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Hyperbolic homeomorphisms and bishadowing

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Abstract. Hyperbolic homeomorphisms on compact manifolds are shown to have both inverse shadowing and bishadowing properties with respect to a class of δ-methods which are represented by continuous mappings from the manifold into the space of bi-infinite sequences in the manifold with the product topology. Topologically stable homeomorphisms and expanding mappings are also considered.

1. Introduction. The shadowing or pseudo-orbit tracing property of a dynamical system is often used to justify the validity of computer simulations of the system, asserting that there is a true orbit of the system close to the computed pseudo-orbit. The property was first established for systems generated by hyperbolic diffeomorphisms and later for those generated by hyperbolic homeomorphisms [1, 7, 9, 10, 11].

In numerical calculations an inverse form of the shadowing concept is also of some interest: can every orbit of the system be shadowed by a numerical trajectory calculated by the specific computational routines and procedures under consideration? A composite concept of bishadowing, combining both direct and inverse shadowing, was proposed by Diamond *et al.* [6, 4, 5] and shown to hold for systems generated by semi-hyperbolic Lipschitz mappings with the pseudo-orbits being true orbits of nearby continuous mappings. An appropriate choice of the class of admissible pseudo-orbits is crucial here, for Corless and Pilyugin [2] have shown that diffeomorphisms satisfying a strong transversality condition are not inverse shadowing if this class is too large.

Shadowing and hyperbolicity, or some modification of hyperbolicity, in a dynamical system are closely entwined. Indeed, it is now known [9] that hyperbolic homeomorphisms on a compact manifold as defined by Mañé [7] are characterized equivalently by expansivity and the shadowing property. In this paper it will be shown that such mappings also have inverse shad-

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owing and bishadowing properties with respect to a class of pseudo-orbits corresponding to "methods" which are represented by continuous mappings from the manifold into the space of bi-infinite sequences in the manifold with the product topology. Definitions are presented in Section 2 and the result formulated as a theorem and proved in Section 3. Finally, in Section 4 it is briefly shown that the result also extends to topologically stable mappings and to expanding mappings.

2. Shadowing, inverse shadowing and bishadowing. Let (X,d) be a compact metric space with metric d and let $f : X \to X$ be a homeomorphism of X onto itself.

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an *orbit* of f if $x_{n+1} = fx_n$ for all $n \in \mathbb{Z}$, and a *δ-pseudo-orbit* of f if

$$d(fx_n, x_{n+1}) \le \delta \quad \text{for all } n \in \mathbb{Z}.$$

We say that the homeomorphism f has the *shadowing property* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{y_n\}_{n\in\mathbb{Z}}$ is ε-shadowed by an orbit $\{f^n x\}_{n\in\mathbb{Z}}$ of f for some $x \in X$, i.e.

$$d(f^n x, y_n) \le \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

Recall that a homeomorphism f is *expansive* if there exists a constant $\zeta > 0$ such that $d(f^n x, f^n y) \le \zeta$ for all $n \in \mathbb{Z}$ implies that $x = y$. Hyperbolic homeomorphisms [7] are expansive and also have the shadowing property. In fact, they are characterized by such properties, that is, a homeomorphism is hyperbolic if and only if it is expansive and has the shadowing property [9].

The concepts of *inverse shadowing* and *bishadowing* are relatively recent [2, 6, 4, 5, 12]. Let $X^{\mathbb{Z}}$ denote the compact space (with the product topology) of all two-sided sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ with components $x_n \in X$ and let $\mathbf{\Phi}_f(\delta) \subset X^{\mathbb{Z}}$ be the set of all δ-pseudo-orbits of a homeomorphism f for a given $\delta > 0$. We call a mapping $\varphi : X \to \mathbf{\Phi}_f(\delta)$ a *method of accuracy δ* of f, or just a *δ-method* for short, and note that the image of the space X under a δ-method is a complete family of δ-pseudo-trajectories in the terminology of Corless and Pilyugin [2, 12].

EXAMPLE 1. The orbits of a one-to-one mapping $g : X \to X$ with $D_\infty(g,f) < \delta$, where $D_\infty(g,f) = \sup_{x\in X} d(gx, fx)$, is a δ-method with φ defined by $\varphi(x) = \mathrm{orbit}_g(x)$ for each $x \in X$.

A δ-method need not, however, be generated by a single mapping.

