Certain partial differential subordinations on some Reinhardt domains in $\mathbb{C}^n$

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Abstract. We obtain an extension of Jack–Miller–Mocanu's Lemma for holomorphic mappings defined in some Reinhardt domains in $\mathbb{C}^n$. Using this result we consider first and second order partial differential subordinations for holomorphic mappings defined on the Reinhardt domain B_{2p} with $p \geq 1$.

1. Preliminaries. Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the euclidian inner product

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i$$

and the norm $\|z\| = \sqrt{\langle z, z \rangle}$, for all $z, w \in \mathbb{C}^n$.

The open euclidian ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r and the open unit euclidian ball is abbreviated by $B_1 = B$. For $n = 1$, the open ball B_r is denoted by U_r and the open unit disc is abbreviated by $U_1 = U$. The *Reinhardt domain*

$$\left\{ z \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^{2p} < r \right\}$$

is denoted by $B_{2p}(r)$, where $p \geq 1$, and the Reinhardt domain $B_{2p}(1)$ is abbreviated by $B_{2p}(1) = B_{2p}$.

The origin $(0, \ldots, 0)'$ is always denoted by 0.

As usual, by $L(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of all continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. The letter I will always represent the identity operator in $L(\mathbb{C}^n, \mathbb{C}^n)$.

The class of holomorphic mappings from a domain $G \subseteq \mathbb{C}^n$ into $\mathbb{C}^n$ is denoted by $H(G)$.

A mapping $f \in H(G)$ is said to be *locally biholomorphic* on G if its Fréchet derivative

$$Df(z) = \left(\frac{\partial f_j}{\partial z_k}(z) \right)_{1 \leq j,k \leq n}$$

as an element of $L(\mathbb{C}^n, \mathbb{C}^n)$ is nonsingular at each point $z \in G$.

A mapping $f \in H(G)$ is called *biholomorphic* on G if the inverse mapping f^{-1} exists, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$.

Let $D^2 f(z)$ be the Fréchet derivative of second order for $f \in H(G)$; then $D^2 f(z)$ is a continuous bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$ and its restriction $D^2 f(z)(u, \cdot)$ to $u \times \mathbb{C}^n$ belongs to $L(\mathbb{C}^n, \mathbb{C}^n)$.

The symbol "$'$" means the transpose of elements and matrices defined on $\mathbb{C}^n$. If $A = (a_{ij})_{1 \leq i,j \leq n}$, then $\overline{A} = (\overline{a}_{ij})_{1 \leq i,j \leq n}$, where $a_{ij} \in \mathbb{C}$.

If φ is a C^2 complex function defined on a domain $\Omega \subset \mathbb{C}^n$, then we use the following notations:

$$\frac{\partial \varphi}{\partial z}(z) = \left(\frac{\partial \varphi}{\partial z_1}(z), \ldots, \frac{\partial \varphi}{\partial z_n}(z) \right)',$$

$$\frac{\partial^2 \varphi}{\partial z^2}(z) = \left[\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(z) \right]_{1 \leq i,\, j \leq n}, \qquad \frac{\partial^2 \varphi}{\partial \overline{z} \partial z}(z) = \left[\frac{\partial^2 \varphi}{\partial \overline{z}_i \partial z_j}(z) \right]_{1 \leq i,\, j \leq n}$$

for all $z = (z_1, \ldots, z_n)' \in \Omega$.

If $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$ and $p \geq 1$, let

$$\|z\|_{2p} = \Big[\sum_{j=1}^{n} |z_j|^{2p} \Big]^{1/2p}.$$

Very recently the authors of [KO2] obtained an extension of Jack–Miller–Mocanu's Lemma [MM1] for holomorphic mappings defined on the unit ball of $\mathbb{C}^n$ with the supremum norm, and also extended the same result to holomorphic mappings defined on the unit ball with an arbitrary norm. Applications to subordinations were considered.

Another extension of Jack–Miller–Mocanu's Lemma was given in [KO4] for holomorphic mappings defined on some domains in $\mathbb{C}^n$ whose boundary is a real C^2 hypersurface. Using this result, the author of [KO1] gave some applications to subordinations of holomorphic mappings defined on some bounded domains for which the kernel function becomes infinite on the boundary.

In this paper we continue these investigations, giving another extension of Jack–Miller–Mocanu's Lemma to holomorphic mappings defined on the Reinhardt domain B_{2p} and we obtain some partial differential subordinations of first and second kind on B_{2p} with $p \geq 1$.

We shall use the following result.

LEMMA 1.1 [MM1], [MM2]. *Let $g(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$ be continuous on $\overline{U}_{r_0}$, and holomorphic on $U_{r_0} \cup \{z_0\}$, where $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$ and $n \geq 1$. If*

$$|g(z_0)| = \max\{|g(z)| : |z| \leq r_0\},$$

then

(i)
$$z_0 g'(z_0) = m g(z_0)$$

and

(ii)
$$\mathrm{Re}[1 + z_0 g''(z_0)/g'(z_0)] \geq m,$$

where $m \geq n \geq 1$.

