot) + \overline{G}u_\varepsilon(\cdot))(t), \qquad t \ge 0. \tag{2.14}$$

Again, it may be shown in a standard way that the following assertion holds true.

2.4. Proposition. *If* $x(0) \in D(A)$, $Ex(\cdot) + Gx(\cdot) \in L^1((-\tau,0); D(A))$, $ED(A) \subset D(A)$ *and* (2.13) *has a solution* $u_\varepsilon \in W^{1,1}(0,T;X) \cap L^1(0.T; D(A))$

for some $T > 0$ *and* $\varepsilon \in (0, \varepsilon_0]$ *then the first equation in* (2.1) *is satisfied pointwise a.e. in* $(0, T)$.

3. Fundamental solution. We start with an investigation of the fundamental solution $U_\varepsilon(t)$ of (2.4).

3.1. Lemma. *Let assumption* (2.3) *be satisfied and let*

$$\varrho(t) = Me^{-\alpha t}, \quad t \ge 0, \text{ with some constants } M > 0,\ \alpha > |E|. \tag{3.1}$$

Assume further that E *commutes with* $(\lambda I + A)^{-1}$ *for some* λ *with* $\operatorname{Re}\lambda > -\alpha$. *Then for any* $\varepsilon > 0$ *there exists a generalized solution* $U_\varepsilon \in L^\infty_{\mathrm{loc}}((-\tau, \infty); L(X))$ *of* (2.4), *and it satisfies*

$$\begin{aligned} &|U_\varepsilon(t)| \le M\left(1 - \frac{|E|e^{\beta\tau}}{\alpha - \varepsilon\beta}\right)^{-1} e^{-\beta t}, \\ &\varepsilon^{-1}\int_0^\infty e^{\beta t}|U_\varepsilon(t)|\,dt \le M\alpha(\alpha - \varepsilon\beta)^{-1}(\alpha - e^{\beta\tau}|E|)^{-1}, \end{aligned} \tag{3.2}$$

$$\textit{for all } t \ge 0,\ \varepsilon \in (0, \varepsilon_0],\ \beta \in [0, \beta_0(\varepsilon)) \supset [0, \beta_0),$$

where $\beta_0(\varepsilon) := \sup\{\beta \in (0, \infty) : e^{\beta\tau}|E| < \alpha - \varepsilon\beta\}$, $\beta_0 := \beta_0(\varepsilon_0) > 0$, $\beta_0(0+) = \tau^{-1}\log(\alpha/|E|)$.

Proof. A formal application of the Fourier transform to the function U_ε (extended by zero for $t \le -\tau$) suggests that we consider a solution of (2.4) in the form

$$U_\varepsilon(t) = \begin{cases} 0, & t < 0, \\ \displaystyle\sum_{n=0}^{[t/\tau]} \frac{(t - n\tau)^n}{\varepsilon^n n!} T\left(\frac{t - n\tau}{\varepsilon}\right) E^n, & t \ge 0,\ \varepsilon > 0, \end{cases} \tag{3.3}$$

where $[s]$ stands for the integral part of s. Let $R(\lambda) = (\lambda I + A)^{-1}$. Then $R(\lambda) \in L(X)$ and, for each μ with $\operatorname{Re}\mu > -\alpha$, $R(\mu) = f_\mu(R(\lambda))$ with a suitable analytic function f_μ. By the functional calculus for bounded linear operators, $ER(\mu) = Ef_\mu(R(\lambda)) = f_\mu(R(\lambda))E = R(\mu)E$. To show that E commutes with $T(s)$ for each $s \ge 0$ we use the Yosida approximation

$$A_n = n^2 R(-\alpha + n) - (\alpha + n)I, \quad n = 1, 2, \ldots;$$

then $T(s)x = \lim_{n\to\infty} \exp(-sA_n)x$ for all $x \in X$ and all $s \ge 0$ (see [5, Section 1.3]) and the commutativity follows. It can then be routinely verified that the function U_ε given by (3.3) is a generalized solution of (2.4). We are going to use formula (3.3) to derive the estimates (3.2). Let $\beta \in [0, \beta_0(\varepsilon))$. Setting

$$v_\varepsilon(t) = e^{\beta t}U_\varepsilon(t), \quad t \ge 0,\ \varepsilon \in (0, \varepsilon_0], \tag{3.4}$$

