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An attraction result and an index theorem for continuous flows on $\mathbb{R}^n \times [0,\infty)$

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Abstract. We study the behavior of a continuous flow near a boundary. We prove that if φ is a flow on $E = \mathbb{R}^{n+1}$ for which $\partial E = \mathbb{R}^n \times \{0\}$ is an invariant set and $S \subset \partial E$ is an isolated invariant set, with non-zero homological Conley index, then there exists an x in $E \setminus \partial E$ such that either $\alpha(x)$ or $\omega(x)$ is in S. We also prove an index theorem for a flow on $\mathbb{R}^n \times [0,\infty)$.

1. Introduction. The aim of this paper is to present generalizations of two theorems proved by Capietto and Garay (Ths. 1 and 2 of [Ca-Ga]). These theorems apply only to flows generated by vector fields, whereas our approach works for any continuous flow. It is based on the time-duality of the Conley index proved by Mrozek and Srzednicki in [Mr-Srz]. Assume $E^+ = \mathbb{R}^n \times [0,\infty)$, $\partial E = \mathbb{R}^n \times \{0\}$ and φ is a flow on E^+ such that ∂E is invariant. We are interested in the behavior of φ in a small vicinity of ∂E. In many applications, subsets of ∂E which are ω-limit sets of points lying in $E^+ \setminus \partial E$ play an important role. The motivation for the considered problem comes from permanence theory. For more details, we refer the reader to [Ho], [Ho1], [Ho-Si] and [Ca-Ga]. We also prove that if φ is a flow on $E = \mathbb{R}^{n+1}$ for which ∂E is invariant and $S \subset \partial E$ is an isolated invariant set with respect to the flow φ on E, with non-zero homological Conley index, then there exists x in $E \setminus \partial E$ such that either $\alpha(x)$ or $\omega(x)$ is in the set S.

2. Isolating blocks and the Conley index. We first give a brief account of the Conley index. Let X be a locally compact, metric space and T denote $\mathbb{R}$ or one of its subgroups of the form $\mathbb{Z}t_0$ for some $t_0 > 0$. By a *dynamical system* on X we mean a continuous function

$$\varphi : X \times T \ni (x,t) \to xt \in X$$

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such that $x0 = x$ and $x(s+t) = (xs)t$. The *backward dynamical system* is defined as the map

$$X \times T \ni (x,t) \to x(-t) \in X.$$

We call φ *a flow* if $T = \mathbb{R}$, otherwise φ is called a *discrete dynamical system*. If φ is a flow, then the restriction of φ to $X \times \mathbb{Z}t$ is a discrete dynamical system. If $f : X \to X$ is a homeomorphism then its iterates define a discrete dynamical system

$$X \times \mathbb{Z} \ni (x,n) \to f^n(x) \in X.$$

A set $S \subset X$ is called *invariant* if $ST = S$. If $N \subset X$, then the set $S(N) = \{x \in N : xT \subset N\}$ is the maximal invariant set contained in N. The set N is called an *isolating neighborhood* if $S(N) \subset \operatorname{int} N$. An invariant set S is said to be *isolated* if there exists an isolating neighborhood N such that $S = S(N)$. If φ is a flow and $S \subset X$ is compact, then by Th. 1 of [Mr2], S is an isolated invariant set with respect to φ iff S is an isolated invariant set with respect to $\varphi_t = \varphi(*,t)$ for all $t > 0$.

The definition of the Conley index is based on the notion of the index pair (or isolating block for a flow). The pair $P = (P_1, P_2)$ of closed subsets of a neighborhood N isolating S is called an *index pair* if the following three conditions are satisfied:

(1) $x \in P_i$, $x[0,t] \subset N \Rightarrow x[0,t] \subset P_i$ for $i = 1,2$;

(2) if $x \in P_i$, $t > 0$, and xt is not in N then there exists $t_1 < t$ such that $x[0,t_1] \subset N$ and $xt_1 \in P_2$;

(3) $S \subset \operatorname{int}(P_1 \setminus P_2)$.