2. Main results

THEOREM 2.1. *Let $r \in (0,1)$, $z_0 \in \overline{B}_{2p}(r)$, where $p \geq 1$, let $f \in H(B_{2p})$ with $f(0) = 0$, f locally biholomorphic at z_0 and $f \not\equiv 0$. If z_0 is defined by*

$$(2.1) \qquad \|f(z_0)\|_{2p} = \max\{\|f(z)\|_{2p} : z \in \overline{B}_{2p}(r)\},$$

then there exists $0 < s \leq [\|z_0\|_{2p}/\|f(z_0)\|_{2p}]^{2p}$ such that

$$(2.2) \quad (|z_1^0|^{2(p-1)} \bar{z}_1^0, \ldots, |z_n^0|^{2(p-1)} \bar{z}_n^0)[Df(z_0)]^{-1}$$
$$= s(|f_1(z_0)|^{2(p-1)} \overline{f_1(z_0)}, \ldots, |f_n(z_0)|^{2(p-1)} \overline{f_n(z_0)}),$$

where $z_0 = (z_1^0, \ldots, z_n^0)'$, and

$$(2.3) \qquad \mathrm{Re}\left\{ (Df(z_0)v)' \frac{\partial^2 \varphi}{\partial w^2}(w_0) Df(z_0)v \right\}$$

$$+ (\overline{Df(z_0)v})' \frac{\partial^2 \varphi}{\partial \overline{w} \partial w}(w_0) Df(z_0)v$$

$$\leq \frac{1}{s}\left\{ \overline{v}' \frac{\partial^2 \varphi}{\partial \overline{z} \partial z}(z_0)v + \mathrm{Re}\left[v' \frac{\partial^2 \varphi}{\partial z^2}(z_0)v \right] \right.$$

$$\left. - \mathrm{Re}\left[\left(\frac{\partial \varphi}{\partial z}(z_0) \right)' [Df(z_0)]^{-1} D^2 f(z_0)(v,v) \right] \right\}$$

for all $v = (v_1, \ldots, v_n)' \in \mathbb{C}^n \setminus \{0\}$, with $\mathrm{Re} \sum_{j=1}^n \bar{z}_j^0 v_j |z_j^0|^{2(p-1)} = 0$, where $\varphi(z) = \sum_{j=1}^n |z_j|^{2p}$, for all $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$ and $w_0 = f(z_0)$.

P r o o f. Let $T_{z_0}(\partial B_{2p}(r_0))$ be the real tangent space to $\partial B_{2p}(r_0)$ at z_0 and let $v \in T_{z_0}(\partial B_{2p}(r_0))$ be an arbitrary tangent vector, where $r_0 = \sum_{j=1}^n |z_j^0|^{2p}$. Since $\partial B_{2p}(r_0)$ is a C^2 hypersurface, it is well known that there exist an $\varepsilon > 0$ and a twice differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to \partial B_{2p}(r_0)$ such that $\gamma(0) = z_0$ and $\frac{d\gamma}{dt}(0) = v$. Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be given by $\varphi(z) =$

$\sum_{j=1}^{n} |z_j|^{2p}$ for all $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$, and $\beta : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by $\beta(t) = (\varphi \circ f \circ \gamma)(t)$, $t \in (-\varepsilon, \varepsilon)$. Then β is a C^2 function on $(-\varepsilon, \varepsilon)$ and using (2.1) we obtain

$$\beta(0) = \sum_{j=1}^{n} |f_j(z_0)|^{2p} = \max\{\beta(t) : t \in (-\varepsilon, \varepsilon)\}.$$

Hence, $\beta'(0) = 0$ and $\beta''(0) \leq 0$.

On the other hand, a straightforward calculation yields

$$(2.4) \qquad \beta'(t) = 2\operatorname{Re}\left\{\left(\frac{\partial\varphi}{\partial w}(w(t))\right)' Df(\gamma(t))\frac{d\gamma}{dt}(t)\right\}, \quad t \in (-\varepsilon, \varepsilon),$$

where $w(t) = f(\gamma(t))$, so at $t = 0$ we deduce that

$$0 = \beta'(0) = 2\operatorname{Re}\left\{\left(\frac{\partial\varphi}{\partial w}(f(z_0))\right)' Df(z_0)v\right\}$$

$$= 2\operatorname{Re}\{p(|f_1(z_0)|^{2(p-1)}\overline{f_1(z_0)}, \ldots, |f_n(z_0)|^{2(p-1)}\overline{f_n(z_0)})Df(z_0)v\}$$

$$= 2p\operatorname{Re}\langle[\overline{Df(z_0)}]'(|f_1(z_0)|^{2(p-1)}f_1(z_0), \ldots, |f_n(z_0)|^{2(p-1)}f_n(z_0))', v\rangle.$$

Since this relation is satisfied for every tangent vector v, we conclude that

$$p[\overline{Df(z_0)}]'(|f_1(z_0)|^{2(p-1)}f_1(z_0), \ldots, |f_n(z_0)|^{2(p-1)}f_n(z_0))'$$

is a normal vector to $\partial B_{2p}(r_0)$ at z_0, hence it is collinear with the outward normal vector to $\partial B_{2p}(r_0)$ at z_0.