U_ε is a generalized solution of (2.4) if and only if v_ε satisfies

$$\begin{aligned} \varepsilon v_\varepsilon'(t) + (A - \varepsilon\beta I)v_\varepsilon(t) &= e^{\beta\tau} E v_\varepsilon(t-\tau), \qquad t \ge 0, \\ v_\varepsilon(0) &= I, \\ v_\varepsilon(s) &= 0, \qquad\qquad\qquad s \in (-\tau, 0). \end{aligned} \tag{3.5}$$

A consideration analogous to that for U_ε above leads to the formula

$$v_\varepsilon(t) = \begin{cases} 0, & t < 0, \\ \displaystyle\sum_{n=0}^{[t/\tau]} \frac{(t-n\tau)^n}{\varepsilon^n n!} e^{\beta(t-n\tau)} T\left(\frac{t-n\tau}{\varepsilon}\right) e^{n\beta\tau} E^n, & t \ge 0,\ \varepsilon > 0. \end{cases} \tag{3.6}$$

We estimate the nth term of the sum in (3.6):

$$\begin{aligned} a_n(t) &:= \left| \frac{(t-n\tau)^n}{\varepsilon^n n!} e^{\beta(t-n\tau)} T\left(\frac{t-n\tau}{\varepsilon}\right) e^{n\beta\tau} E^n \right| \\ &\le M \frac{(t-n\tau)^n}{\varepsilon^n n!} \exp\left[\frac{\varepsilon\beta - \alpha}{\varepsilon}(t-n\tau)\right] e^{n\beta\tau} |E|^n. \end{aligned} \tag{3.7}$$

Taking logarithm of $a_n(t)$ and using the estimate

$$\log(n!) = \sum_{k=2}^{n} \log k \ge \int_1^n \log\nu \, d\nu = n\log n - n,$$

we obtain

$$\begin{aligned} \log a_n(t) \le\ & \log M + n \log\left(\frac{|E|}{\alpha - \varepsilon\beta}\right) \\ & + \log[\sup_{s\ge 0}\{s^n e^{-s}\}] + n\beta\tau - (n\log n - n) \\ =\ & \log M + n\left[\beta\tau + \log\left(\frac{|E|}{\alpha - \varepsilon\beta}\right)\right]. \end{aligned}$$

Hence we get

$$a_n(t) \le M e^{-\kappa n}, \tag{3.8}$$

where $\kappa := -\beta\tau - \log(|E|/(\alpha - \varepsilon\beta))$, which is positive by assumption. Consequently, by (3.6)–(3.8) we have

$$\begin{aligned} |v_\varepsilon(t)| &\le \sum_{n=0}^{[t/\tau]} a_n(t) \le M \sum_{n=0}^{[t/\tau]} e^{-\kappa n} \\ &\le M \sum_{n=0}^{\infty} e^{-\kappa n} = \frac{M}{1 - e^{-\kappa}} = M\left(1 - \frac{|E| e^{\beta\tau}}{\alpha - \varepsilon\beta}\right)^{-1}, \end{aligned}$$

and (3.4) yields the first inequality in (3.2).

Now we prove the second inequality in (3.2). Setting

$$s = t/\varepsilon, \quad \sigma = \tau/\varepsilon, \quad v(s) = e^{\beta t} U_\varepsilon(t), \tag{3.9}$$

we obtain

$$\begin{aligned} v'(s) + (A - \varepsilon\beta I)v(s) &= e^{\beta\tau} E v(s-\sigma), \quad && s \geq 0, \\ v(0) &= I, && \\ v(s) &= 0, && s \in (-\sigma, 0). \end{aligned} \tag{3.10}$$

A similar reasoning to the above leads to the formula

$$v(s) = \begin{cases} 0, & s < 0, \\ \displaystyle\sum_{n=0}^{[s/\sigma]} \frac{(s-n\sigma)^n}{n!} e^{\varepsilon\beta(s-n\sigma)} e^{n\beta\tau} T(s-n\sigma) E^n, & s \geq 0. \end{cases} \tag{3.11}$$