This definition was introduced in the continuous case by Conley in [Co] and in the discrete case by Mrozek in [Mr1]. Assume H^* is the Alexander–Spanier cohomology functor with real coefficients. We recall that in the continuous case $H^*(P_1,P_2)$ depends only on the isolated invariant set S and it is by definition the *cohomological Conley index* of S. In the discrete case it was proved in [Mr1] that $L(H^*(P_1,P_2), I_{(P_1,P_2)})$ depends only on S, where L is the Leray functor and $I_{(P_1,P_2)}$ is the index map (introduced in [Mr1]). By Mrozek's results (see [Mr2]) if φ is flow on X, $f = \varphi_t$ for some $t > 0$ and S is an isolated invariant set with respect to φ, then the distinguished isomorphism in the Conley index of S with respect to f is the identity and the cohomological Conley index of an isolated invariant set of a flow φ coincides with the corresponding index with respect to the discrete dynamical system φ_t for any $t > 0$.

Now we describe the notion of an isolating block (for a flow). Recall that a set $\Sigma \subset X$ is called a *δ-section* provided $\Sigma(-\delta,\delta)$ is an open set in X and the map

$$\Sigma \times (-\delta,\delta) \ni (x,t) \to xt \in \Sigma(-\delta,\delta)$$

is a homeomorphism. Let B be a compact subset of X. B is called an *isolating block* if there exists a $\delta > 0$ and two δ-sections Σ^+ and Σ^- such that

(i) $\mathrm{cl}(\Sigma^+ \times (-\delta,\delta)) \cap \mathrm{cl}(\Sigma^- \times (-\delta,\delta)) = \emptyset$,

(ii) $B \cap (\Sigma^+(-\delta,\delta)) = (B \cap \Sigma^+)[0,\delta)$,
$B \cap (\Sigma^-(-\delta,\delta)) = (B \cap \Sigma^-)(-\delta,0]$,

(iii) $\forall x \in \partial B \setminus (\Sigma^+ \cup \Sigma^-)\ \exists \mu < 0 < \nu : x\mu \in \Sigma^+,\ x\nu \in \Sigma^-,\ x[\mu,\nu] \subset \partial B$.

We put $B^+ = B \cap \Sigma^+$, $B^- = B \cap \Sigma^-$, $a^+ = \{x \in B^+ : x[0,\infty) \subset B\}$ and $a^- = \{x \in B^- : x(-\infty,0] \subset B\}$.

In particular, if B is an isolating block, and B^- is the "exit" set, then (B, B^-) is an index pair.

THEOREM 1. *If S is an isolated invariant set, then each isolating neighborhood of S contains a block, which is a neighborhood of S. If B_1 and B_2 are two blocks which isolate S then the homotopy types of the pointed spaces $(B_1/B_1^-, [B_1^-])$ and $(B_2/B_2^-, [B_2^-])$ coincide.*

For the proof see [Ch], [Co].

The homotopy type uniquely determined by Theorem 1 is denoted by $h(S)$ and is called the *homotopy Conley* index of S. If H denotes an arbitrary homology or cohomology functor, then $H(h(S)) \cong H(B, B^-)$. This is proved in [Ryb. p. 57]. By $h^*(S)$ we denote the Conley index of S with respect to the backward flow. Obviously, $H(h^*(S)) \cong H(B, B^+)$.

COROLLARY 2. *If S is an isolated invariant set in a Euclidean space (or a half-space), then*

$$\check{H}(h(S)) \cong H^*(h(S)),$$

where $\check{H}$, H^* denote the Čech and singular cohomology functors respectively (with real coefficients).

Proof. Szymczak [Sz] showed that if f is a discrete dynamical system in a Euclidean space then there exists an index pair (P_1, P_2) for S such that P_i is an ENR. Since for ENR pairs the Čech, singular and Alexander–Spanier cohomologies are isomorphic, our assertion follows from Mrozek's results (see [Mr2, Cor., p. 311]).

We need the notion of the index of rest points of a flow introduced by Srzednicki [Srz]. Let X be an ENR and φ be a flow on X. Assume that U is an open subset of X such that there are no rest points of φ on ∂U.

