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On highly nonintegrable functions and homogeneous polynomials

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Abstract. We construct a sequence of homogeneous polynomials on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ which are big at each point of the unit sphere $\mathbb{S}$. As an application we construct a holomorphic function on $\mathbb{B}_d$ which is not integrable with any power on the intersection of $\mathbb{B}_d$ with any complex subspace.

1. Introduction. Let $\mathbb{S}$ denote the unit sphere in the complex space $\mathbb{C}^d$. In the paper [5] a sequence $(p_n(z))_{n=0}^{\infty}$ of homogenous polynomials in $\mathbb{C}^d$ was constructed such that $|p_n(z)| \leq 1$ for all n and all $z \in \mathbb{S}$ and $\int_{\mathbb{S}} |p_n(z)|^2 \, d\sigma(z) \geq c > 0$ for all n. Such polynomials can be used to produce holomorphic functions in $\mathbb{B}_d$ (the unit ball of $\mathbb{C}^d$) with "bad" behaviour on almost all slices (cf. [5], Remark 1.10). The "almost all" restriction is caused by the fact that each $p_n(z)$ has zeros on $\mathbb{S}$ (unless $d = 1$, which is a trivial case), and to conclude something on all slices one has to control the location of the sets where $p_n(z)$ is small. On the other hand, from the function theory point of view it is interesting to have results for all slices (see e.g. [2]). In this note we construct a sequence of homogeneous polynomials which allows us to control behaviour on all slices. Our arguments in this note are modifications of some arguments from [5], [7] and [1]. As an application we construct a holomorphic function in the unit ball $\mathbb{B}_d$ which is not integrable with any power on any slice.

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1.1. *Geometric notions*. In the complex d-dimensional space $\mathbb{C}^d$ we will always consider the natural scalar product $\langle \cdot, \cdot \rangle$. On the unit sphere $\mathbb{S}$ we

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will consider a unitarily invariant pseudo-metric $\varrho(z_1, z_2)$ defined as

$$\varrho(z_1, z_2) := \sqrt{1 - |\langle z_1, z_2\rangle|}. \tag{1}$$

It is clear that $\varrho(z_1, z_2) = 0$ if and only if $z_1 = \lambda z_2$ for some $\lambda \in \mathbb{C}$ (and clearly $|\lambda| = 1$). As usual, we denote by $B(z; r)$ the open ball with center z and radius r, i.e.

$$B(z_0; r) := \{z \in \mathbb{S} : \varrho(z_0, z) < r\}.$$

There is a natural, unitarily invariant (Lebesgue) measure on $\mathbb{S}$. We normalize it so that the measure of the whole sphere $\mathbb{S}$ equals 1 and we denote this measure by σ. Using (1.4.5) of [4] we easily compute that

$$\sigma\big(B(z; r)\big) = \big(2r^2 - r^4\big)^{d-1}. \tag{2}$$

This clearly gives

$$r^{2d-2} \le \sigma(B(z; r)) \le 2^{d-1} r^{2d-2}. \tag{3}$$

Clearly for small r's the constant on the right hand side can be made as close to 1 as we wish. A subset $A \subset \mathbb{S}$ is called *α-separated* if $\varrho(z_1, z_2) > \alpha$ for all distinct elements z_1 and z_2 of A. It is clear that for $\alpha > 0$ each α-separated subset of $\mathbb{S}$ is finite. We will consider maximal α-separated sets. We always mean maximal in the sense of inclusion of sets.

2. Some homogeneous polynomials.

All homogeneous polynomials of degree n constructed in this paper will have the form

$$p(z) = \sum_{j=1}^{s} \langle z, \zeta_j\rangle^n \tag{4}$$

for some finite subset $\{\zeta_1, \ldots, \zeta_s\}$ of $\mathbb{S}$. In order to be able to control values of the polynomial p we will usually assume that the set $\{\zeta_1, \ldots, \zeta_s\}$ is α-separated for some α. The natural and useful degree of separation for polynomials of degree n is $1/\sqrt{n}$. We start with two lemmas on separated sets.