Then we have

$$\begin{aligned} J(\varepsilon) &:= \varepsilon^{-1} \int_0^\infty e^{\beta t} |U_\varepsilon(t)|\, dt = \int_0^\infty |v(s)|\, ds \\ &\leq \sum_{m=0}^\infty \int_{m\sigma}^{(m+1)\sigma} \Big| \sum_{n=0}^m \frac{(s-n\sigma)^n}{n!} e^{\varepsilon\beta(s-n\sigma)} e^{n\beta\tau} T(s-n\sigma) E^n \Big| ds \\ &\leq M \sum_{m=0}^\infty \sum_{n=0}^m \frac{e^{n\beta\tau}|E|^n}{n!} \int_{m\sigma}^{(m+1)\sigma} (s-n\sigma)^n e^{-(\alpha-\varepsilon\beta)(s-n\sigma)}\, ds \\ &= M \sum_{m=0}^\infty \sum_{n=0}^m \frac{e^{n\beta\tau}|E|^n}{n!} \int_{(m-n)\sigma}^{(m+1-n)\sigma} s^n e^{-(\alpha-\varepsilon\beta)s}\, ds. \end{aligned}$$

Since

$$\int s^n e^{-\delta s}\, ds = -\frac{1}{\delta} e^{-\delta s} \sum_{l=0}^n \frac{n! s^{n-l}}{(n-l)!\delta^l},$$

we find that

$$\begin{aligned} J(\varepsilon) \leq &- \frac{M}{\alpha - \varepsilon\beta} \sum_{m=0}^\infty \sum_{n=0}^m e^{n\beta\tau}|E|^n \\ &\times \sum_{l=0}^n \frac{1}{(\alpha-\varepsilon\beta)^l (n-l)!} [s^{n-l} e^{-(\alpha-\varepsilon\beta)s}]_{s=(m-n)\sigma}^{(m+1-n)\sigma} \\ \leq &\frac{M}{\alpha-\varepsilon\beta} \sum_{m=0}^\infty \sum_{n=0}^m e^{n\beta\tau}|E|^n \sum_{l=0}^n \frac{\sigma^{n-l}}{(\alpha-\varepsilon\beta)^l (n-l)!} \\ &\times [(m-n)^{n-l} e^{-(\alpha-\varepsilon\beta)(m-n)\sigma} - (m+1-n)^{n-l} e^{-(\alpha-\varepsilon\beta)(m+1-n)\sigma}] \end{aligned}$$

$$= \frac{M}{\alpha-\varepsilon\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{l=0}^{n}\frac{e^{n\beta\tau}|E|^n\sigma^{n-l}}{(\alpha-\varepsilon\beta)^l(n-l)!}(m-n)^{n-l}e^{-(\alpha-\varepsilon\beta)(m-n)\sigma}$$

$$- \frac{M}{\alpha-\varepsilon\beta}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\sum_{l=0}^{n}\frac{e^{n\beta\tau}|E|^n\sigma^{n-l}}{(\alpha-\varepsilon\beta)^l(n-l)!}(m-n)^{n-l}e^{-(\alpha-\varepsilon\beta)(m-n)\sigma}$$

$$= \frac{M}{\alpha-\varepsilon\beta}+\frac{M}{\alpha-\varepsilon\beta}\sum_{m=1}^{\infty}\sum_{l=0}^{m}\frac{e^{m\beta\tau}|E|^m\sigma^{m-l}}{\alpha^m(m-l)!}(m-m)^{m-l}e^{-(\alpha-\varepsilon\beta)(m-m)\sigma}$$

$$= \frac{M}{\alpha-\varepsilon\beta}+\frac{M}{\alpha-\varepsilon\beta}\sum_{m=1}^{\infty}\frac{e^{m\beta\tau}|E|^m}{\alpha^m} = M\alpha(\alpha-\varepsilon\beta)^{-1}(\alpha-e^{\beta\tau}|E|)^{-1},$$

and the second inequality in (3.2) follows immediately. ∎

4. Uniform stability. In this last section we present a uniform stability theorem for problem (2.1).

4.1. Theorem. *Let the assumptions of Lemma* 3.1 *hold*, *together with the following additional condition*:

(v) *there exists* $r_0 > 0$ *and a continuous nondecreasing function* $\lambda : [0, r_0) \to \mathbb{R}^+$ *with* $\lambda(0) = 0$ *such that for any* $r \in (0, r_0)$ *we have*

$$\max\{|F(u)-F(v)|, |G(u)-G(v)|\} \le \lambda(r)|u-v| \quad \textit{for } u, v \in B_r(0;X).$$

Then there exists $R > 0$ *such that if*

$$\|x\|_{L^\infty(-\tau,0)} + |x(0)| \le R, \tag{4.1}$$

then the corresponding generalized solution $u_\varepsilon(t)$ *of* (2.1) *exists and satisfies*

$$|u_\varepsilon(t)| \le C(\beta)(\|x\|_{L^\infty(-\tau,0)}+|x(0)|)e^{-\beta t} \quad \textit{for } t \ge 0 \textit{ and } \varepsilon \in (0, \varepsilon_0], \tag{4.2}$$

with a constant $C(\beta)$ *independent of the function* x, *and* β *in the same range as in Lemma* 3.1.