EXAMPLE 2. Consider two functions $g_i : X \to X$ with $D_\infty(g_i, f) \le \delta$, $i = 1, 2$, and for each point $x \in X$ define a two-sided sequence $\varphi(x) \in \mathbf{\Phi}_f(\delta)$ by

$$\varphi(x) = \{\ldots, f^{-2}x, f^{-1}x, x, g_{\alpha_1}x, g_{\alpha_2}g_{\alpha_1}x, g_{\alpha_3}g_{\alpha_2}g_{\alpha_1}x, \ldots\},$$

with $\alpha_1, \alpha_2, \alpha_3, \ldots \in \{1, 2\}$, where x is the 0th component. Such a δ-method φ arises when a two-step process is used to simulate the positive orbits of the dynamical system generated by f.

Let $\mathcal{T}$ be a family of methods containing the 0-methods, i.e. methods generated by the homeomorphism f itself.

DEFINITION 1. A homeomorphism f is said to be

(i) *inverse shadowing* with respect to the class $\mathcal{T}$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}$ and any point $y \in X$ there exists a point $x \in X$ for which

$$d(f^n y, \varphi(x)_n) \le \varepsilon \quad \text{for all } n \in \mathbb{Z};$$

(ii) *bishadowing* with respect to the class $\mathcal{T}$ if for any $\varepsilon > 0$ there is a $\delta \ge 0$ such that for any δ-method $\varphi \in \mathcal{T}$ and any δ-pseudo-orbit $\{y_n\}$ there exists a point $x \in X$ for which

$$d(y_n, \varphi(x)_n) \le \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

Note that the definitions of shadowing and inverse shadowing are included in that of bishadowing by considering 0-methods and 0-pseudo-orbits, respectively, while an application of the triangle inequality shows that shadowing and inverse shadowing combined imply bishadowing.

In numerical applications it is important to know just how δ depends on ε in the above definitions. In most proofs of the Shadowing Lemma this dependence often appears, implicitly at least, in the form $\delta \le K\varepsilon$ for some constant K provided $\varepsilon > 0$ is sufficiently small. Such a relationship was used explicitly in [6, 4, 5] by Diamond *et al.* to define inverse shadowing and bishadowing for semi-hyperbolic mappings, which they called α-robustness and (α, β)-bishadowing, respectively. We shall restate their definitions here in a slightly more general setting. Call

$$D_f(\varphi) = \sup_{\substack{x \in X \\ k \in \mathbb{Z}}} d(f\varphi(x)_k, \varphi(x)_{k+1})$$

the 1-*step deviation* of a δ-method φ with respect to a homeomorphism f, so $D_f(\varphi) \le \delta$, and note that $D_f(\varphi) \le D_\infty(g, f)$ if the method φ is generated by a single mapping g as in Example 1, while $D_f(\varphi) \le \max(D_\infty(g_1, f), D_\infty(g_2, f))$ if φ is generated by two mappings g_1 and g_2 as in Example 2.

DEFINITION 2. Let $\alpha, \beta > 0$ be fixed and let $\mathcal{T}$ be a family of methods. A homeomorphism f is said to be

(i) (α, β)-*shadowing* if for every δ-pseudo-orbit $\{y_n\}$ with $\delta \le \beta$ there exists a point $x \in X$ such that

$$d(f^n x, y_n) \le \alpha\delta \quad \text{for all } n \in \mathbb{Z};$$

(ii) (α,β)-*inverse shadowing*, or (α,β)-*robust*, with respect to the class $\mathcal{T}$ if for every β-method $\varphi \in \mathcal{T}$ and every point $y \in X$ there exists a point $x \in X$ such that

$$d(f^n y, \varphi(x)_n) \le \alpha D_f(\varphi) \quad \text{for all } n \in \mathbb{Z};$$

(iii) (α,β)-*bishadowing* with respect to the class $\mathcal{T}$ if for every δ-pseudo-orbit $\{y_n\}$ and every β-method $\varphi \in \mathcal{T}$ with

$$\delta + D_f(\varphi) \le \beta,$$

there exists a point $x \in X$ such that

$$d(y_n, \varphi(x)_n) \le \alpha\,(\delta + D_f(\varphi)) \quad \text{for all } n \in \mathbb{Z}.$$

Clearly, if a homeomorphism is (α_1,β_1)-shadowing and (α_2,β_2)-inverse shadowing with respect to some class of methods $\mathcal{T}$, then it is (α,β)-bishadowing with respect to the same class $\mathcal{T}$ for $\alpha = \max(\alpha_1,\alpha_2)$ and $\beta = \min(\beta_1,\beta_2)$. Moreover, (α,β)-bishadowing implies both (α,β)-shadowing and (α,β)-inverse shadowing.