LEMMA 1. *Suppose that $\{\zeta_1, \ldots, \zeta_s\}$ is a $C/\sqrt{N}$-separated subset of $\mathbb{S}$. For $z \in \mathbb{S}$ let*

$$A_k(z) := \left\{i : \frac{kC}{2\sqrt{N}} \le \varrho(z, \zeta_i) \le \frac{(k+1)C}{2\sqrt{N}}\right\}.$$

Then for $k = 1, 2, \ldots$ the set $A_k(z)$ has at most $2^{d-1}(k+2)^{2d-2}$ elements. The set $A_0(z)$ has at most one element.

P r o o f. The assertion about A_0 is clear. Since the balls $B(\zeta_j; C/(2\sqrt{N}))$ are disjoint and

$$\bigcup_{i\in A_k(z)} B\Big(\zeta_i;\frac{C}{2\sqrt{N}}\Big)\subset B\Big(z;\frac{(k+2)C}{2\sqrt{N}}\Big)$$

we get

$$\begin{aligned}\#A_k(z)&\le \#\Big\{i:\varrho(z,\zeta_i)<\frac{(k+1)C}{2\sqrt{N}}\Big\}\\&\le\frac{\sigma\big(B\big(z;\frac{(k+2)C}{2\sqrt{N}}\big)\big)}{\sigma\big(B\big(z;\frac{C}{2\sqrt{N}}\big)\big)}\\&\le\frac{2^{d-1}\big(\frac{(k+2)C}{2\sqrt{N}}\big)^{2d-2}}{\big(\frac{C}{2\sqrt{N}}\big)^{2d-2}}=2^{d-1}(k+2)^{2d-2}. \ \blacksquare\end{aligned}$$

LEMMA 2. *If $A\subset\mathbb{S}$ is $\alpha/\sqrt{N}$-separated then for each $\beta>\alpha$ there exists an integer $K=K(\alpha,\beta)$ such that A can be partitioned into K disjoint $\beta/\sqrt{N}$-separated sets.*

P r o o f. Let us select from A a maximal $\beta/\sqrt{N}$-separated subset A_1. Next from $A\setminus A_1$ we select a maximal $\beta/\sqrt{N}$-separated subset A_2. We continue in this way till we exhaust A. Let A_s be the last non-empty set in this procedure. Take $\zeta\in A_s$. Since A_{s-1} is a maximal $\beta/\sqrt{N}$-separated subset of $A\setminus\bigcup_{j=1}^{s-2}A_j$ we see that $\zeta\notin A_{s-1}$, so $B(\zeta;\beta/\sqrt{N})\cap A_{s-1}\neq\emptyset$. Analogously $B(\zeta;\beta/\sqrt{N})\cap A_{s-2}\neq\emptyset$ etc. So we see that $B(\zeta;\beta/\sqrt{N})$ contains at least s distinct elements of A. Looking at the measures of balls as in Lemma 1 we see that $B\big(\zeta;\frac{\beta+\alpha/2}{\sqrt{N}}\big)$ contains s disjoint balls of radius $\alpha/(2\sqrt{N})$. From (3) we obtain

$$s\Big(\frac{\alpha}{2\sqrt{N}}\Big)^{2d-2}\le 2^{d-1}\Big(\frac{\beta+\alpha/2}{\sqrt{N}}\Big)^{2d-2}$$

so $s\le 2^{3d-3}(\beta/\alpha+1/2)^{2d-2}$. This gives the required decomposition. $\blacksquare$

Now we are ready to state some estimates for polynomials (4).

PROPOSITION 1. *There exists a constant C (rather large) such that for all integers N large enough, for each $C/\sqrt{N}$-separated subset $\{\zeta_1,\dots,\zeta_s\}$ of $\mathbb{S}$ and each integer k with $N\le k\le 2N$ the polynomial*

$$p(z):=\sum_{j=1}^{s}\langle z,\zeta_j\rangle^k$$

satisfies

(i) $|p(z)|\le 2$ *for all* $z\in\mathbb{S}$,

(ii) $|p(z)| \geq 0.5$ *for each* $z \in \mathbb{S}$ *such that* $\varrho(z,\zeta_j) \leq 1/(4\sqrt{N})$ *for some* $j = 1,\ldots,s$.