DEFINITION. The *index of rest points* $\mathrm{I}(U,\varphi)$ of φ in U is given by

$$\mathrm{I}(U,\varphi) = \lim_{\varepsilon \to 0^+} \mathrm{ind}(\varphi_t, U),$$

where ind denotes the fixed point index.

We refer the reader to [Srz] for the main properties of the index of rest points.

Definition. Assume that S is an isolated invariant set. The *index of rest points in* S is defined as $\mathrm{I}(S,\varphi)=\mathrm{I}(N,\varphi)$, where N is any isolating neighborhood for S.

The excision property implies that $\mathrm{I}(S,\varphi)$ is well-defined.

Remark 3. In [Srz] Srzednicki proved that if for an isolated invariant set S there exists a block B such that B and B^- are ENR's then $\mathrm{I}(S,\varphi)=\chi(h(S))$, where $\chi(h(S))$ is the Euler characteristic of $h(S)$ with respect to the singular homology. We do not know whether there exists a block (B,B^-) consisting of ENR's, but by Th. 3 of [Mr3] and Cor. 3 of [Mr4] we have the following:

Corollary 4. *If S is an isolated invariant set for the flow φ on the ENR space X, then* $\mathrm{I}(S,\varphi)=\chi(h(S))$.

Remark 5. In case of a flow on a Euclidean space Szymczak's result implies that we may use the Čech or singular cohomology to compute $\mathrm{ind}(S,\varphi)$.

We recall that a nonempty compact invariant set S is called *positively asymptotically stable* (PAS) if

(i) for each open neighborhood U of S there is an open neighborhood $V\subset U$ of S such that $V[0,\infty)\subset U$,

(ii) there is an open neighborhood W of S such that $\omega(x)\subset S$ for any $x\in W$.

The maximal set W which fulfils the condition (ii) is open and invariant. It is called the *region of attraction* of S. If we change the sign $+$ to $-$ and $\omega(x)$ to $\alpha(x)$ in (i) and (ii), we obtain the definition of a negatively asymptotically stable set NAS. Note that asymptotically stable sets are isolated.

3. Main result.

Let $\varphi:\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}^{n+1}$ be a continuous flow such that the set $\mathbb{R}^n\times\{0\}$ is invariant. For brevity, we write $E=\mathbb{R}^{n+1}$, $\partial E=\mathbb{R}^n\times\{0\}$.

Theorem 6. *Assume that $S\subset\partial E$ is an isolated invariant set for φ such that $H(h_{\partial E}(S))\neq 0$ (H denotes the singular homology functor). Then there exists an $x\in E\setminus\partial E$ such that either $\alpha(x)\subset S$ or $\omega(x)\subset S$.*

Proof. Let B_E be any block for S with respect to the flow φ. Then $B_{\partial E}=B_E\cap\partial E$ is a block for S with respect to $\varphi|_{\partial E}$ (note that S is automatically an isolated invariant set for $\varphi|_{\partial E}$). Suppose, contrary to our claim, that $a^+_{\partial E}=a^+_E$ and $a^-_{\partial E}=a^-_E$. Consider the following diagram, in which all vertical maps are induced by inclusions and the rows are the Churchill exact sequences for S with respect to φ and $\varphi|_{\partial E}$, respectively:

$$\begin{array}{ccccccc} \to & \check{H}^q(B_E, B_E^+) & \to & \check{H}^q(S) & \to & \check{H}^q(a_E^+) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & \check{H}^q(B_{\partial E}, B_{\partial E}^+) & \to & \check{H}^q(S) & \to & \check{H}^q(a_{\partial E}^+) & \to \end{array}$$

It is a commutative diagram (this is easy to check by the construction of the Churchill exact sequence; see Lemma 4.3, Prop. 4.6 and Th. 4.7 of [Ch]). The Five Lemma shows that $\check{H}(h_E^*(S)) \cong \check{H}(h_{\partial E}^*(S))$. In the same manner we can see that $\check{H}(h_E(S) \cong \check{H}(h_{\partial E}(S))$. From Corollary 2 we have

$$\dim H_i(h_E(S)) = \dim H_i(h_{\partial E}(S)). \tag{1}$$

Now, from the time-duality of the Conley index (see [Mr-Srz]), we obtain

$$H_{n+1-i}(h_E(S)) \cong \check{H}^i(h_E^*(S)) \cong \check{H}^i(h_{\partial E}^*(S)) \cong H_{n-i}(h_{\partial E}(S)). \tag{2}$$

Combining (1) with (2) we conclude that for all $i \in \mathbb{Z}$,

$$\dim H_i(h_{\partial E}(S)) = \dim H_{i-1}(h_{\partial E}(S)),$$

hence $\dim H_i(h_{\partial E}(S)) = 0$ for all i and this contradicts our assumption.

