

## Classifications and existence of nonoscillatory solutions of second order nonlinear neutral differential equations

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**Abstract.** A class of neutral nonlinear differential equations is studied. Various classifications of their eventually positive solutions are given. Necessary and/or sufficient conditions are then derived for the existence of these eventually positive solutions. The derivations are based on two fixed point theorems as well as the method of successive approximations.

**1. Introduction.** Neutral nonlinear differential equations of the form

$$(1) \quad (r(t)(x(t) - p(t)x(t - \tau)))' + f(t, x(t - \delta)) = 0$$

have been studied by a number of authors [2]–[5]. In particular, Lu in [4] studied the equation

$$(x(t) - p(t)x(t - \tau))'' + f(t, x(t - \delta)) = 0$$

and obtained classifications of its nonoscillatory solutions as well as necessary and/or sufficient conditions for their existence.

Here, we will assume that  $\tau$  is a positive number,  $\delta$  is a nonnegative number,  $p(t) \in C([t_0, \infty), \mathbb{R}_+)$  and  $0 \leq p(t) \leq p < 1$  for  $t \geq t_0$  and  $r(t) \in C([t_0, \infty), (0, \infty))$ . We further assume that for any  $t \geq t_0$ ,  $f$  is continuous as a function of  $x \in \mathbb{R}$  and  $xf(t, x) > 0$  for  $x \neq 0$ ,  $t \geq t_0$ .

In this paper we consider the classification of all nonoscillatory solutions of (1) in the cases

$$\int_{t_0}^{\infty} \frac{1}{r(u)} du < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{r(u)} du = \infty$$

and give necessary and/or sufficient conditions for their existence.

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1991 *Mathematics Subject Classification*: Primary 34C15.

*Key words and phrases*: oscillation, neutral nonlinear differential equation, asymptotic behavior.

Supported by the Science Foundation of Gansu Province of China for the Young.

Hereafter, the term *solution* of (1) is always used to denote a real function  $x(t)$  satisfying (1) for which  $\sup_{t \geq t_1} |x(t)| > 0$  for any  $t_1 \geq t_0$ . We assume that (1) always has such solutions. A solution of (1) is called *nonoscillatory* if it is eventually positive or eventually negative. Otherwise it is called *oscillatory*. Equation (1) is called *oscillatory* if all its solutions are oscillatory. We use the functions  $R(s, t)$  and  $R(s)$  defined by

$$R(s, t) = \int_s^t \frac{1}{r(u)} du \quad \text{and} \quad R(s) = \int_s^{\infty} \frac{1}{r(u)} du,$$

where  $s \geq t_0$ . We also let  $R_0 = \lim_{t \rightarrow \infty} R(t_0, t)$ .

We shall say that *condition* (H) is met if the following hold:

(H<sub>1</sub>)  $xf(t, x) > 0$  for  $x \neq 0$ ,  $t \geq t_0$ , and  $f(t, x_1) \geq f(t, x_2)$  for  $x_1 \geq x_2 > 0$  or  $x_1 \leq x_2 < 0$ ,  $t \geq t_0$ ;

(H<sub>2</sub>)  $0 \leq p(t) < p < 1$  for  $t \geq t_0$ .

Let  $x(t)$  be a solution of (1). We will define a function  $y(t)$  associated with  $x(t)$  by

$$(2) \quad y(t) = x(t) - p(t)x(t - \tau), \quad t \geq t_0.$$

Note that if  $x(t)$  is eventually positive, then the fact that

$$(r(t)y'(t))' = -f(n, x(t - \tau)) < 0$$

for all large  $t$  implies that  $y'(t)$  is of constant positive or constant negative sign eventually. This fact, in turn, implies that  $y(t)$  is eventually positive or eventually negative.

In the following sections, we first give some lemmas which are interesting in their own right and then discuss the existence and asymptotic behavior of nonoscillatory solutions of the differential equation (1) for  $R_0 < \infty$  and  $R_0 = \infty$  respectively.

**2. Preparatory lemmas.** In this section, we will give some lemmas which are important in proving our results in Sections 3 and 4.

LEMMA 1. *Suppose that (H) holds. Let  $x(t)$  be an eventually positive (negative) solution of (1) and let  $y(t)$  be defined by (2). Then  $y'(t)$  is eventually of constant sign.*

LEMMA 2. *Suppose that (H) holds. Let  $x(t)$  be an eventually positive (negative) solution of (1) and let  $y(t)$  be defined by (2).*

(i) *If  $R_0 = \infty$  and  $\lim_{t \rightarrow \infty} x(t) = 0$ , then  $y(t)$  is eventually increasing (decreasing) and negative (positive) and  $\lim_{t \rightarrow \infty} y(t) = 0$ .*

(ii) If  $\limsup_{t \rightarrow \infty} x(t) > 0$  (resp.  $\limsup_{t \rightarrow \infty} (-x(t)) > 0$ ), then  $y(t)$  is eventually positive (respectively negative).

Proof. (i) Let  $x(t)$  be an eventually positive solution of (1). If  $\lim_{t \rightarrow \infty} x(t) = 0$ , then  $\lim_{t \rightarrow \infty} y(t) = 0$ . Suppose to the contrary that  $y(t) > 0$  for all large  $t$ . Then  $y'(t) < 0$  for all large  $t$ . From  $(r(t)y'(t))' < 0$ , we obtain

$$(3) \quad y(t) \leq y(s) + r(s)y'(s) \int_s^t \frac{1}{r(u)} du = y(s) + r(s)y'(s)R(s, t)$$

for  $t \geq s$ , where  $s \geq t_0$  such that  $x(t) > 0$  for  $t \geq s$ . Since  $r(t)y'(t)$  is eventually decreasing, we can choose  $t_1$  so large that  $r(t_1)y'(t_1) < 0$ , thus,  $r(s)y'(s) < r(t_1)y'(t_1) = -a_1$  for  $s \geq t_1$ . Therefore

$$(4) \quad y(t) \leq y(s) - a_1R(s, t), \quad a_1 > 0.$$

If  $R_0 = \infty$ , taking the limit as  $t \rightarrow \infty$  on both sides of the last inequality and in view of  $\lim_{t \rightarrow \infty} y(t) = 0$ , we see that

$$y(s) \geq \infty,$$

which is a contradiction.