Proof. Note that if $\varrho(z,\zeta_j) \geq \alpha/\sqrt{N}$ and $N \leq k \leq 2N$ then

$$|\langle z,\zeta_j\rangle^k| \leq (1-\alpha^2/N)^k \leq e^{-\alpha^2 k/N} \leq e^{-\alpha^2}. \tag{5}$$

Consider the sets $A_k(z)$ defined in Lemma 1. From Lemma 1 we obtain

$$\begin{aligned}|p(z)| &\leq \sum_{j=1}^{s} |\langle z,\zeta_j\rangle|^k \leq \sum_{k=0}^{\infty} \sum_{i\in A_k(z)} |\langle z,\zeta_i\rangle|^k \\ &\leq 1 + \sum_{k=1}^{\infty} e^{-(kC/2)^2} 2^{d-1}(k+2)^{2d-2}.\end{aligned}$$

It is clear that we can fix a $C > 0.5$ such that

$$\sum_{k=1}^{\infty} e^{-(kC/2)^2} 2^{d-1}(k+2)^{2d-2} \leq 0.1.$$

Such a choice of C clearly ensures (i).

For a fixed j and $z \in \mathbb{S}$ such that $\varrho(z,\zeta_j) < 1/(4\sqrt{N})$ we have, for $i \neq j$,

$$\varrho(z,\zeta_i) \geq \frac{C}{\sqrt{N}} - \frac{1}{4\sqrt{N}} \geq \frac{1}{4\sqrt{N}}. \tag{6}$$

This shows that

$$|\langle z,\zeta_j\rangle^k| \geq \left(1 - \frac{1}{16N}\right)^k \geq \left(1 - \frac{1}{16N}\right)^{2N}$$

so for N large enough we have

$$|\langle z,\zeta_j\rangle^k| \geq (1/3)^{1/8} \geq 0.87. \tag{7}$$

Analogously to the argument for (i) we see from (6) that

$$\sum_{i\neq j} |\langle z,\zeta_i\rangle^k| \leq \sum_{k=1}^{\infty} \sum_{i\in A_k(z)} |\langle z,\zeta_i\rangle^k| \leq 0.1. \tag{8}$$

Since

$$|p(z)| \geq |\langle z,\zeta_j\rangle^k| - \sum_{i\neq j} |\langle z,\zeta_i\rangle^k|,$$

from (7) and (8) we obtain (ii). ■

Now we are ready for the main technical result of this note.

THEOREM 1. *There exists an integer* $k = k(d)$ *and a sequence* $p_n(z)$ *of homogeneous polynomials of degree* n (*for* n *large enough*) *such that*

(i) $|p_n(z)| \leq 2$ *for all* $z \in \mathbb{S}$,

(ii) *for each* s (*large enough*), $\sum_{n=ks}^{k(s+1)-1} |p_n(z)| \geq 0.5$ *for all* $z \in \mathbb{S}$.

P r o o f. Let k be the integer given by Lemma 2 for $\alpha = 0.25$ and $\beta = C$ where C is the constant given by Proposition 1. For $N = sk$ (and such that the estimate of Proposition 1 holds) fix a maximal $1/(4\sqrt{N})$-separated subset $A \subset \mathbb{S}$ and using Lemma 2 divide it into k disjoint $C/\sqrt{N}$-separated subsets $A_0, A_1, \ldots, A_{k-1}$. For $n = sk + j$ we define

$$p_n(z) := \sum_{\zeta \in A_j} \langle z, \zeta \rangle^n.$$

From Proposition 1 we infer that $|p_n(z)| \le 2$ (so (i) holds) and $|p_n(z)| \ge 0.5$ for