Let N_{z_0} be the outward normal vector to $\partial B_{2p}(r_0)$ at z_0. Then

$$N_{z_0} = \frac{\partial\varphi}{\partial\overline{z}}(z_0) = p(|z_1^0|^{2(p-1)}z_1^0, \ldots, |z_n^0|^{2(p-1)}z_n^0)'.$$

Hence we can find a real number δ such that

$$(2.5) \quad (|f_1(z_0)|^{2(p-1)}f_1(z_0), \ldots, |f_n(z_0)|^{2(p-1)}f_n(z_0))[\overline{Df(z_0)}]$$

$$= \delta(|z_1^0|^{2(p-1)}z_1^0, \ldots, |z_n^0|^{2(p-1)}z_n^0).$$

Now, since f is locally biholomorphic at z_0, it follows that $[Df(z_0)]^{-1}$ exists and from (2.5) we conclude that $\delta \neq 0$ and

$$(|z_1^0|^{2(p-1)}\overline{z}_1^0, \ldots, |z_n^0|^{2(p-1)}\overline{z}_n^0)[Df(z_0)]^{-1}$$

$$= s(|f_1(z_0)|^{2(p-1)}\overline{f_1(z_0)}, \ldots, |f_n(z_0)|^{2(p-1)}\overline{f_n(z_0)}),$$

where $s = 1/\delta$.

Consider the function

$$g(\zeta) = \sum_{j=1}^{n} f_j(\zeta z_0/\|z_0\|_{2p})|f_j(z_0)|^{2(p-1)}\overline{f_j(z_0)},$$

for all $\zeta \in \mathbb{C}$, $|\zeta| \le \|z_0\|_{2p}$. Then $g \in H(U_{\|z_0\|_{2p}} \cup \{\zeta_0\}) \cap C(\overline{U}_{\|z_0\|_{2p}})$, where $\zeta_0 = \|z_0\|_{2p}$. Also, $g(0) = 0$ and, using Hölder's inequality, it is not difficult to see that

$$|g(\zeta_0)| = \sum_{j=1}^{n} |f_j(z_0)|^{2p} = \max\{|g(\zeta)| : |\zeta| \le |\zeta_0|\};$$

hence, from Lemma 1.1, there exists $m \ge 1$ such that

(2.6) $$\zeta_0 g'(\zeta_0) = mg(\zeta_0).$$

On the other hand,

(2.7) $$\zeta_0 g'(\zeta_0) = \sum_{j=1}^{n} |f_j(z_0)|^{2(p-1)} \overline{f_j(z_0)} \sum_{k=1}^{n} \frac{\partial f_j}{\partial z_k}(z_0) z_k^0,$$

so, combining (2.6) and (2.7), we deduce that

(2.8) $$(|f_1(z_0)|^{2(p-1)} \overline{f_1(z_0)}, \dots, |f_n(z_0)|^{2(p-1)} \overline{f_n(z_0)}) Df(z_0) z_0$$
$$= m \sum_{j=1}^{n} |f_j(z_0)|^{2p}.$$

Multiplying both sides of (2.2) by $z = z_0$, and using (2.8), we obtain

(2.9) $$\frac{1}{s} \sum_{j=1}^{n} |z_j^0|^{2p} = m \sum_{j=1}^{n} |f_j(z_0)|^{2p}.$$

Since $m \ge 1$, $z_0 \ne 0$ and $f(z_0) \ne 0$, we deduce that

$$0 < s \le \sum_{j=1}^{n} |z_j^0|^{2p} / \sum_{j=1}^{n} |f_j(z_0)|^{2p}.$$

It remains to show (2.3). To do this we use (2.4) again. Differentiating it with respect to t, we obtain

$$\beta''(t) = 2\operatorname{Re}\left\{\left(Df(\gamma(t))\frac{d\gamma}{dt}(t)\right)' \frac{\partial^2 \varphi}{\partial w^2}(w(t)) Df(\gamma(t))\frac{d\gamma}{dt}(t)\right\}$$

$$+ 2\overline{\left(Df(\gamma(t))\frac{d\gamma}{dt}(t)\right)'} \frac{\partial^2 \varphi}{\partial \overline{w} \partial w}(w(t)) Df(\gamma(t))\frac{d\gamma}{dt}(t)$$

$$+ 2\operatorname{Re}\left[\left(\frac{\partial \varphi}{\partial w}(w(t))\right)' D^2 f(\gamma(t))\left(\frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t)\right)\right]$$

$$+ 2\operatorname{Re}\left[\left(\frac{\partial \varphi}{\partial w}(w(t))\right)' Df(\gamma(t))\frac{d^2\gamma}{dt^2}(t)\right],$$

where $w(t) = f(\gamma(t))$, for all $t \in (-\varepsilon, \varepsilon)$, so at $t = 0$, we deduce that

$$(2.10) \quad \operatorname{Re}\left\{(Df(z_0)v)'\frac{\partial^2\varphi}{\partial w^2}(w_0)Df(z_0)v\right\}$$

$$+ (\overline{Df(z_0)v})'\frac{\partial^2\varphi}{\partial\overline{w}\partial w}(w_0)Df(z_0)v$$

$$+ \operatorname{Re}\left\{\left(\frac{\partial\varphi}{\partial w}(w_0)\right)' D^2f(z_0)(v,v) + \left(\frac{\partial\varphi}{\partial w}(w_0)\right)' Df(z_0)\frac{d^2\gamma}{dt^2}(0)\right\} \le 0,$$

where $w_0 = f(z_0)$.

Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be given by $\alpha(t) = (\varphi \circ \gamma)(t)$ for all $t \in (-\varepsilon, \varepsilon)$. Since $\gamma(t) \in \partial B_{2p}(r_0)$ when $t \in (-\varepsilon, \varepsilon)$, it follows that $\alpha(t) = r_0$ for all $t \in (-\varepsilon, \varepsilon)$, hence $\alpha'(t) \equiv \alpha''(t) \equiv 0$ on $(-\varepsilon, \varepsilon)$.

Since

$$\alpha'(t) = 2\operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(\gamma(t))\right)'\frac{d\gamma}{dt}(t)\right]$$

and

$$\alpha''(t) = 2\left\{\operatorname{Re}\left[\left(\frac{d\gamma}{dt}(t)\right)'\frac{\partial^2\varphi}{\partial z^2}(\gamma(t))\frac{d\gamma}{dt}(t)\right]\right.$$

$$\left. + \left(\overline{\frac{d\gamma}{dt}(t)}\right)'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(\gamma(t))\frac{d\gamma}{dt}(t) + \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(\gamma(t))\right)'\frac{d^2\gamma}{dt^2}(t)\right]\right\},$$

for $t = 0$ we deduce that

$$0 = \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(z_0)\right)'v\right] = p\operatorname{Re}\sum_{j=1}^{n}|z_j^0|^{2(p-1)}\overline{z}_j^0 v_j$$

and also

$$(2.11) \quad \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(z_0)\right)'\frac{d^2\gamma}{dt^2}(0)\right] = -\operatorname{Re}\left[v'\frac{\partial^2\varphi}{\partial z^2}(z_0)v\right] - \overline{v}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(z_0)v.$$

Using (2.2), (2.10) and (2.11), we obtain

$$\operatorname{Re}\left\{(Df(z_0)v)'\frac{\partial^2\varphi}{\partial w^2}(w_0)Df(z_0)v\right\} + (\overline{Df(z_0)v})'\frac{\partial^2\varphi}{\partial\overline{w}\partial w}(w_0)Df(z_0)v$$

$$\le \frac{1}{s}\left\{\overline{v}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(z_0)v + \operatorname{Re}\left[v'\frac{\partial^2\varphi}{\partial z^2}(z_0)v\right]\right.$$

$$\left. - \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(z_0)\right)'[Df(z_0)]^{-1}D^2f(z_0)(v,v)\right]\right\},$$

i.e. the relation (2.3). The proof is complete.

R e m a r k 2.1. The result of Theorem 2.1 has the following geometric interpretation: since f is locally biholomorphic at z_0, there exists a neigh-

borhood V of z_0 such that f is biholomorphic on V. Then $f(\partial B_{2p}(r_0) \cap V)$ is a real hypersurface in $\mathbb{C}^n$ and an outward normal vector at $w_0 = f(z_0)$ is given by $N_{w_0} = \frac{\partial \phi}{\partial \bar{w}}(w_0)$, where $\phi(w) = \varphi(f^{-1}(w))$, for all $w \in f(V)$, i.e.

$$N_{w_0} = p(\overline{[Df(z_0)]^{-1}})'(|z_1^0|^{2(p-1)}z_1^0, \ldots, |z_n^0|^{2(p-1)}z_n^0)'.$$

Using (2.1), we deduce that the vectors N_{w_0} and

$$p(|f_1(z_0)|^{2(p-1)}f_1(z_0), \ldots, |f_n(z_0)|^{2(p-1)}f_n(z_0))'$$

are collinear and have the same sense.

For $p = 1$ in Theorem 2.1, we obtain the following result:

COROLLARY 2.1 [KO4]. *Let $z_0 \in B \setminus \{0\}$, and let $f \in H(B)$ be such that $f(0) = 0$, f is locally biholomorphic at z_0 and $f \not\equiv 0$. If*

$$\|f(z_0)\| = \max\{\|f(z)\| : \|z\| \le \|z_0\|\},$$

then there exists $0 < s \le \|z_0\|^2/\|f(z_0)\|^2$ such that

$$([Df(z_0)]^{-1})'(\bar{z}_0) = s\overline{f(z_0)}$$

and

$$\|v\|^2 - \operatorname{Re}\{\bar{z}_0'[Df(z_0)]^{-1}D^2f(z_0)(v,v)\} \ge s\|Df(z_0)v\|^2,$$

for all $v \in \mathbb{C}^n \setminus \{0\}$ with $\operatorname{Re}\langle z_0, v \rangle = 0$.

P r o o f. In this case it is sufficient to put $\varphi(z) = \sum_{k=1}^n |z_k|^2$ in Theorem 2.1 for all $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$. Then $\frac{\partial \varphi}{\partial z}(z) = \bar{z}$, $\frac{\partial^2 \varphi}{\partial z^2}(z) = 0$ (the null matrix) and $\frac{\partial^2 \varphi}{\partial \bar{z} \partial z}(z) = I$, for all $z \in \mathbb{C}^n$, hence the asserted relations are immediate.