3. Main result. Since it has been shown by Corless and Pilyugin [2] that diffeomorphisms satisfying a strong transversality condition are not inverse shadowing with respect to the class of all possible methods, we restrict attention here to the class $\mathcal{T}_c$ of all methods φ that are continuous as functions from X to $X^{\mathbb{Z}}$. Note that the methods in Examples 1 and 2 are in the class $\mathcal{T}_c$ when g is a homeomorphism and when g_1 and g_2 are continuous.

Our main result is

THEOREM 1. *Let X be a compact manifold and let $f : X \to X$ be a hyperbolic homeomorphism. Then f is bishadowing with respect to the class of methods $\mathcal{T}_c$. If in addition f is (α,β)-shadowing, then it is also (α,β_1)-bishadowing for some sufficiently small $\beta_1 > 0$.*

Proof. To prove the first statement it is enough to show that f is inverse shadowing. Let $\zeta > 0$ be the expansiveness threshold constant for f, fix $\varepsilon \le \zeta/2$ and choose δ corresponding to this ε in the shadowing property of f.

Given any δ-method $\varphi \in \mathcal{T}_c$ we construct a map $h = h_\varphi : X \to X$ as follows. For each point $x \in X$ the sequence $\varphi(x) \in X^{\mathbb{Z}}$ is a δ-pseudo-orbit. Let $h(x)$ be a point in X which ε-shadows this δ-pseudo-orbit, i.e. we have

$$d(f^n h(x), \varphi(x)_n) \le \varepsilon \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

In fact, $h(x)$ is uniquely determined. To see this let $y \in X$ be a point for which the orbit also ε-shadows the δ-pseudo-orbit $\varphi(x)$, i.e.

$$d(f^n y, \varphi(x)_n) \le \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

By the triangle inequality and the choice of ε we have

$$d(f^n y, f^n h(x)) \le 2\varepsilon \le \zeta \quad \text{for all } n \in \mathbb{Z}.$$

Hence from the expansiveness of f it follows that $y = h(x)$.

To show that the mapping h is continuous if ε is sufficiently small we shall use the following property of an expansive homeomorphism [12, 14]. For $\eta > 0$ and a natural number N write

$$B_\eta = \{(x,y) \in X \times X : d(x,y) \le \eta\}$$

and

$$V_N = \{(x,y) \in X \times X : d(f^n x, f^n y) \le \zeta \text{ for all } |n| \le N\}.$$

Lemma 1. *For every positive η there exists an N such that $V_N \subset B_\eta$.*

Assume $\varepsilon \le \zeta/3$. Let $0 < \eta \le \zeta/3$ and choose N as in Lemma 1. Since the method $\varphi \in \mathcal{T}_c$ is continuous, there is a $\mu > 0$ such that $d(x,y) \le \mu$ implies $d(\varphi(x)_n, \varphi(y)_n) \le \eta$ for all $|n| \le N$. For $d(x,y) \le \mu$ and $|n| \le N$ we then have

$$\begin{aligned} d(f^n h(x), f^n h(y)) &\le d(f^n h(x), \varphi(x)_n) \\ &\qquad + d(\varphi(x)_n, \varphi(y)_n) + d(\varphi(y)_n, f^n h(y)) \\ &\le \varepsilon + \eta + \varepsilon \le \zeta, \end{aligned}$$

so $(h(x), h(y)) \in V_N \subset B_\eta$. Hence, h is continuous.

Setting $n = 0$ in (1) we also have $d(h(x), x) \le \varepsilon$ for all $x \in X$. From [8] it is known that a continuous map sufficiently close to the identity map on a compact manifold is surjective. Consequently, for ε sufficiently small and δ chosen as above, the mapping $h = h_\varphi$ is onto for any δ-method $\varphi \in \mathcal{T}_c$. For any point $y \in X$ we can thus find an $x \in X$ such that $h(x) = y$. From inequality (1) we then have

$$d(f^n y, \varphi(x)_n) = d(f^n h(x), \varphi(x)_n) \le \varepsilon$$

for all $n \in \mathbb{Z}$, which proves the inverse shadowing of f.