(ii) Let  $x(t)$  be an eventually positive solution of (1) and  $\limsup_{t \rightarrow \infty} x(t) > 0$ . Then  $y(t) > 0$ . If not, we have  $y(t) < 0$  for all large  $t$ . If  $x(t)$  is unbounded, then there exists a sequence  $\{t_k\}$  which tends to infinity and is such that

$$x(t_k) = \max_{t \leq t_k} x(t)$$

and  $\lim_{k \rightarrow \infty} x(t_k) = \infty$ . Then, from (2), we have

$$y(t_k) = x(t_k) - p(t_k)x(t_k - \tau) \geq x(t_k)(1 - p).$$

From the above inequality we obtain  $\lim_{k \rightarrow \infty} y(t_k) = \infty$ . This is a contradiction. If  $x(t)$  is bounded, then there is a sequence  $\{t_k\}$  which tends to infinity and is such that

$$\lim_{k \rightarrow \infty} x(t_k) = \limsup_{t \rightarrow \infty} x(t) = L > 0.$$

Since  $\lim_{k \rightarrow \infty} x(t_k - \tau) \leq L$ , we have

$$0 \geq \lim_{k \rightarrow \infty} y(t_k) \geq L(1 - p) > 0.$$

This is also a contradiction and the proof is complete.

LEMMA 3. Suppose that (H) holds. Let  $x(t)$  be an eventually positive (negative) solution of (1) and let  $y(t)$  be defined by (2). If  $R_0 = \infty$ , then  $y'(t) > 0$  (respectively  $y'(t) < 0$ ) for all large  $t$ .

Proof. Suppose  $x(t)$  is an eventually positive solution of (1) and let  $y(t)$  be defined by (2). We assert that  $y'(t) > 0$  for all large  $t$ . If  $\lim_{t \rightarrow \infty} x(t) = 0$ ,

in view of part (i) of Lemma 2, the conclusion holds. If  $\limsup_{t \rightarrow \infty} x(t) > 0$ , by part (ii) of Lemma 2, we have  $y(t) > 0$  for all large  $t$ . Suppose to the contrary that  $y'(t)$  is eventually negative. Note that in view of (1),  $(r(t)y'(t))' = -f(t, x(t - \delta)) < 0$  for all large  $t$ . Thus there is  $t_1 \geq t_0$  such that  $y(t) > 0$ ,  $y'(t) < 0$  and  $(r(t)y'(t))' < 0$  for all  $t \geq t_1$ . Then

$$y(t) \leq y(t_1) + r(t_1)y'(t_1) \int_{t_1}^t \frac{1}{r(s)} ds \rightarrow -\infty,$$

which is a contradiction. The proof is complete.

The following lemma is independent of equation (1), and is obtained by Lu in [4].

LEMMA 4. *Suppose that  $0 \leq p(t) \leq p < 1$ ,  $x(t) > 0$  (resp.  $x(t) < 0$ ) and  $y(t)$  is defined by (2).*

(i) *If  $\lim_{t \rightarrow \infty} p(t) = p_0$  (with  $0 \leq p_0 < 1$ ) and  $\lim_{t \rightarrow \infty} y(t) = a$  (with  $|a| < \infty$ ), then  $\lim_{t \rightarrow \infty} x(t) = a/(1 - p_0)$ .*

(ii) *If  $\lim_{t \rightarrow \infty} y(t) = \infty$  (resp.  $-\infty$ ), then  $\lim_{t \rightarrow \infty} x(t) = \infty$  (resp.  $-\infty$ ).*

**3. The case  $R_0 < \infty$ .** We have already remarked that if  $x(t)$  is an eventually positive solution of (1), then  $y(t)$  and  $y'(t)$  are also of constant sign eventually. These sign regularities provide additional asymptotic information as will be seen in the following two lemmas.

LEMMA 5. *Suppose that (H) holds. If  $x(t)$  is a nonoscillatory solution of (1), then  $y(t)$  defined by (2) is eventually increasing or decreasing and  $\lim_{t \rightarrow \infty} y(t) = L$  exists, where  $L$  is a finite constant.*

Proof. Suppose  $x(t)$  is an eventually positive solution of (1). If  $\lim_{t \rightarrow \infty} x(t) = 0$ , then  $\lim_{t \rightarrow \infty} y(t) = 0$ . If  $\limsup_{t \rightarrow \infty} x(t) > 0$ , by part (ii) of Lemma 2, we have  $y(t) > 0$  for all large  $t$ . Thus, there exists  $t_1 \geq t_0$  such that  $x(t - \delta) > 0$  and  $y(t) > 0$  for  $t \geq t_1$ . It follows from (1) that  $(r(t)y'(t))' < 0$ , that is,

$$(5) \quad y(t) < y(s) + r(s)y'(s) \int_s^t \frac{du}{r(u)}$$

for  $t \geq s$ ,  $s \geq t_1$ . If there exists a  $t_2 \geq t_1$  such that  $y'(t_2) \leq 0$ , then it follows from (5) that  $y(t) < y(s)$  for  $t > s \geq t_2$ . This means that  $y(t)$  is eventually decreasing. If  $y(t)$  is eventually decreasing, it follows from  $y(t) > 0$  that  $\lim_{t \rightarrow \infty} y(t) = L$  exists and  $|L| < \infty$ . If there does not exist an  $s \geq t_1$  such that  $y'(s) \leq 0$ , then  $y'(s) > 0$  for all  $s \geq t_1$ . This means  $y(t)$  is eventually increasing. From (5), we have

$$(6) \quad y(t) < y(s) + r(s)y'(s)R(s, t).$$

Since  $R_0 < \infty$  and  $y'(t) > 0$ , we see that  $y(t)$  is bounded. Therefore  $\lim_{t \rightarrow \infty} y(t) = L$  exists and  $|L| < \infty$ .

Similarly, we can discuss the case when  $x(t)$  is an eventually negative solution of (1). The proof is complete.

LEMMA 6. *Suppose that (H) holds. If  $x(t)$  is a nonoscillatory solution of (1) and  $y(t)$  is defined by (2), then there exist two positive constants  $a_1, a_2$  and  $t_1 \geq t_0$  such that either*

$$a_1R(t) \leq y(t) \leq a_2 \quad \text{or} \quad -a_2 \leq y(t) \leq -a_1R(t)$$

for all  $t \geq t_1$ .

PROOF. Let  $x(t)$  be an eventually positive solution of (1). By Lemma 1,  $y(t)$  is eventually of constant sign. We have four cases to consider:

- (i)  $y(t) > 0$  and  $y'(t) > 0$  eventually;
- (ii)  $y(t) > 0$  and  $y'(t) < 0$  eventually;
- (iii)  $y(t) < 0$  and  $y'(t) > 0$  eventually;
- (iv)  $y(t) < 0$  and  $y'(t) < 0$  eventually.

We shall only consider cases (i) and (ii) in detail, since the other two cases are similar.

(i) If  $y(t)$  is eventually increasing, then (6) holds. From Lemma 5, there exists a positive constant  $a_2$  such that  $y(t) \leq a_2$ . Since we are assuming  $y(t)$  is positive and increasing and since  $R(t) \rightarrow 0$  as  $t \rightarrow \infty$ , there exists a positive constant  $a_1$  and  $t_1 \geq t_0$  such that  $y(t) \geq a_1R(t)$  for all  $t \geq t_1$ .