A direct consequence of Theorem 2.1 is given in the next result:

THEOREM 2.2. *Let $f \in H(B_{2p})$, where $p \ge 1$, and let g be a biholomorphic mapping on $B_{2p}(r)$ for some $r > 1$, with $f(0) = g(0)$. If there exist $z_0 \in \partial B_{2p}(r_0)$, with $0 < r_0 < 1$, and $\zeta_0 \in \partial B_{2p}$ such that f is locally biholomorphic at z_0,*

$$(2.12) \qquad f(z_0) = g(\zeta_0) \quad and \quad f(B_{2p}(r_0)) \subset g(B_{2p}),$$

then there exists $0 < s \le [\|z_0\|_{2p}]^{2p}$ such that the following relations hold at $z = z_0$:

$$(2.13) \quad (|z_1^0|^{2(p-1)}\bar{z}_1^0, \ldots, |z_n^0|^{2(p-1)}\bar{z}_n^0)[Df(z_0)]^{-1}$$
$$= s(|\zeta_1^0|^{2(p-1)}\bar{\zeta}_1^0, \ldots, |\zeta_n^0|^{2(p-1)}\bar{\zeta}_n^0)[Dg(\zeta_0)]^{-1},$$

and

$$(2.14) \quad \operatorname{Re}\left\{w'\frac{\partial^2\varphi}{\partial z^2}(\zeta_0)w\right\} + \overline{w}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(\zeta_0)w$$

$$\leq \frac{1}{s}\left\{\operatorname{Re}\left[v'\frac{\partial^2\varphi}{\partial z^2}(z_0)v\right] + \overline{v}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(z_0)v\right.$$

$$- \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(z_0)\right)'[Df(z_0)]^{-1}D^2f(z_0)(v,v)\right]$$

$$\left. + \operatorname{Re}\left[\left(\frac{\partial\varphi}{\partial z}(z_0)\right)'[Df(z_0)]^{-1}D^2g(\zeta_0)(w,w)\right]\right\},$$

for all $v \in \mathbb{C}^n \setminus \{0\}$ *with* $\operatorname{Re}\left\langle\frac{\partial\varphi}{\partial\overline{z}}(z_0),v\right\rangle = 0$ *and* $w = [Dg(\zeta_0)]^{-1}Df(z_0)v$, *with* $z_0 = (z_1^0,\ldots,z_n^0)'$, $\zeta_0 = (\zeta_1^0,\ldots,\zeta_n^0)'$ *and* $\varphi(z) = \sum_{k=1}^n |z_k|^{2p}$, $z \in \mathbb{C}^n$.

P r o o f. Let $h : \overline{B}_{2p}(r_0) \to \overline{B}_{2p}$, and $h(z) = (g^{-1} \circ f)(z)$, for all $z \in \overline{B}_{2p}(r_0)$. Since g is biholomorphic on $B_{2p}(r)$, for some $r > 1$, we have $h \in H(\overline{B}_{2p}(r_0))$, $h(0) = 0$, $h(z_0) = \zeta_0$, and from (2.12) we infer that $h(\overline{B}_{2p}(r_0)) \subseteq \overline{B}_{2p}$. Also, h is locally biholomorphic at z_0, hence we deduce that

$$1 = \|h(z_0)\|_{2p} = \max\{\|h(z)\|_{2p} : z \in \overline{B}_{2p}(r_0)\}.$$

Using Theorem 2.1, there exists $0 < s \leq [\|z_0\|_{2p}]^{2p}$ such that

$$(|z_1^0|^{2(p-1)}\overline{z}_1^0,\ldots,|z_n^0|^{2(p-1)}\overline{z}_n^0)[Dh(z_0)]^{-1}$$
$$= s(|\zeta_1^0|^{2(p-1)}\overline{\zeta}_1^0,\ldots,|\zeta_n^0|^{2(p-1)}\overline{\zeta}_n^0)[Dg(\zeta_0)]^{-1},$$

i.e. the relation (2.13).

On the other hand, since $Df(z_0) = Dg(\zeta_0)Dh(z_0)$ and since $D^2f(z_0)$ is a continuous bilinear operator, we easily obtain

$$(2.15) \qquad D^2f(z_0)(v,v) = D^2f(\zeta_0)(Dh(z_0)v, Dh(z_0)v)$$
$$+ Dg(\zeta_0)D^2h(z_0)(v,v)$$

for all $v \in \mathbb{C}^n$. Multiplying both sides of (2.15) by

$$p(|z_1^0|^{2(p-1)}\overline{z}_1^0,\ldots,|z_n^0|^{2(p-1)}\overline{z}_n^0)[Df(z_0)]^{-1}$$

and using (2.13), we obtain

$$(2.16) \quad p(|z_1^0|^{2(p-1)}\overline{z}_1^0,\ldots,|z_n^0|^{2(p-1)}\overline{z}_n^0)[Df(z_0)]^{-1}D^2f(z_0)(v,v)$$
$$= p(|z_1^0|^{2(p-1)}\overline{z}_1^0,\ldots,|z_n^0|^{2(p-1)}\overline{z}_n^0)[Df(z_0)]^{-1}D^2g(\zeta_0)(w,w)$$
$$+ p(|z_1^0|^{2(p-1)}\overline{z}_1^0,\ldots,|z_n^0|^{2(p-1)}\overline{z}_n^0)[Dh(z_0)]^{-1}D^2h(z_0)(v,v),$$

where $w = Dh(z_0)v$. From (2.16) we get

(2.17) $\left(\dfrac{\partial\varphi}{\partial z}(z_0)\right)'[Df(z_0)]^{-1}D^2f(z_0)(v,v)$

$$= \left(\dfrac{\partial\varphi}{\partial z}(z_0)\right)'[Df(z_0)]^{-1}D^2g(\zeta_0)(w,w)$$

$$+ \left(\dfrac{\partial\varphi}{\partial z}(z_0)\right)'[Dh(z_0)]^{-1}D^2h(z_0)(v,v).$$

Now, it is enough to take the real part in (2.17) and to use the relation (2.3) to obtain (2.14).