$$z \in \bigcup_{\zeta \in A_j} B\Big(\zeta; \frac{1}{4\sqrt{N}}\Big).$$

Since $A = \bigcup_{l=0}^{k-1} A_l$ is a maximal $1/(4\sqrt{N})$-separated subset of $\mathbb{S}$ we infer that

$$\bigcup_{j=0}^{k-1} \bigcup_{\zeta \in A_j} B\Big(\zeta; \frac{1}{4\sqrt{N}}\Big) = \bigcup_{\zeta \in A} B\Big(\zeta; \frac{1}{4\sqrt{N}}\Big) = \mathbb{S}.$$

This gives (ii). ■

R e m a r k 1. The sets A_j used in the above proof need not be maximal $C/\sqrt{N}$-separated subsets of $\mathbb{S}$. If we enlarge them to get such subsets, say $\widetilde{A}_j$, then there are signs ε_ζ^n such that the polynomials

$$\widetilde{p}_n(z) = \sum_{\zeta \in \tilde{A}_j} \varepsilon_\zeta^n \langle z, \zeta \rangle^n$$

will satisfy

$$\int_{\mathbb{S}} |\widetilde{p}_n(z)|^2 \, d\sigma(z) > c > 0$$

for all n and some C. This follows from the arguments following Lemma 2.7 of [5]. Clearly those polynomials will also satisfy (i) and (ii) of Theorem 1.

R e m a r k 2. The possibility of generalizing arguments from [5] to yield results like our Theorem 1 was known to A. B. Aleksandrov. In his paper [1] he states (Theorem 4) that there is a K (depending only on the dimension d) such that for each n there are homogeneous polynomials $p_n^s(z)$ of degree n, where $s = 1, \ldots, K$, such that for some constants $C \ge c > 0$ we have $C \ge \sum_{s=1}^K |p_n^s(z)| \ge c > 0$ for all $s \in \mathbb{S}$. It is easy to modify our proof of Theorem 1 to get this fact.

3. An application. As an easy application of Theorem 1 let us show the following fact:

The function

$$\sum_n n^{\ln n} p_n(z) =: f(z)$$

is a holomorphic function in $\mathbb{B}_d$ *such that for each hyperplane* $\Pi \subset \mathbb{C}^d$ *and any* $p > 0$,

(9)
$$\int_{\Pi\cap\mathbb{B}_d} |f(z)|^p \, d\nu(z) = \infty$$

where $d\nu$ *is the volume measure on* $\Pi \cap \mathbb{B}_d$.

Since $|p_n(z)| \le 2|z|^n$ and the series $\sum n^{\ln n}|z|^n$ converges for $|z| < 1$ we see that $f(z)$ is a holomorphic function in $\mathbb{B}_d$. Hence we easily see that (9) is equivalent to

(10)
$$\int_{z\in\Pi,\, 0.5<|z|<1} |f(z)|^p \, d\nu(z) = \infty.$$

Writing (10) in polar coordinates (see e.g. 1.4.3 in [4]) we see that in order to show (9) it suffices to consider complex lines Π only. It is also clear that only small p's matter. Thus we must show that for each $w \in \mathbb{S}$ and each $1 > p > 0$ the function $g_w(\lambda) := f(\lambda w)$ defined for $\lambda \in \mathbb{C}$ and $|\lambda| < 1$ satisfies

(11)
$$\int_{|\lambda|<1} |g_w(\lambda)|^p \, d\nu(\lambda) = \infty.$$

But it is known (cf. [3] or [6]) that if a function $g(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ on the unit disc satisfies

$$\int_{|\lambda|<1} |g(z)|^p \, d\nu(\lambda) < \infty$$

then

(12)
$$|a_n| = o(n^{2/p-1}).$$

But $g_w(\lambda)$ has the power series expansion

$$g_w(\lambda) = \sum_n n^{\ln n} p_n(w)\lambda^n$$

so we infer from Theorem 1 that (12) does not hold. This shows our claim.

This example improves a bit upon Theorem 1 of [2].

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