Proof. Let $\beta \in (0, \beta_0(\varepsilon))$, where $\beta_0(\varepsilon)$ is defined as in Lemma 3.1, $\varepsilon \in (0, \varepsilon_0]$. Define $w(t) = e^{\beta t}$ for $t \ge 0$, and let

$$H_\varepsilon(u)(t) := U_\varepsilon(t)x(0) + \varepsilon^{-1}\int_0^t U_\varepsilon(t-s)g(s)\,ds + \varepsilon^{-1}\int_0^t U_\varepsilon(t-s)[Fu(s)+\overline{G}u(s)]\,ds \tag{4.3}$$

for $u \in L^\infty_w(0,\infty;X)$, $t \ge 0$ with $x \in L^\infty(0,\tau)$, $x(0) \in X$ given, and g and $\overline{G}$ as in (2.13), (2.14). By Definition 2.3 and (2.14) it is sufficient to prove that if (4.1) is satisfied with $R > 0$ small enough, then for each $\varepsilon \in (0, \varepsilon_0]$ the mapping H_ε has a fixed point in $L^\infty_w(0,\infty;X)$. As in the proof of Theorem 3.3 of [3] we make use of the Banach contraction principle in a sufficiently small ball $B_r(0;L^\infty_w(0,\infty;X))$, where $r>0$. Then by (3.2) for $u\in B_r(0,L^\infty_w(0,\infty;X))$ we have

$$\begin{aligned}
e^{\beta t}|H_\varepsilon(u)(t)| &\le e^{\beta t}|U_\varepsilon(t)|\cdot|x(0)| + \varepsilon^{-1}\int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)|\,ds\,\|g\|_w \\
&\quad + 2\varepsilon^{-1}\int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)|\,ds\,\lambda(r)\|u\|_w \\
&\le M\left(1-\frac{|E|e^{\beta\tau}}{\alpha-\varepsilon\beta}\right)^{-1}R \\
&\quad + M\alpha(\alpha-\varepsilon\beta)^{-1}(\alpha-e^{\beta\tau}|E|)^{-1}(\lambda(R)+|E|)\|x\|_{L^\infty(-\tau,0)}e^{\beta\tau} \\
&\quad + 2\lambda(r)M\alpha(\alpha-\varepsilon\beta)(\alpha-e^{\beta\tau}\,|E|)^{-1}r \\
&\le \mathrm{const}\cdot(R+\lambda(r)r)\le r,
\end{aligned}$$

the last inequality holding when R and r are sufficiently small.

Similarly we have

$$\begin{aligned}
e^{\beta t}|H_\varepsilon(u)(t)-H_\varepsilon(v)(t)| &\le 2\varepsilon^{-1}\int_0^t e^{\beta(t-s)}|U_\varepsilon(t-s)|\,ds\,\lambda(r)\|u-v\|_w \\
&\le \mathrm{const}\cdot\lambda(r)\|u-v\|_w,
\end{aligned}$$

and $r>0$ can be chosen so that const $\lambda(r)<1$. So we have proved that, for sufficiently small numbers $R>0$ and $r>0$, H_ε maps the ball $B_r(0;L^\infty_w(0,\infty;X))$ into itself and is a contraction. The Banach contraction principle implies that, for any $\varepsilon>0$ and x satisfying (4.1), there exists a unique fixed point u_ε of H_ε in $B_r(0;L^\infty_w(0,\infty;X))$. This is clearly the generalized solution of (2.1) satisfying (4.2). ■