(ii) If  $y(t)$  is eventually decreasing, then, from (6),

$$y(s) \geq y(t) - r(s)y'(s)R(s, t).$$

By Lemma 5,  $\lim_{t \rightarrow \infty} y(t) = L \geq 0$ . Taking the limit as  $t \rightarrow \infty$  on both sides of the last inequality, we can see that

$$y(s) \geq L - r(s)y'(s)R(s).$$

Since  $r(t)y'(t)$  is eventually decreasing, we can choose  $t_1$  so large that  $r(t_1)y'(t_1) < 0$ ; then  $r(s)y'(s) \leq r(t_1)y'(t_1) = -a_1$  for all  $s \geq t_1$ . Therefore

$$y(s) \geq a_1R(s)$$

for  $s \geq t_1$ , where  $a_1 > 0$  is independent of  $s$ . Changing  $s$  to  $t$ , we can see that

$$y(t) \geq a_1R(t), \quad a_1 > 0$$

for  $t \geq t_1$ .

Similarly, we can prove the case when  $x(t)$  is an eventually negative solution of (1). This completes the proof.

The following result is one of the main classification theorems.

THEOREM 1. Suppose that (H) holds and that  $\lim_{t \rightarrow \infty} p(t) = p_0$ , where  $0 \leq p_0 < 1$ . Then any nonoscillatory solution of (1) must belong to one of the following four types:

$$\begin{aligned}
 S(b, a, c): \quad & x(t) \rightarrow b = \frac{a}{1 - p_0} \neq 0, \quad y(t) \rightarrow a \neq 0, \\
 & r(t)y'(t) \rightarrow c \quad (t \rightarrow \infty), \\
 S(b, a, \infty): \quad & x(t) \rightarrow b = \frac{a}{1 - p_0} \neq 0, \quad y(t) \rightarrow a \neq 0, \\
 & r(t)y'(t) \rightarrow \pm\infty \quad (t \rightarrow \infty), \\
 S(0, 0, c): \quad & x(t) \rightarrow 0, \quad y(t) \rightarrow 0, \quad r(t)y'(t) \rightarrow c \neq 0 \quad (t \rightarrow \infty), \\
 S(0, 0, \infty): \quad & x(t) \rightarrow 0, \quad y(t) \rightarrow 0, \quad r(t)y'(t) \rightarrow \pm\infty \quad (t \rightarrow \infty),
 \end{aligned}$$

where  $a, b, c$  are some finite constants.

PROOF. Suppose that  $x(t)$  is a nonoscillatory solution of (1). By Lemmas 1 and 5,  $y(t)$  is eventually of constant sign and  $\lim_{t \rightarrow \infty} y(t) = L$ , where  $L$  is a finite constant. So there are only two possibilities:  $\lim_{t \rightarrow \infty} y(t) = a \neq 0$  or  $\lim_{t \rightarrow \infty} y(t) = 0$ , where  $a$  is a finite constant. According to Lemma 4,  $\lim_{t \rightarrow \infty} x(t) = b \neq 0$  or  $\lim_{t \rightarrow \infty} x(t) = 0$ .

In addition, by our assumption that  $xf(t, x) > 0$  for  $x \neq 0$  we see from (1) that  $r(t)y'(t)$  is eventually decreasing or increasing. Again there are only two possibilities:  $\lim_{t \rightarrow \infty} r(t)y'(t) = c$  or  $\lim_{t \rightarrow \infty} r(t)y'(t) = \pm\infty$ , where  $c$  is a finite constant.

From the above discussion, we see that  $x(t)$  must belong to one of the four types as stated, except that we have not yet shown that for the case  $S(0, 0, c)$ , the constant  $c \neq 0$ . We do this next.

Suppose  $x(t)$  is a nonoscillatory solution of (1) which belongs to  $S(0, 0, c)$ , that is,  $\lim_{t \rightarrow \infty} x(t) = 0$ ,  $\lim_{t \rightarrow \infty} y(t) = 0$ , and  $\lim_{t \rightarrow \infty} r(t)y'(t) = c$ . Then  $c \neq 0$ . In fact, consider functions  $y(t)$  and  $R(t)$ . From our assumption, it is easy to see that

$$\lim_{t \rightarrow \infty} y(t) = 0, \quad \lim_{t \rightarrow \infty} R(t) = 0, \quad R'(t) < 0,$$

and

$$\lim_{t \rightarrow \infty} \frac{y'(t)}{R'(t)} = \lim_{t \rightarrow \infty} -r(t)y'(t) = -c.$$

By l'Hopital's theorem,  $\lim_{t \rightarrow \infty} (y(t)/R(t))$  exists, and

$$\lim_{t \rightarrow \infty} \frac{y(t)}{R(t)} = \lim_{t \rightarrow \infty} \frac{y'(t)}{R'(t)} = -c.$$

If  $y(t)$  is eventually positive, then, by Lemma 6, there exist two positive

constants  $a_1$  and  $a_2$  such that  $a_1R(t) \leq y(t) \leq a_2$ . Thus  $y(t)/R(t) \geq a_1$ ; that is,  $a_1 \leq -c$ . It follows that  $c \neq 0$ . If  $y(t)$  is eventually negative, then by Lemma 6, there exist two positive constants  $a_1$  and  $a_2$  such that  $-a_2 \leq y(t) \leq -a_1R(t)$ . Thus  $y(t)/R(t) \leq -a_1$ , which means  $-c \leq -a_1$ . Once more it follows that  $c \neq 0$ , which completes the proof.

Next we derive two existence theorems.

**THEOREM 2.** *Assume that (H) holds and  $\lim_{t \rightarrow \infty} (R(t - \tau)/R(t)) = 1$ . A necessary and sufficient condition for (1) to have a nonoscillatory solution  $x(t)$  which belongs to  $S(0, 0, c)$  is that*

$$(7) \quad \int_{t_0}^{\infty} |f(t, \lambda R(t - \delta))| dt < \infty$$

for some  $\lambda \neq 0$ .