4. Index theorem. This section was inspired by recent work of Capietto and Garay. Let $E = \mathbb{R}^{n+1}$, $E^+ = \mathbb{R}^n \times [0,\infty)$, $E^- = \mathbb{R}^n \times (-\infty, 0]$ and $\partial E = \mathbb{R}^n \times \{0\}$. Assume that $\varphi : E^+ \times \mathbb{R} \to E^+$ is a continuous flow on E^+ (note that ∂E is automatically invariant). Following [Ca-Ga] we use the notion of a saturated set. A compact isolated invariant set $S \subset \partial E$ is called an invariant set of *type* A (or *saturated*) for the flow φ if there is a neighborhood N of S in E^+ such that $\mathrm{d}(xt_2, \partial E) < \mathrm{d}(xt_1, \partial E)$ whenever $x \in N \setminus \partial E$, $t_1, t_2 \in \mathbb{R}$ and $x[t_1, t_2] \subset N$. Similarly, a set S is of *type* R if it is of type A with respect to the backward flow.

Remark 7. A set of type A or R is an isolated invariant set with respect to the flow φ on E^+.

Definition. The *stable set* $W^+(S)$ of an isolated invariant set S is defined to be

$$\{x \in E : \omega(x) \neq \emptyset,\ \omega(x) \subset S\}$$

and the *unstable set* $W^-(S)$ is defined similarly in terms of $\alpha(x)$.

Note that we assume no special structure of S, $W^+(S)$ or $W^-(S)$, but when E is a smooth manifold and S is hyperbolic, well-known results show that $W^+(S)$ and $W^-(S)$ have a (local) manifold structure.

Definition. Let $S \subset \partial E$ be an isolated invariant set with respect to the flow φ on E^+. S is called a set of *type* A_1 iff $W^-(S) \subset \partial E$. The notion of the set of *type* R_1 is defined by reversal of time.

Proposition 8. (1) *The set S of type A (resp. R) is also of type A_1 (resp. R_1).*

(2) *If $S \subset \partial E$ is of type A_1 then $\check{H}(h_{E^+}(S)) \cong \check{H}(h_{\partial E}(S))$.*

Proof. (1) Suppose that there exists an $x_0 \in W^-(S) \setminus \partial E$. Since $\alpha(x_0)$ is contained in S there is a sequence $t_n \to -\infty$ such that $\mathrm{d}(x_0 t_n, \partial E) \to 0$. Let B be an isolating block for S in E^+. We may assume that $x_0 t_n \in B$ for all n. Suppose that $x_0(-\infty, t_k] \subset B$ for some k. The set S is of type A, so for all $n \geq k$ we have

$$\mathrm{d}(x_0 t_n, \partial E) > \mathrm{d}(x_0 t_k, \partial E) = \varepsilon > 0,$$

a contradiction. So, we can choose a sequence $t_n^* \to -\infty$ such that $x_0 t_n^* \in \partial B$ and this contradicts $\alpha(x_0) \subset S$.

(2) As in the proof of Theorem 6, $a_{E^+}^- = a_{\partial E}^-$ gives our statement by the Five Lemma.

Remark 9. If φ is a continuous flow on a locally compact, metric space X and $S \subset X$ is an isolated invariant set then by the same method as in the proof of Prop. 8 we can show that $\check{H}(h_X(S)) \cong \check{H}(h_{W^-(S)}(S))$.