The proof of the second statement of the theorem is similar, so we leave the details to the reader; here we only note that we can choose $\beta_1 > 0$ to satisfy

$$\alpha\beta_1 \le \min\left\{\alpha\beta, \frac{\zeta}{3}, \varepsilon_0\right\},$$

where ε_0 is such that any continuous mapping $h : X \to X$ with $D_\infty(h, \mathrm{id}_X) \le \varepsilon_0$ is onto. ■

The assumption that the space is a manifold cannot be easily removed. We adapt the following counterexample from [14].

Example 3. Consider $X = \{0,1\}^{\mathbb{Z}}$ with a metric d defined by $d(x,y) = 2^{-n}$ if $x_0 = y_0$, where $n = \max\{k \ge 0 : \forall |i| < k,\ x_i = y_i\}$, and by $d(x,y) = 1$

otherwise. Let f be the shift homeomorphism on X, i.e. with $(fx)_n = x_{n+1}$, and consider a sequence of methods φ_m determined by mappings g_m defined by

$$g_m(x)_i = \begin{cases} x_i & \text{for } |i| > m, \\ x_{i+1} & \text{for } i = -m, \ldots, m-1, \\ x_m & \text{for } i = m, \end{cases}$$

for $m = 1, 2, \ldots$ These mappings g_m are homeomorphisms with $g_m^{2m+1} = \mathrm{id}_X$, while φ_m is a 2^{-m}-method with $\varphi_m \in \mathcal{T}_{\mathrm{c}}$ for each $m = 1, 2, \ldots$

The shift mapping f here is not inverse shadowing. To see this, let $\varepsilon = 1/2$ and pick δ arbitrarily and fix it. For m with $2^{-m} < \delta$ the method φ_m is a δ-method. Define $x \in X$ by $x_0 = 1$ and $x_i = 0$ for $i \neq 0$, and let $y \in X$. If $y_0 \neq 1$, then inequality (1) is violated for $n = 0$. On the other hand, if if $y_0 = 1$ then after $2m+1$ iterations the sequence $f^{2m+1}x$ has 0 as its 0th component, while the sequence $g^{2m+1}y = y$ has 1 as its 0th component and so $d(f^{2m+1}x, \varphi_m(y)_{2m+1}) = 1 > \varepsilon$.

The shift mapping f is, however, expansive (proof immediate) and shadowing [14]; in fact, it is $(2,1)$-shadowing.

4. Extensions. A homeomorphism f on a compact metric space X is *topologically stable* [12] if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any homeomorphism g on X with $D_\infty(g, f) \leq \delta$ there exists a continuous surjective map h on X with $D_\infty(h, \mathrm{id}_X) \leq \varepsilon$ and $f \circ h = h \circ g$. Let $\mathcal{T}_{\mathrm{h}}$ be the class of methods generated by homeomorphisms, choose δ corresponding to ε as in the above definition and a δ-method $\varphi \in \mathcal{T}_{\mathrm{h}}$ and fix a point $y \in X$. If g is a homeomorphism generating the method φ and $x \in X$ is a point such that $h(x) = y$, then

$$d(\varphi(x)_n, f^n y) = d(g^n x, f^n h(x)) = d(g^n x, h(g^n x)) \leq D_\infty(f, \mathrm{id}_X) \leq \varepsilon$$

for all $n \in \mathbb{Z}$. Hence f has the inverse shadowing property with respect to methods in the class $\mathcal{T}_{\mathrm{h}}$.

On the other hand, it is known [12, 14] that topological stability implies shadowing on a compact manifold, which we can combine with the preceding to conclude that topological stability on a compact manifold implies bishadowing with respect to the class $\mathcal{T}_{\mathrm{h}}$. Note, however, that $\mathcal{T}_{\mathrm{h}}$ is somewhat smaller than the class $\mathcal{T}_{\mathrm{c}}$ in Theorem 1.

Theorem 1 is also true for expanding maps if we restrict ourselves to positive orbits and positive pseudo-orbits, i.e. sequences $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, \ldots\}$. A continuous surjection $f : X \to X$ of a compact metric space X is called an *expanding map* if it is positively expansive and open [1, 13].

THEOREM 2. *An expanding map on a compact manifold is bishadowing with respect to the class* $\mathcal{T}_{\mathrm{c}}$.

The proof follows from the fact that an expanding map is shadowing (see [11] for a proof based on the inverse limit space technique) and from the proof of Theorem 1 with some obvious changes to show that the map is also inverse shadowing, and hence bishadowing.

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