**PROOF.** *Necessity.* Let  $x(t)$  be any nonoscillatory positive solution of (1) which belongs to  $S(0, 0, c)$ , i.e.,  $\lim_{t \rightarrow \infty} x(t) = 0$ ,  $\lim_{t \rightarrow \infty} y(t) = 0$  and  $\lim_{t \rightarrow \infty} r(t)y'(t) = c \neq 0$ . Without loss of generality we may assume that  $c < 0$ . Since  $r(t)y'(t)$  is monotone, there exist two positive constants  $\lambda_1, \lambda_2$  and  $t_1 \geq t_0$  such that  $-\lambda_1 \leq r(t)y'(t) \leq -\lambda_2$  for  $t \geq t_1$ . It follows that

$$-\lambda_1 R(t, s) \leq y(s) - y(t) \leq -\lambda_2 R(t, s)$$

for  $s > t, t \geq t_1$ . Let  $s \rightarrow \infty$ ; then  $-\lambda_1 R(t) \leq -y(t) \leq -\lambda_2 R(t)$ . That is,  $\lambda_2 R(t) \leq y(t) \leq \lambda_1 R(t)$ . On the other hand, by (1),

$$r(t)y'(t) = r(t_1)y'(t_1) - \int_{t_1}^t f(s, x(s - \delta)) ds.$$

Since  $\lim_{t \rightarrow \infty} r(t)y'(t) = c < 0$ , we have

$$\int_{t_1}^{\infty} |f(s, x(s - \delta))| ds < \infty.$$

Furthermore, since (H) holds, by  $y(t) \leq x(t)$ , we have

$$f(s, x(s - \delta)) \geq f(s, y(s - \delta)) \geq f(s, \lambda_2 R(s - \delta)).$$

This means that

$$\int_{t_1}^{\infty} |f(s, \lambda_2 R(s - \delta))| ds < \infty.$$

*Sufficiency.* Suppose that (7) holds for  $\lambda > 0$ . A similar argument can be applied if  $\lambda < 0$ . Since  $\lim_{t \rightarrow \infty} (R(t - \tau)/R(t)) = 1$ , we may choose  $p < A < 1$

and  $t_1 \geq t_0$  such that  $p(t)R(t - \tau)/R(t) \leq A$  and

$$(8) \quad \int_{t_1}^{\infty} f(s, \lambda R(s - \tau)) ds < a(1 - A),$$

where  $a = \lambda/2$ . Consider the equation

$$(9) \quad x(t) = p(t)x(t - \tau) + R(t) \left[ (1 - A)a + \int_{t_1}^t f(s, x(s - \delta)) ds \right] \\ + \int_t^{\infty} R(s)f(s, x(s - \delta)) ds$$

for  $t \geq t_1 + \max\{\delta, \tau\}$ . It is easy to see that a solution of (9) must also be a solution of (1). We shall show that (9) has a nonoscillatory solution  $x(t)$  which belongs to  $S(0, 0, c)$  by means of the method of successive approximations. Consider the sequence  $\{x^k(t)\}$  of successive approximating sequences defined as follows:

$$x^1(t) = 0, \quad t_0 \leq t \leq t_1, \\ x^{i+1}(t) = (Fx^i)(t), \quad t \geq t_1, \quad i = 1, 2, \dots,$$

where  $F$  is defined by

$$(10) \quad (Fx)(t) = p(t)x(t - \tau) + R(t) \left[ (1 - A)a + \int_{t_1}^t f(s, x(s - \delta)) ds \right] \\ + \int_t^{\infty} R(s)f(s, x(s - \delta)) ds$$

for  $t \geq t_1 + \max\{\delta, \tau\}$ , and  $(Fx)(t) = (Fx)(t_1 + \max\{\delta, \tau\})$  for  $t_1 \leq t \leq t_1 + \max\{\delta, \tau\}$ . In view of  $(H_1)$ , it is easy to see that

$$0 \leq x^i(t) \leq x^{i+1}(t), \quad t \geq t_1, \quad i = 1, 2, \dots$$

On the other hand,

$$x^2(t) = (Fx^1)(t) = a(1 - A)R(t) \leq 2aR(t), \quad t \geq t_1,$$

and inductively, we have

$$(Fx^i)(t) \leq 2p(t)aR(t - \tau) + (1 - A)aR(t) + (1 - A)aR(t) \\ \leq 2AaR(t) + 2(1 - A)aR(t) = 2aR(t)$$

for  $i \geq 2$ . Thus, by means of Lebesgue's dominated convergence theorem, we see that  $Fx = x$ . Furthermore, it is clear that  $x(t)$ , and hence its associated function  $y(t)$ , converges to zero (since  $R(t)$  does). Finally, in view of (9), we



see that

$$\begin{aligned} -(1 - A)a > r(t)y'(t) &= -(1 - A)a - \int_{t_1}^t f(s, x(s - \delta)) ds \\ &> -(1 - A)a - \int_{t_1}^t f(s, \lambda R(s - \delta)) ds \\ &> -(1 - A)a - (1 - A)a = -2(1 - A)a, \end{aligned}$$

which implies  $\lim_{t \rightarrow \infty} r(t)y'(t) = c \neq 0$  as required. The proof is complete.

**THEOREM 3.** *Assume that (H) holds and that  $\lim_{t \rightarrow \infty} p(t) = p_0$ , where  $0 \leq p_0 < 1$ . A necessary and sufficient condition for (1) to have a nonoscillatory solution  $x(t)$  which belongs to  $S(b, a, c)$  or  $S(b, a, \infty)$  is that*

$$(11) \quad \int_{t_2}^{\infty} \frac{1}{r(s)} \int_{t_1}^s |f(u, \lambda)| du ds < \infty$$

for some  $\lambda \neq 0$  and  $t_2 \geq t_1 \geq t_0$ .

**Proof.** *Necessity.* Let  $x(t)$  be any nonoscillatory positive solution of (1) such that  $\lim_{t \rightarrow \infty} x(t) = b > 0$ . By (2), we have  $\lim_{t \rightarrow \infty} y(t) = a = b(1 - p_0) > 0$ , which means that  $y(t)$  is eventually positive and monotonically tends to  $a$ . Thus there exist two positive constants  $c_1, c_2$  and  $t_1 \geq t_0$  such that  $c_1 \leq y(t) \leq c_2$  for  $t \geq t_1$ . It follows from (1) that

$$y(t) = y(s) + r(t_1)y'(t_1) \int_s^t \frac{1}{r(u)} du - \int_s^t \frac{1}{r(u)} \int_{t_1}^u f(v, x(v - \delta)) dv du.$$

Taking the limit as  $t \rightarrow \infty$  on both sides of the last equality, we obtain

$$a = y(s) + r(t_1)y'(t_1) \int_s^{\infty} \frac{1}{r(u)} du - \int_s^{\infty} \frac{1}{r(u)} \int_{t_1}^u f(v, x(v - \delta)) dv du.$$

This means that

$$(12) \quad 0 \leq \int_s^{\infty} \frac{1}{r(u)} \int_{t_1}^u f(v, x(v - \delta)) dv du < \infty.$$

By (H), we have  $f(t, c_1) \leq f(t, y(t - \delta)) \leq f(t, x(t - \delta))$ . It follows from (12) that

$$\int_s^{\infty} \frac{1}{r(u)} \int_{t_1}^u f(v, c_1) dv du < \infty.$$

*Sufficiency.* Suppose that (11) holds for  $\lambda > 0$ . A similar argument can

be applied if  $\lambda < 0$ . Choose  $t_1 \geq t_0$  so large that

$$(13) \quad \int_s^\infty \frac{1}{r(u)} \int_{t_1}^u f(v, \lambda) \, dv \, du < (1 - p)a,$$

where  $a = \lambda/2$ . Consider the equation

$$(14) \quad x(t) = p(t)x(t - \tau) + (1 - p)a + \int_t^\infty \frac{1}{r(u)} \int_{t_1}^u f(v, x(v - \delta)) \, dv \, du.$$