Remark 10. Consider the equation

$$\dot{x} = x, \qquad \dot{y} = -y$$

on the Euclidean plane. Let φ be the flow generated by this equation. The saddle point $(0,0)$ is of type A_1 with respect to φ restricted to the upper half-space E^+. It is easy to compute that $\check{H}^0(h_{\partial E^+}^*(\{(0,0)\})) \cong \mathbb{R}$ and $\check{H}^i(h_{E^+}^*(\{(0,0)\})) = 0$ for all $i \in \mathbb{Z}$. Hence, for a set S of type A_1 it is not necessarily true that $\check{H}(h_{E^+}^*(S)) \cong \check{H}(h_{\partial E}^*(S))$.

Proposition 11. *Assume $S \subset \partial E$ is of type A_1 and $H(h_{\partial E}(S)) \neq 0$. Then there exists an $x \in E^+ \setminus \partial E$ such that $\omega(x) \subset S$.*

Proof. We define a map $\psi : E \times \mathbb{R} \to E$ such that ψ restricted to $E^+ \times \mathbb{R}$ equals φ and if $x \in E^-$ then $\psi(x,t) = s(\varphi(s(x),t))$, where $s : E \ni (x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, x_n, -x_{n+1}) \in E$. Obviously ψ is a flow on E. Let B_E be any isolating block for S with respect to ψ. Since $W^-(S) \subset \partial E$, we have $a_E^- = a_{\partial E}^-$. As in the proof of Theorem 6, this shows that $a_{\partial E}^+$ is not a strong deformation retract of the set a_E^+. Hence there exists an $x \in B_E \setminus \partial E$ such that $\omega(x) \subset S$.

Remark 12. (1) An analogue of this result for a set of type R_1 is also valid (if we change $\omega(x)$ to $\alpha(x)$).

(2) Proposition 11 was first proved by Hofbauer (see [Ho]) in the setting of a flow induced by a C^1 vector field. In [Ca-Ga] it was proved for dynamical systems induced by a C^0 vector field and for a set of type A. Capietto and

Garay conjectured that it is also valid for any continuous flow, but their approach does not work in the general case.

COROLLARY 13. *Assume* $S \subset \partial E$ *is of type* A_1. *Then*

(1) *if* S *is an NAS set with respect to* $\varphi|_{\partial E}$, *then there is an* $x \in E^+ \setminus \partial E$ *such that* $\omega(x) \subset S$,

(2) *if* S *is a PAS set in* ∂E, *then* S *is a PAS set with respect to the flow* φ *on* E^+.

Proof. (1) It is easy to check that $H(h_{\partial E}(S)) \neq 0$.

(2) Let B be any block for S in E^+. By Th. 2.1 of [Srz], S is a PAS iff $a^-_{E^+} = \emptyset$. We know that $a^-_{E^+} = a^-_{\partial E} = \emptyset$, because S is a PAS set in ∂E.

We use the following:

LEMMA 14. (1) *If* $S \subset \partial E$ *is an isolated invariant set*, *then*

$$\chi(h_{\partial E}(S)) = (-1)^n \chi(h^*_{\partial E}(S)).$$

(2) *If* S *is of type* A_1, *then*

(a) $\chi(h_{\partial E}(S)) = \chi(h_{E^+}(S))$,

(b) $\chi(h^*_{E^+}(S)) = 0$.

Proof. (1) It is a consequence of the time-duality of the Conley index.

(2) We first prove (a). Since $a^-_{\partial E} = a^-_{E^+}$, $\check{H}(h_{\partial E}(S)) \cong \check{H}(h_{E^+}(S))$ and by Corollary 2 we get $\chi(h_{\partial E}(S)) = \chi(h_{E^+}(S))$. To prove (b) we consider a flow $\psi : E \times \mathbb{R} \to E$ defined as in the proof of Proposition 11. Assume B_E is a block for S with respect to ψ. We have the following Mayer–Vietoris exact sequence (see [Do]):

$$\ldots \to \check{H}^q(B_E, B_E^+) \to \check{H}^q(B_{E^+}, B^+_{E^+}) \oplus \check{H}^q(B_{E^-}, B^+_{E^-}) \to \check{H}^q(B_{\partial E}, B^+_{\partial E}) \to \ldots$$

where $B_X = B_E \cap X$. Note that this exact sequence exists because the triads (B_E, B_{E^+}, B_{E^-}), $(B^+_E, B^+_{E^+}, B^+_{E^-})$ are Čech excisive. Then we have

$$\chi(h^*_{E^+}(S)) + \chi(h^*_{E^-}(S)) = \chi(h^*_{\partial E}(S)) + \chi(h^*_E(S)).$$