For the next result we use the following definition:

DEFINITION 2.1. Let $f, g \in H(B_{2p})$. We say that f is *subordinate* to g (written $f \prec g$) if there exists $\omega \in H(B_{2p})$ such that $\omega(0) = 0$, $\omega(B_{2p}) \subset B_{2p}$ and $f(z) = (g \circ \omega)(z)$, for all $z \in B_{2p}$.

R e m a r k 2.2. It is clear that if $f \prec g$, then $f(0) = 0$ and $f(B_{2p}) \subset g(B_{2p})$. But, if g is biholomorphic on B_{2p}, we deduce that $f \prec g$ iff $f(0) = g(0)$ and $f(B_{2p}) \subset g(B_{2p})$. Also, by a version of the Schwarz Lemma (see, for example, Theorem 1 in [GW] or Theorem 6 in [C], p. 55), if $f \prec g$, then $f(B_{2p}(r)) \subset g(B_{2p}(r))$ for all $0 < r < 1$.

THEOREM 2.3. *Let f, g be locally biholomorphic mappings on B_{2p} and suppose that g is biholomorphic on $B_{2p}(r)$, for some $r > 1$, and $f(0) = g(0)$. If f is not subordinate to g, then there exist $0 < r_0 < 1$, $0 < s < 1$, $z_0 \in \partial B_{2p}(r_0)$ and $\zeta_0 \in \partial B_{2p}$ such that the relations (2.12)–(2.14) hold.*

P r o o f. Since $f(0) = g(0)$ and $f \not\prec g$, it follows that $f(B_{2p}) \not\subset g(B_{2p})$. It is easy to see that there exists an $\varepsilon > 0$ sufficiently small such that $f(B_{2p}(\varepsilon)) \subset g(B_{2p})$. Hence, if we put

$$r_0 = \sup\{r > 0 : f(B_{2p}(r)) \subset g(B_{2p})\},$$

then $0 < r_0 < 1$ and $f(\overline{B}_{2p}(r_0)) \subseteq g(\overline{B}_{2p}) \setminus g(B_{2p})$. So, we can find points $z_0 \in \partial B_{2p}(r_0)$ and $\zeta_0 \in \partial B_{2p}$ with $f(z_0) = g(\zeta_0)$. Now it is sufficient to apply Theorem 2.1.

DEFINITION 2.2. Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$ and $D \subseteq \mathbb{C}^n$ be two domains, and let g be a biholomorphic mapping on $B_{2p}(r)$, for some $r > 1$, where $p \geq 1$. Suppose that $(g(0), 0) \in \Omega$ and $Q(g) \subset \Omega$, where

$$Q(g) = \bigcup_{\substack{0 < s < 1 \\ \zeta \in \partial B_{2p}}} Q_s(g, \zeta)$$

and

$$Q_s(g, \zeta) = \{(u, v) \in \mathbb{C}^n \times \mathbb{C}^n : u = g(\zeta),$$
$$v = s([Dg(\zeta)]^{-1})'(\overline{\zeta}_1|\zeta_1|^{2(p-1)}, \ldots, \overline{\zeta}_n|\zeta_n|^{2(p-1)})'\},$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)'$. The class $X_{2p}(\Omega, D, g)$ consists of those mappings $\psi : \Omega \times B_{2p} \to \mathbb{C}^n$ which satisfy the following conditions:

$$(2.18) \qquad\qquad \psi(g(0), 0; 0) \in D$$

and

$$(2.19) \qquad\qquad \psi(u, v; z) \notin D$$

for all $(u, v) \in Q(g)$ and $z \in B_{2p}$.

Using the above definition and Theorem 2.3, we obtain

THEOREM 2.4. *Let f be a locally biholomorphic mapping on B_{2p} and g be a biholomorphic mapping on $B_{2p}(r)$ for some $r > 1$, with $f(0) = g(0)$. If*

$$(f(z), ([Df(z)]^{-1})'(\bar{z}_1|z_1|^{2(p-1)}, \ldots, \bar{z}_n|z_n|^{2(p-1)})') \in \Omega$$

and if there exists a mapping $\psi \in X_{2p}(\Omega, D, g)$ such that

$$\psi(f(z), ([Df(z)]^{-1})'(\bar{z}_1|z_1|^{2(p-1)}, \ldots, \bar{z}_n|z_n|^{2(p-1)})'; z) \in D,$$

for all $z \in B_{2p}$, then $f \prec g$.