It is easy to verify that a solution of (14) must also be a solution of (1). Consider the Banach space  $\Phi$  of all bounded real functions  $x(t)$  with norm  $\sup_{t \geq t_1} |x(t)|$ , endowed with the usual pointwise ordering  $\leq$ : for  $x, y \in \Phi$ ,  $x \leq y$  is equivalent to  $x(t) \leq y(t)$  for  $t \geq t_1$ . Then  $\Phi$  is partially ordered. Define a subset  $\Omega$  of  $\Phi$  as follows:

$$\Omega = \{x \in \Phi : (1 - p)a \leq x(t) \leq 2a, \, t \geq t_1\}.$$

For any subset  $B \subset \Omega$ , it is obvious that  $\inf B \in \Omega$  and  $\sup B \in \Omega$ . We also define an operator  $F : \Omega \rightarrow \Phi$  as

$$(Fx)(t) = p(t)x(t - \tau) + (1 - p)a + \int_t^\infty \frac{1}{r(u)} \int_{t_1}^u f(v, x(v - \delta)) \, dv \, du, \quad t \geq t_1.$$

The mapping  $F$  satisfies the assumptions of Knaster's fixed point theorem [1]. Namely, it satisfies the following:

- (i)  $F$  maps  $\Omega$  into itself. Indeed, if  $x \in \Omega$ , by (13), we have  $(1 - p)a \leq (Fx)(t) \leq 2p(t)a + (1 - p)a + (1 - p)a \leq 2pa + 2(1 - p)a = 2a$ .
- (ii) By (H),  $F$  is nondecreasing. That is, for any  $x, y \in \Omega$ ,  $x \leq y$  implies that  $Fx \leq Fy$ .

By Knaster's fixed point theorem [1], there exists an  $x \in \Omega$  such that  $Fx = x$ , that is,  $x(t)$  is a nonoscillatory solution of (1) which belongs to  $S(b, a, c)$  or  $S(b, a, \infty)$ . Note that  $b \neq 0$ . Since  $x \in \Omega$  and by (2) and Lemma 4,  $\lim_{t \rightarrow \infty} y(t) = a$  and  $\lim_{t \rightarrow \infty} x(t) = a/(1 - p_0) = b \neq 0$  must exist. This completes the proof.

**4. The case  $R_0 = \infty$ .** When  $R_0 = \infty$ , the asymptotic behaviors of the nonoscillatory solutions of (1) are quite different from those obtained in Section 3.

LEMMA 7. *Suppose that (H) holds. If  $x(t)$  is an eventually positive (negative) solution of (1) and  $\limsup_{t \rightarrow \infty} x(t) > 0$  (resp.  $\limsup_{t \rightarrow \infty} (-x(t)) > 0$ ), then there exist two positive constants  $a_1, a_2$  and  $t_1 \geq t_0$  such that the function  $y(t)$  defined by (2) is monotonically increasing (decreasing) and*

$$(15) \quad a_1 \leq y(t) \leq a_2 R(t_1, t) \quad (\text{resp. } -a_2 R(t_1, t) \leq y(t) \leq -a_1)$$

for all  $t \geq t_1$ .

**Proof.** Let  $x(t)$  be an eventually positive solution of (1) and  $\limsup_{t \rightarrow \infty} x(t) > 0$ . Then, by part (ii) of Lemma 2, we have  $y(t) > 0$  eventually. Thus, there exists  $s \geq t_0$  such that  $x(t) > 0$  and  $y(t) > 0$  for all  $t \geq s$ . It follows from (1) that  $(r(t)y'(t))' < 0$  and thus  $r(t)y'(t)$  is eventually decreasing. By Lemma 3, and in view of  $R_0 = \infty$ , we have  $y'(t) > 0$  eventually. Thus,  $y(t)$  is eventually increasing and  $r(t)y'(t)$  is positive. By (3) we then see that (15) holds. This completes the proof.

**THEOREM 4.** *Suppose that (H) holds and  $\lim_{t \rightarrow \infty} p(t) = p_0$ , where  $0 \leq p_0 < 1$ . Then any nonoscillatory solution of (1) must belong to one of the following five types:*

$$S(0, 0, 0): \quad x(t) \rightarrow 0, \quad y(t) \rightarrow 0, \quad r(t)y'(t) \rightarrow 0 \quad (t \rightarrow \infty),$$

$$S(0, 0, c): \quad x(t) \rightarrow 0, \quad y(t) \rightarrow 0, \quad r(t)y'(t) \rightarrow c \neq 0 \quad (t \rightarrow \infty),$$

$$S(b, a, 0): \quad x(t) \rightarrow b = \frac{a}{1 - p_0} \neq 0, \quad y(t) \rightarrow a \neq 0, \\ r(t)y'(t) \rightarrow 0 \quad (t \rightarrow \infty),$$

$$S(\infty, \infty, c): \quad x(t) \rightarrow \infty, \quad y(t) \rightarrow \infty, \quad r(t)y'(t) \rightarrow c \neq 0 \quad (t \rightarrow \infty),$$

$$S(\infty, \infty, 0): \quad x(t) \rightarrow \infty, \quad y(t) \rightarrow \infty, \quad r(t)y'(t) \rightarrow 0 \quad (t \rightarrow \infty),$$

where  $a, b, c$  are some finite constants.

**Proof.** For any nonoscillatory solution of (1), without loss of generality, we may suppose that  $x(t)$  is an eventually positive solution. If  $\lim_{t \rightarrow \infty} x(t) = 0$ , then, by part (i) of Lemma 2,  $\lim_{t \rightarrow \infty} y(t) = 0$  and  $y'(t) > 0$  for all large  $t$ . Since  $(r(t)y'(t))' < 0$  for all large  $t$ , we have

$$\lim_{t \rightarrow \infty} r(t)y'(t) = c \geq 0.$$

If  $\limsup_{t \rightarrow \infty} x(t) > 0$ , in view of part (ii) of Lemma 2, we have  $y(t) > 0$  for all large  $t$ . By Lemma 7, we know that  $y(t)$  is eventually increasing, and  $r(t)y'(t)$  is positive and decreasing. Thus there exist only the following possibilities:

$$\lim_{t \rightarrow \infty} y(t) = a, \quad 0 < a < \infty, \quad \text{or} \quad \lim_{t \rightarrow \infty} y(t) = \infty,$$

and

$$\lim_{t \rightarrow \infty} r(t)y'(t) = c \geq 0.$$

Since  $r(t)y'(t)$  is eventually decreasing and  $\lim_{t \rightarrow \infty} r(t)y'(t) = c$ , we have

$$r(t)y'(t) \geq c, \quad \text{i.e.,} \quad y'(t) \geq \frac{c}{r(t)}$$

for all large  $t$ . Integrating the inequality, we obtain

$$(16) \quad y(t) \geq y(s) + c \int_s^t \frac{1}{r(u)} du.$$

If  $c > 0$  and  $R_0 = \infty$ , it follows that

$$\lim_{t \rightarrow \infty} y(t) = \infty.$$

By Lemma 4, we have  $\lim_{t \rightarrow \infty} x(t) = \infty$ . If  $c = 0$  and  $\lim_{t \rightarrow \infty} y(t) = a$  (resp.  $\infty$ ), then, by Lemma 4, it follows that  $\lim_{t \rightarrow \infty} x(t) = b = a/(1 - p_0)$  (resp.  $\infty$ ). Therefore  $x(t)$  must belong to one of the five types as stated. The proof is complete.