Since $\chi(h^*_{E^-}(S)) = \chi(h^*_{E^+}(S))$, by the time duality of the Conley index we have

$$2\chi(h^*_{E^-}(S)) = (-1)^n \chi(h_{\partial E}(S)) + (-1)^{n+1} \chi(h_E(S)) = 0.$$

(The last equality follows from the fact that $a^-_E = a^-_{\partial E}$ implies that $\chi(h_E(S)) = \chi(h_{\partial E}(S))$.)

COROLLARY 15. (1) *If* S *is of type* A_1, *then* $\mathrm{I}(S, \varphi|_{\partial E}) = \mathrm{I}(S, \varphi)$.

(2) *If* S *is of type* R_1, *then* $\mathrm{I}(S, \varphi) = 0$.

Assume $\varphi : E^+ \times \mathbb{R} \to E^+$ is a continuous flow. Let $\mathrm{R}(\varphi)$ denote the set of rest points in E^+. We also assume that

$$\mathrm{R}(\varphi) \subset \Big(\bigcup_{i=1}^{p} K_i\Big) \cup \Big(\bigcup_{j=1}^{g} L_j\Big) \cup \Big(\bigcup_{k=1}^{r} M_k\Big),$$

where K_i, L_j, M_k are pairwise disjoint, $K_i \subset \partial E$ are sets of type A_1, $L_j \subset \partial E$ are sets of type R_1, and $M_k \subset E^+ \setminus \partial E$ are isolated invariant sets.

THEOREM 16. *Assume that φ is a dissipative flow* (*i.e. there exists a compact subset K of E^+ such that $x[0,\infty) \cap K \neq \emptyset$ for every $x \in E^+$*). *Then*

$$\sum_{i=1}^{p} \mathrm{I}(K_i, \varphi) + \sum_{k=1}^{r} \mathrm{I}(M_k, \varphi) = 1$$

and

$$\sum_{k=1}^{r} \mathrm{I}(M_k, \varphi) = \sum_{j=1}^{g} \mathrm{I}(L_j, \varphi|_{\partial E}).$$

Proof. Since φ is dissipative, by Th. 2.4 of [Srz] there exists a PAS set S such that E^+ is its region of attraction. If B is an isolating block for S then by Cor. 4.5 of [Srz] we have

$$\mathrm{I}(\operatorname{int} B, \varphi) = \chi(B) = \chi(E^+) = 1$$

and

$$\mathrm{I}(\operatorname{int} B \cap \partial E, \varphi|_{\partial E}) = \chi(\partial E) = 1.$$

The additivity property of the rest point index implies that

$$1 = \sum_{i=1}^{p} \mathrm{I}(K_i, \varphi) + \sum_{j=1}^{p} \mathrm{I}(L_j, \varphi) + \sum_{k=1}^{g} \mathrm{I}(M_k, \varphi)$$

and

$$1 = \sum_{i=1}^{p} \mathrm{I}(K_i, \varphi|_{\partial E}) + \sum_{j=1}^{g} \mathrm{I}(L_j, \varphi|_{\partial E}).$$

From Cor. 15, $\mathrm{I}(K_i, \varphi|_{\partial E}) = \mathrm{I}(K_i, \varphi)$ and $\mathrm{I}(L_j, \varphi) = 0$.

Remark 17. (1) In the case a flow induced by a C^0 vector field and sets of types A and R, the above theorem was proved by Capietto and Garay, but their proof fails in the general case (see Remark 3 of [Ca-Ga]).

(2) If it is true that for any isolated invariant set on a topological manifold (with or without boundary) there exists a block which is an ENR, then the similar conclusion can be drawn for a flow on a manifold with invariant boundary.

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