4.2. Example. As an example of application let us consider the following problem:

$$(4.4)\quad \begin{aligned}
&\varepsilon\frac{\partial u_\varepsilon}{\partial t}(x,t)-\sum_{j,k=1}^N\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u_\varepsilon}{\partial x_k}(x,t)\right)-bu_\varepsilon(x,t) \\
&\qquad\qquad = f(u_\varepsilon(x,t))+g(u_\varepsilon(x,t-\tau)), \\
&\qquad\qquad x\in\Omega\subset\mathbb{R}^N,\ t>0,\ \varepsilon\in(0,\varepsilon_0]\ (\varepsilon_0>0), \\
&u_\varepsilon(x,t)=0,\quad x\in\partial\Omega,\ t>0, \\
&u_\varepsilon(x,s)=\varphi(x,s),\quad x\in\Omega,\ s\in(-\tau,0]\ (\tau>0).
\end{aligned}$$

Here Ω is a bounded domain with C^2-boundary $\partial\Omega$; $a_{jk}\in C^2(\overline{\Omega})$, $a_{jk}=a_{kj}$ for $j,k=1,\dots,n$; $\sum_{j,k=1}^N a_{jk}\xi_j\xi_k\ge c_0|\xi|^2$ for $\xi\in\mathbb{R}^N$ with $c_0>0$; $b\in\mathbb{R}$;

$f, g : \mathbb{R}^N \to \mathbb{R}$, $f(0) = g(0) = 0$; $\varphi : \Omega \times (-\tau, 0] \to \mathbb{R}$. Moreover, assume that

(v′) f, f', g, g' are locally Lipschitz continuous and there exists $r_0 > 0$ and a continuous function $\lambda = \lambda(r)$, $r \in [0, r_0)$, $\lambda(0) = 0$ such that for any $r \in (0, r_0]$ we have $\max\{|f(u) - f(v)|, |f'(u) - f'(v)|, |g(u) - g(v)|, |g'(u) - g'(v)|\} \le \lambda(r)|u - v|$ for $u, v \in \mathbb{R}$ satisfying $\max\{|u|, |v|\} \le r$.

Let $p > N$ and $X = \mathring{W}^{1,p}(\Omega)$. It is a standard result [5] that the operator $-A$ defined by $Av = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\big(a_{jk}(x)\frac{\partial v}{\partial x_k}\big)$ for $v \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$ generates an exponentially decreasing semigroup on $L^p(\Omega)$. This semigroup is invariant on X and is also exponentially decreasing (see e.g. [3], Proposition 6.1), which means that the assumptions (i) and (iii) of (2.3) are satisfied, and it can easily be shown (see [3], proof of Proposition 6.1) that α in (3.1) can be chosen as

$$\alpha := 4c_0 m \frac{p-1}{p^2}, \tag{4.5}$$

where $m = \inf\{\int_\Omega |\nabla v|^2 dx / \int_\Omega v^2\, dx : v \in \mathring{W}^{1,2}(\Omega),\ v \ne 0\}$. Assuming $b < \alpha$ we meet the demands of (3.1). Finally, it is a routine matter to verify from (v′) the assumption (ii) and (iv) of (2.3) and the assumption (v) of Theorem 4.1, since $X \hookrightarrow L^\infty(\Omega)$. Then Theorem 4.1 has the following consequence:

COROLLARY 4.3. *Under the above assumptions there exists $R > 0$ such that if $\|\varphi(\cdot,\cdot)\|_{L^\infty(-\tau,0;W^{1,p}(\Omega))} + \|\varphi(\cdot,0)\|_{W^{1,p}(\Omega)} \le R$ then the corresponding generalized solution of the problem* (4.4) *exists with values in $\mathring{W}^{1,p}(\Omega)$ and satisfies*

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} \le C(\beta)(\|\varphi(\cdot,\cdot)\|_{L^\infty(-\tau,0;W^{1,p}(\Omega))} + \|\varphi(\cdot,0)\|_{W^{1,p}(\Omega)})e^{-\beta t} \quad \text{for } t \ge 0 \text{ and } \varepsilon \in (0, \varepsilon_0],$$

with a constant $C(\beta)$ independent of the function u_0 and β in the same range as in Lemma 3.1, *α being given by* (4.5). ∎

Let us note that the diffusive functional differential equations of the type (4.4) are important in biological models (cf. [4]).

Acknowledgements. The authors are indebted to the referee for his careful reading of the manuscript and valuable remarks which helped to improve the quality of the paper.

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Reçu par la Rédaction le 19.6.1996
Révisé le 17.10.1996