Next we derive several existence criteria for the nonoscillatory solutions of (1).

**THEOREM 5.** *Assume that (H) holds and  $\lim_{t \rightarrow \infty} p(t) = p_0$ , where  $0 \leq p_0 < 1$ . A necessary and sufficient condition for (1) to have a nonoscillatory solution which belongs to  $S(b, a, 0)$  is that*

$$(17) \quad \int_{t_0}^{\infty} \frac{1}{r(u)} \int_u^{\infty} |f(v, \lambda)| dv du < \infty$$

for some  $\lambda \neq 0$ .

**Proof.** *Necessity.* Let  $x(t)$  be a nonoscillatory solution of (1) which belongs to  $S(b, a, 0)$ , i.e.,

$$\lim_{t \rightarrow \infty} x(t) = b, \quad \lim_{t \rightarrow \infty} y(t) = a, \quad \text{and} \quad \lim_{t \rightarrow \infty} r(t)y'(t) = 0.$$

Without loss of generality, we may suppose that  $b > 0$ . By (2), it follows that  $\lim_{t \rightarrow \infty} y(t) = a = b(1 - p_0) > 0$ . Then there exist two positive constants  $c_1, c_2$  and  $t_1 \geq t_0$  such that

$$(18) \quad c_1 \leq y(t) \leq c_2 \quad \text{for } t \geq t_1.$$

On the other hand, by (1),

$$(19) \quad r(t)y'(t) = r(s)y'(s) - \int_s^t f(u, x(u - \delta)) du$$

for  $t \geq s \geq t_1$ . After taking the limit as  $t \rightarrow \infty$  on both sides of (19) and using  $\lim_{t \rightarrow \infty} r(t)y'(t) = 0$ , we obtain

$$(20) \quad r(s)y'(s) = \int_s^{\infty} f(u, x(u - \delta)) du$$

for  $s \geq t_1$ . Now it follows from (20) that

$$(21) \quad y(s) = y(t_1) + \int_{t_1}^s \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du$$

for  $s \geq t_1$ . Let  $s \rightarrow \infty$ . Then

$$a = y(t_1) + \int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du.$$

Therefore

$$(22) \quad \int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du < \infty.$$

By (H),  $f(t, c_1) \leq f(t, y(t - \delta)) \leq f(t, x(t - \delta))$ , so (17) holds.

*Sufficiency.* Without loss of generality we may assume that (17) holds for  $\lambda > 0$ . Then there exists  $t_1 \geq t_0$  such that

$$\int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f(v, \lambda) \, dv \, du < (1 - p)a,$$

where  $a = \lambda/2$ . Now consider the equation

$$(23) \quad x(t) = p(t)x(t - \tau) + (1 - p)a + \int_{t_1}^t \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du$$

for  $t \geq t_1$ . Consider the Banach space  $\Phi$  of all bounded real functions  $x(t)$  with norm  $\sup_{t \geq t_1} |x(t)|$ , endowed with the usual pointwise ordering  $\leq$ : for  $x, y \in \Phi$ ,  $x \leq y$  is equivalent to  $x(t) \leq y(t)$  for  $t \geq t_1$ . Then  $\Phi$  is partially ordered. Define a subset  $\Omega$  of  $\Phi$  as follows:

$$\Omega = \{x \in \Phi : (1 - p)a \leq x(t) \leq 2a, \, t \geq t_1\}.$$

If  $x(t) \in \Phi$ , let also

$$(Fx)(t) = p(t)x(t - \tau) + (1 - p)a + \int_{t_1}^t \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du, \quad t \geq t_1.$$

Then, by using Knaster's fixed point theorem and Lemma 7, we can show that there exists a nonoscillatory solution of (23), and thus of (1), which belongs to  $S(b, a, 0)$ . We omit the details. The proof is complete.

THEOREM 6. Assume that (H) holds. A necessary and sufficient condition for (1) to have a nonoscillatory solution which belongs to  $S(\infty, \infty, c)$  is that

$$(24) \quad \int_{t_1}^{\infty} |f(u, \lambda R(t_1, u - \delta))| du < \infty$$

for some  $\lambda \neq 0$  and  $t_1 \geq t_0$ .

PROOF. *Necessity.* Suppose that  $x(t)$  is an eventually positive solution of (1) which belongs to  $S(\infty, \infty, c)$ . That is,

$$(25) \quad \lim_{t \rightarrow \infty} x(t) = \infty, \quad \lim_{t \rightarrow \infty} y(t) = \infty, \quad \lim_{t \rightarrow \infty} r(t)y'(t) = c > 0.$$

Then it follows from (15) and (16) that there exist two positive constants  $c_1, c_2$  and  $t_1 \geq t_0$  such that

$$(26) \quad c_1 R(t_1, t) \leq y(t) \leq c_2 R(t_1, t)$$

for  $t \geq t_1$ . On the other hand, by (1),

$$(27) \quad r(t)y'(t) = r(t_1)y'(t_1) - \int_{t_1}^t f(u, x(u - \delta)) du.$$

Let  $t \rightarrow \infty$  on both sides of (27); we obtain

$$(28) \quad \int_{t_1}^{\infty} |f(u, x(u - \delta))| du < \infty.$$

Since (H) holds,  $f(t, c_1 R(t_1, t - \delta)) \leq f(t, y(t - \delta)) \leq f(t, x(t - \delta))$  and we can conclude from (28) that (24) holds for some  $\lambda \neq 0$  and  $t_1 \geq t_0$ .