P r o o f. If the subordination $f \prec g$ does not hold, then, in view of Theorem 2.3, there exist $z_0 \in B_{2p}$, $\zeta_0 \in \partial B_{2p}$ and $0 < s \leq [\|z_0\|_{2p}]^{2p} < 1$ such that the relations (2.12)–(2.14) hold.

Let $u = f(z_0)$ and $v = ([Df(z_0)]^{-1})'(\bar{z}_1^0|z_1^0|^{2(p-1)}, \ldots, \bar{z}_n^0|z_n^0|^{2(p-1)})'$, where $z_0 = (z_1^0, \ldots, z_n^0)'$. Then $(u, v) \in Q_s(g, \zeta_0) \subseteq Q(g)$, hence, according to Definition 2.2, we deduce that $\psi(u, v; z_0) \notin D$, contrary to the hypothesis. So, $f \prec g$.

We close this paper with the following applications. The first of them concerns a boundedness result; then we give some applications concerning subordinations by convex mappings defined on the Reinhardt domain B_{2p}.

THEOREM 2.5. *Let f be a locally biholomorphic mapping on B_{2p} with $f(0) = 0$, let a and b be two functions defined on B_{2p} with $|a(z)| - |b(z)| \geq 1$, for all $z \in B_{2p}$. If*

$$\|a(z)f(z) + b(z)(\overline{[Df(z)]^{-1}})'(z_1|z_1|^{2(p-1)}, \ldots, z_n|z_n|^{2(p-1)})'\|_{2p} < 1,$$

for all $z \in B_{2p}$, then $\|f(z)\|_{2p} < 1$ on B_{2p}.

P r o o f. Suppose that the inequality $\|f(z)\|_{2p} < 1$ does not hold at all points of B_{2p}; then taking into account the fact that $f(0) = 0$, we can easily find a real number $r \in (0, 1)$ and $z_0 \in B_{2p} \setminus \{0\}$ such that

$$\|f(z_0)\|_{2p} = 1 = \max\{\|f(z)\|_{2p} : z \in \bar{B}_{2p}(r)\}.$$

By Theorem 2.1, there exists $0 < s \leq [\|z_0\|_{2p}]^{2p}$ such that

$$(\overline{[Df(z_0)]^{-1}})'(z_1^0|z_1^0|^{2(p-1)}, \ldots, z_n^0|z_n^0|^{2(p-1)})'$$
$$= s(f_1(z_0)|f_1(z_0)|^{2(p-1)}, \ldots, f_n(z_0)|f_n(z_0)|^{2(p-1)})',$$

where $z_0 = (z_1^0, \ldots, z_n^0)'$. Hence, we obtain

$$\|a(z_0)f(z_0) + b(z_0)(\overline{[Df(z_0)]^{-1}})'(z_1^0|z_1^0|^{2(p-1)}, \ldots, z_n^0|z_1^0|^{2(p-1)})'\|_{2p}$$
$$= \|a(z_0)f(z_0) + sb(z_0)(f_1(z_0)|f_1(z_0)|^{2(p-1)}, \ldots, f_n(z_0)|f_n(z_0)|^{2(p-1)})'\|_{2p}.$$

Since $\|\cdot\|_{2p}$ is a norm, the above equality implies

$$\|a(z_0)f(z_0) + b(z_0)(\overline{[Df(z_0)]^{-1}})'(z_1^0|z_1^0|^{2(p-1)}, \ldots, z_n^0|z_n^0|^{2(p-1)})'\|_{2p}$$
$$\geq |a(z_0)| - s|b(z_0)| \cdot \|(f_1(z_0)|f_1(z_0)|^{2(p-1)}, \ldots, f_n(z_0)|f_n(z_0)|^{2(p-1)})'\|_{2p}.$$

On the other hand, since $\|f(z_0)\|_{2p} = 1$, we deduce that

$$\|(f_1(z_0)|f_1(z_0)|^{2(p-1)}, \ldots, f_n(z_0)|f_n(z_0)|^{2(p-1)})'\|_{2p}^{2p}$$
$$= \sum_{j=1}^{n} |f_j(z_0)|^{2p}|f_j(z_0)|^{4p(p-1)} \leq \sum_{j=1}^{n} |f_j(z_0)|^{2p} = 1,$$

so

$$\|a(z_0)f(z_0) + b(z_0)(\overline{[Df(z_0)]^{-1}})'(z_1^0|z_1^0|^{2(p-1)}, \ldots, z_n^0|z_n^0|^{2(p-1)})'\|_{2p}$$
$$\geq |a(z_0)| - s|b(z_0)| \geq |a(z_0)| - |b(z_0)| \geq 1;$$

but this inequality contradicts the hypothesis.

Hence, $\|f(z)\|_{2p} < 1$, for all $z \in B_{2p}$. The proof is complete.

THEOREM 2.6. *Let g be a biholomorphic mapping on $B_{2p}(r)$, for some $r > 1$, such that $g(0) = 0$ and $g(B_{2p})$ is a convex domain in $\mathbb{C}^n$. Let ϕ be a holomorphic function on a domain $E \supset g(B_{2p})$ with $\mathrm{Re}\,\phi(g(z)) > 0$, for all $z \in B_{2p}$. Also, let A be a function defined on B_{2p} with $A(z) \geq 1$, for all $z \in B_{2p}$. If f is a locally biholomorphic mapping on B_{2p}, with $f(0) = 0$, $f(B_{2p}) \subset E$ and if*

$$(2.20) \quad A(z)f(z)$$
$$+ \phi(f(z))(\overline{[Df(z)]^{-1}})'(z_1|z_1|^{2(p-1)}, \ldots, z_n|z_n|^{2(p-1)})' \in g(B_{2p}),$$

for all $z \in B_{2p}$, then $f \prec g$.