*Sufficiency.* Without loss of generality, we may assume that (24) holds for  $\lambda > 0$  and  $t_1 \geq t_0$ . Then there exists  $t_2 \geq t_1$  such that

$$\int_{t_2}^{\infty} f(u, \lambda R(t_1, u - \delta)) du < (1 - p)a,$$

where  $a = \lambda/2$ . Now consider the equation

$$(29) \quad x(t) = p(t)x(t - \tau) + (1 - p)aR(t_1, t) + \int_{t_2}^t R(t_2, u)f(u, x(u - \delta)) du \\ + R(t_1, t) \int_t^{\infty} f(u, x(u - \delta)) du$$

for  $t \geq t_2$ . We introduce the Banach space  $\Phi$  of all bounded real functions  $x(t)$  which satisfy

$$\sup_{t \geq t_2} \frac{|x(t)|}{R(t_1, t)} < \infty,$$

with the norm

$$\|x\| = \sup_{t \geq t_2} \frac{|x(t)|}{R(t_1, t)}.$$

$\Phi$  is considered to be endowed with the usual pointwise ordering  $\leq$ : for  $x, y \in \Phi$ ,  $x \leq y$  is equivalent to  $x(t) \leq y(t)$  for  $t \geq t_2$ . Then  $\Phi$  is partially ordered. Define a subset  $\Omega$  of  $\Phi$  and an operator  $F : \Omega \rightarrow \Phi$  as follows:

$$\Omega = \{x \in \Phi : (1 - p)aR(t_1, t) \leq x(t) \leq 2aR(t_1, t), t \geq t_2\},$$

and for  $x(t) \in \Phi$ ,

$$\begin{aligned} (Fx)(t) &= p(t)x(t - \tau) + (1 - p)aR(t_1, t) \\ &\quad + \int_{t_2}^t R(t_2, u)f(u, x(u - \delta)) du \\ &\quad + R(t_1, t) \int_t^\infty f(u, x(u - \delta)) du, \quad t \geq t_2. \end{aligned}$$

Similarly to the proof of Theorem 3, we can show that there exists a nonoscillatory solution of (29), and hence also of (1), which belongs to  $S(\infty, \infty, c)$ . This completes the proof.

**THEOREM 7.** *Assume that (H) holds. A sufficient condition for (1) to have a nonoscillatory solution which belongs to  $S(\infty, \infty, 0)$  is that*

$$(30) \quad \int_{t_0}^\infty R(t_0, u)|f(u, \lambda_1 R(t_1, u - \delta))| du = \infty$$

and

$$(31) \quad \int_{t_0}^\infty |f(u, \lambda_2 R(t_1, u - \delta))| du < \infty$$

for some  $\lambda_1, \lambda_2$  such that  $\lambda_1 \lambda_2 > 0$  and  $t_1 \geq t_0$ .

**Proof.** Without loss of generality, we may assume  $\lambda_1 > 0, \lambda_2 > 0$ . Take  $t_2 \geq t_1$  such that for  $t \geq t_2$ ,

$$(32) \quad \frac{\lambda_1}{R(t_1, t)\lambda_2} + p(t) + \frac{1}{\lambda_2} \int_{t_2}^\infty f(u, \lambda_2 R(t_1, u - \delta)) du < 1.$$

Define an operator  $F$  as

$$\begin{aligned} (Fx)(t) &= \frac{\lambda_1}{R(t_1, t)} + \frac{R(t_1, t - \tau)}{R(t_1, t)} p(t)x(t - \tau) \\ &\quad + \frac{1}{R(t_1, t)} \int_{t_1}^t R(t_1, u) f(u, R(t_1, u - \delta)x(u - \delta)) du \\ &\quad + \int_t^\infty f(u, R(t_1, u - \delta)x(u - \delta)) du, \quad t \geq t_2, \end{aligned}$$

and

$$(Fx)(t) = Fx(t_2), \quad t_1 \leq t \leq t_2,$$

and let  $x^1(t) = 0$ ,  $x^{i+1}(t) = (Fx^i)(t)$ ,  $t \geq t_1$ ,  $i = 1, 2, \dots$ . By (H) and induction, it is easy to see that  $0 \leq x^i(t) \leq x^{i+1}(t)$ ,  $t \geq t_1$ ,  $i = 1, 2, \dots$ . On the other hand,  $x^2(t) \leq \lambda_2$ ,  $t \geq t_1$ . It follows from (32) that

$$(33) \quad x^{i+1}(t) \leq \lambda_2 \left[ \frac{\lambda_1}{R(t_1, t)\lambda_2} + p(t) + \frac{1}{\lambda_2} \int_{t_2}^\infty f(u, \lambda_2 R(t_1, u - \delta)) du \right] \leq \lambda_2.$$

By induction, we have  $x^i(t) \leq \lambda_2$  for  $t \geq t_1$ ,  $i = 1, 2, \dots$ . Let

$$\lim_{i \rightarrow \infty} x^i(t) = x^*(t), \quad t \geq t_2.$$

Using Lebesgue's dominated convergence theorem, we get

$$x^*(t) = (Fx^*)(t), \quad t \geq t_1.$$

By (32) and (33), it is easy to see that

$$\frac{\lambda_1}{R(t_1, t)} \leq x^*(t) \leq \lambda_2.$$

Set

$$z(t) = R(t_1, t)x^*(t), \quad t \geq t_1.$$

Then we have  $\lambda_1 \leq z(t) \leq \lambda_2 R(t_1, t)$  and

$$\begin{aligned} z(t) &= \lambda_1 + p(t)z(t - \tau) + \int_{t_1}^t R(t_1, u) f(u, z(u - \delta)) du \\ &\quad + R(t_1, t) \int_t^\infty f(u, z(u - \delta)) du, \quad t \geq t_2. \end{aligned}$$

Again set

$$w(t) = z(t) - p(t)z(t - \tau);$$



then, by (30), we have  $\lim_{t \rightarrow \infty} w(t) = \infty$ . From part (ii) of Lemma 2, it follows that  $\lim_{t \rightarrow \infty} z(t) = \infty$ . Since

$$r(t)w'(t) = \int_t^\infty f(u, z(u - \delta)) du,$$

it follows that  $\lim_{t \rightarrow \infty} r(t)w'(t) = 0$ . This means  $z(t)$  is a positive solution of (1) that belongs to  $S(\infty, \infty, 0)$ , which completes the proof.

**THEOREM 8.** *Assume that (H) holds, that  $|p(t) - p(s)| \leq K|t - s|$  and that  $r(t)$  is nondecreasing. Furthermore, suppose that there exist  $K_1 > K_2 > 0$  such that*

$$(34) \quad p(t)e^{K_1\tau} > 1 \geq p(t)e^{K_2\tau},$$

and for large  $t_1$  and  $t \geq t_1$ ,

$$(35) \quad (p(t)e^{K_1\tau} - 1)e^{-K_1t} \geq \int_t^\infty \frac{1}{r(u)} \int_u^\infty f(v, e^{-K_2(v-\delta)}) dv du.$$

Then (1) has a nonoscillatory solution  $x(t) \in S(0, 0, 0)$ .

**Proof.** Set

$$\Omega = \{x \in C[0, \infty) : e^{-K_1t} \leq x(t) \leq e^{-K_2t}, |x(t) - x(s)| \leq L|t - s|, t \geq s \geq t_0\},$$

where  $L \geq \max\{K, K_1\}$ . It is easy to show that  $\Omega$  is nonempty, bounded, convex and closed in  $\Phi$  defined to be all bounded real functions  $x(t)$  with norm  $\sup_{t \geq t_0} |x(t)|$ .

Define an operator  $F$  on  $\Omega$  as

$$(36) \quad (Fx)(t) = \begin{cases} p(t)x(t - \tau) - \int_t^\infty \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) dv du, & t \geq t_2, \\ \exp\left(\frac{\ln((Fx)(t_2))}{t_2}t\right), & t_0 \leq t \leq t_2, \end{cases}$$

where  $t_2 \geq t_1$ , and for given  $\alpha, p < \alpha < 1$ , let

$$(37) \quad \frac{1}{r(t)} \int_{t_2}^\infty f(s, e^{-K_2(s-\delta)}) ds \leq (\alpha - p(t))L, \quad t \geq t_2,$$

and

$$(38) \quad \alpha + e^{-K_2(t-\tau)} \leq 1, \quad t \geq t_2.$$

The mapping  $F$  satisfies the assumptions of Schauder's fixed point theorem. Namely, it satisfies the following:

(i)  $F$  maps  $\Omega$  into  $\Omega$ . For any  $x \in S$ , by (34) and (35) we obtain

$$(Fx)(t) \leq p(t)x(t - \tau) \leq p(t) \exp(-K_2(t - \tau)) \leq \exp(-K_2t),$$

$$\begin{aligned}
(Fx)(t) &\geq p(t) \exp(-K_1(t - \tau)) - \int_t^\infty \frac{1}{r(u)} \int_u^\infty f(v, \exp(-K_2(v - \delta))) \, dv \, du \\
&= \exp(-K_1 t) + (p(t) \exp(K_1 \tau) - 1) \exp(-K_1 t) \\
&\quad - \int_t^\infty \frac{1}{r(u)} \int_u^\infty f(v, \exp(-K_2(v - \delta))) \, dv \, du \\
&\geq \exp(-K_1 t),
\end{aligned}$$

and so

$$(39) \quad K_2 \leq \frac{-\ln((Fx)(t_2))}{t_2} \leq K_1.$$

For any  $s \geq t \geq t_2$ , by (37) and (38) we get

$$\begin{aligned}
|(Fx)(s) - (Fx)(t)| &\leq \left[ (p(s) + \exp(-K_2(t - \tau)))L \right. \\
&\quad \left. + \frac{1}{r(t)} \int_t^\infty f(u, \exp(-K_2(u - \delta))) \, du \right] |s - t| \\
&\leq [\exp(-K_2(t - \tau)) + \alpha]L |s - t| \leq L |s - t|,
\end{aligned}$$

and for  $t_0 \leq t \leq s \leq t_2$ ,

$$|(Fx)(s) - (Fx)(t)| = \left| \exp\left(\frac{\ln((Fx)(t_2))}{t_2} s\right) - \exp\left(\frac{\ln((Fx)(t_2))}{t_2} t\right) \right| \leq L |s - t|.$$

Hence,  $F$  maps  $\Omega$  into  $\Omega$ .

(ii)  $F$  is continuous. Let  $x^i(t) \in \Omega$  such that

$$\lim_{i \rightarrow \infty} \|x^i - x\| = 0.$$

Since  $\Omega$  is closed,  $x \in \Omega$ . Then, by (36), we get

$$\begin{aligned}
|(Fx^i)(t) - (Fx)(t)| &\leq p(t) \|x^i - x\| + \int_{t_2}^\infty \frac{1}{r(u)} \int_u^\infty |f(v, x^i(v - \delta)) - f(v, x(v - \delta))| \, dv \, du \\
&\leq p \|x^i - x\| + \int_{t_2}^\infty \frac{1}{r(u)} \int_u^\infty |f(v, x^i(v - \delta)) - f(v, x(v - \delta))| \, dv \, du.
\end{aligned}$$

By the continuity of  $f$  and Lebesgue's dominated convergence theorem, it follows that

$$(40) \quad \lim_{i \rightarrow \infty} \sup_{t \geq t_2} |(Fx^i)(t) - (Fx)(t)| = 0.$$

We can easily show that

$$(41) \quad \sup_{t_0 \leq t \leq t_2} |(Fx^i)(t) - (Fx)(t)| \leq |\ln((Fx^i)(t_2)) - \ln((Fx)(t_2))|.$$

Using (40) and (41), it is easy to show that

$$\lim_{i \rightarrow \infty} \|Fx^i - Fx\| = 0.$$

(iii)  $F(\Omega)$  is precompact. Let  $x \in \Omega$  and  $s, t \geq t_2$ . Then by (37) we have, for  $s > t$ ,

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| &= \left| p(t)x(t - \tau) - p(s)x(s - \tau) \right. \\ &\quad + \int_s^\infty \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du \\ &\quad \left. - \int_t^\infty \frac{1}{r(u)} \int_u^\infty f(v, x(v - \delta)) \, dv \, du \right| \\ &\leq |p(t)x(t - \tau) - p(s)x(s - \tau)| \\ &\quad + \left| \int_s^t \frac{1}{r(u)} \int_u^\infty f(s, x(v - \delta)) \, dv \, du \right| \\ &\leq |p(t)x(t - \tau) - p(s)x(s - \tau)| \\ &\quad + \left| \int_s^\infty \frac{1}{r(u)} \int_u^\infty f(s, x(v - \delta)) \, dv \, du \right| \\ &\leq 2e^{-K_2(t-\tau)} + (p(t)e^{K_1\tau} - 1)e^{-K_1t}. \end{aligned}$$

Since  $e^{-K_1t} \rightarrow 0$  and  $e^{-K_2(t-\tau)} \rightarrow 0$  as  $t \rightarrow \infty$ , we conclude from the above inequalities that, for any given  $\varepsilon > 0$ , there exists  $t_3 \geq t_2$  such that for all  $x \in \Omega$ , and  $s, t \geq t_3$ ,

$$|(Fx)(t) - (Fx)(s)| < \varepsilon.$$

This means that  $F(\Omega)$  is relatively compact in the topology of the Fréchet space  $C[t_0, \infty)$ .

Similarly to Theorem 2 in [4] and using Schauder's fixed point theorem we can conclude that there exists an  $x \in \Omega$  such that  $x = Fx$ . That is,  $x(t)$  is a positive solution of (36) and thus also of (1). Since  $\lim_{t \rightarrow \infty} x(t) = 0$ , by Lemma 2, it follows that  $\lim_{t \rightarrow \infty} y(t) = 0$  and  $\lim_{t \rightarrow \infty} y'(t) = 0$ . The proof is complete.

We close this paper by remarking that it would not be difficult to extend all the results in this paper to an equation whose nonlinear term has the form

$$f(t, x(t - \delta_1), \dots, x(t - \delta_m)).$$

**Acknowledgements.** I am grateful to Professor Sui Sun Cheng for his assistance.

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*Reçu par la Rédaction le 19.6.1996*