Proof. If the subordination $f \prec g$ does not hold, then using Theorem 2.3, there exist $z_0 \in B_{2p} \setminus \{0\}$, $\zeta_0 \in \partial B_{2p}$ and $0 < s < 1$ such that $f(z_0) = g(\zeta_0)$ and

$$(|z_1^0|^{2(p-1)}\bar{z}_1^0, \ldots, |z_n^0|^{2(p-1)}\bar{z}_n^0)[Df(z_0)]^{-1}$$
$$= s(|\zeta_1^0|^{2(p-1)}\overline{\zeta}_1^0, \ldots, |\zeta_n^0|^{2(p-1)}\overline{\zeta}_n^0)[Dg(\zeta_0)]^{-1},$$

where $z_0 = (z_1^0, \ldots, z_n^0)'$ and $\zeta_0 = (\zeta_1^0, \ldots, \zeta_n^0)'$.

Let
$$\psi(z) = A(z)f(z) + \phi(f(z))(\overline{[Df(z)]^{-1}})'(z_1|z_1|^{2(p-1)}, \ldots, z_n|z_n|^{2(p-1)})';$$
then at $z = z_0$ we obtain

(2.21) $\psi(z_0) = A(z_0)g(\zeta_0)$
$$+ s\phi(g(\zeta_0))(\overline{[Dg(\zeta_0)]^{-1}})'(\zeta_1^0|\zeta_1^0|^{2(p-1)}, \ldots, \zeta_n^0|\zeta_n^0|^{2(p-1)})'.$$

Since $g(B_{2p})$ is a convex domain, hence starlike with respect to zero, and $g(\partial B_{2p})$ is a real hypersurface in $\mathbb{C}^n$, it follows that $g(\partial B_{2p})$ is a starlike hypersurface, so the following relation holds (see for example Theorem 2 of [GW] or Theorem 4 of [S1]):

(2.22) $\operatorname{Re}\langle g(\zeta_0), N_{w_0}\rangle \geq 0,$

where $w_0 = g(\zeta_0)$ and N_{w_0} is the outward normal vector to $g(\partial B_{2p})$ at w_0. But
$$N_{w_0} = p(\overline{[Dg(\zeta_0)]^{-1}})'(\zeta_1^0|\zeta_1^0|^{2(p-1)}, \ldots, \zeta_n^0|\zeta_n^0|^{2(p-1)})',$$
hence, using the relations (2.21) and (2.22), we deduce that

$$\operatorname{Re}\langle \psi(z_0) - g(\zeta_0), N_{w_0}\rangle = (A(z_0) - 1)\operatorname{Re}\langle g(\zeta_0), N_{w_0}\rangle$$
$$+ (s/p)\operatorname{Re}\phi(g(\zeta_0))\|N_{w_0}\|^2.$$

Now, from the hypothesis, we conclude that
$$\operatorname{Re}\langle \psi(z_0) - g(\zeta_0), N_{w_0}\rangle \geq 0.$$

On the other hand, since $g(B_{2p})$ is a convex domain, the above relation implies $\psi(z_0) \notin g(B_{2p})$, contrary to (2.20). The proof is complete.

For $g(z) = z$ in Theorem 2.6, we obtain the following result:

COROLLARY 2.2. *Let A be a function defined on B_{2p} such that $A(z) \geq 1$, for all $z \in B_{2p}$. Let ϕ be a holomorphic function defined on a domain $E \supset B_{2p}$ such that $\operatorname{Re}\phi(z) > 0$ for all $z \in B_{2p}$. If f is a locally biholomorphic mapping defined on B_{2p} with $f(0) = 0$, $f(B_{2p}) \subset E$, and if*

$$\|A(z)f(z) + \phi(f(z))(\overline{[Df(z)]^{-1}})'(z_1|z_1|^{2(p-1)}, \ldots, z_n|z_n|^{2(p-1)})'\|_{2p} < 1$$

for all $z \in B_{2p}$, then $\|f(z)\|_{2p} < 1$ on B_{2p}.

Another interesting result is obtained from Theorem 2.5 if we let $p = 1$.

COROLLARY 2.3. *Let g be a biholomorphic mapping on $\overline{B}$ such that $g(0) = 0$ and $g(B_{2p})$ is a convex domain in $\mathbb{C}^n$, let ϕ be a holomorphic function defined on a domain $E \supset g(B)$ with $\operatorname{Re}\phi(g(z)) > 0$, for all $z \in B$, and let A be a function defined on B with $A(z) \geq 1$, for all $z \in B$. If $f \in H(B)$ is locally biholomorphic on B, $f(0) = 0$, and if*

$$A(z)f(z) + \phi(f(z))(\overline{[Df(z)]^{-1}})'z \in g(B),$$

for all $z \in B$, then $f \prec g$.

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