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On weak solutions of functional-differential abstract nonlocal Cauchy problems

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Abstract. The existence, uniqueness and asymptotic stability of weak solutions of functional-differential abstract nonlocal Cauchy problems in a Banach space are studied. Methods of m-accretive operators and the Banach contraction theorem are applied.

1. Introduction. In this paper we study the existence, uniqueness and asymptotic stability of weak solutions of nonlocal Cauchy problems for a non-linear functional-differential evolution equation. Methods of m-accretive operators and the Banach contraction theorem are applied. The functional-differential problem considered here is of the form

$$u'(t) + A(t)u(t) = f(t, u_t), \quad t \in [0, T], \tag{1.1}$$

$$u_0 = g(u_{T^*}) \in C_0 \subset C, \quad T^* \in [t_0 + r, T], \tag{1.2}$$

where for every $t \in [0, T]$, $A(t) : X \supset D(A(t)) \to X$ is an m-accretive operator, X is a Banach space, $f : [0, T] \times C \to X$, $g : C \to C_0$, $u : [-r, T] \to X$, $u_t \in C$, $t \in [0, T]$, $C := C([-r, 0], X)$, $T > r > 0$ and t_0 is a positive constant. Also, problems of type (1.1)–(1.2) on the interval $[0, \infty)$ are investigated.

The results obtained are generalizations of those given by Kartsatos and Parrott [8] on the existence and uniqueness of a weak solution of the Cauchy problem

$$u'(t) + A(t)u(t) = f(t, u_t), \quad t \in [0, T], \tag{1.3}$$

$$u_0 = \phi \in C_0, \tag{1.4}$$

and on the existence, uniqueness and stability of a weak solution of a problem of type (1.3)–(1.4) on the interval $[0, \infty)$.

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The paper is a continuation of papers [2–4] on the existence and uniqueness of solutions of nonlocal Cauchy problems for evolution equations.

Theorems about the existence, uniqueness and stability of solutions of the abstract evolution Cauchy problem (1.3)–(1.4) in the differential version were studied by Bochenek [1], Crandall and Pazy [5], Evans [6] and Winiarska [9], [10].

2. Preliminaries. Let X be a Banach space with norm $\|\cdot\|$ and let $C := C([-r,0],X)$, where r is a positive number. The Banach space C is equipped with the norm $\|\cdot\|_C$ given by the formula

$$\|\psi\|_C := \sup_{t\in[-r,0]} \|\psi(t)\| \quad \text{for } \psi \in C.$$

Let $T > r$ and let $t_0 \in (a, T-r)$, where $a \geq 0$ will be defined in Section 4.

For a continuous function $w : [-r,T] \to X$, we denote by w_t the function belonging to C and given by the formula

$$w_t(\tau) := w(t+\tau) \quad \text{for } t \in [0,T],\ \tau \in [-r,0].$$

An operator $B : X \supset D(B) \to X$ is said to be *accretive* (see [5]) if

$$\|x_1 - x_2 + \lambda(Bx_1 - Bx_2)\| \geq \|x_1 - x_2\|$$

for every $x_1, x_2 \in D(B)$ and $\lambda > 0$.

An accretive operator $B : X \supset D(B) \to X$ is said to be *m-accretive* (see [6]) if

$$R(I + \lambda B) = X \quad \text{for all } \lambda > 0,$$

where $R(I+\lambda B)$ is the range of $I + \lambda B$.

We will need the following assumption:

Assumption (A_1). For each $t \in [0,T], A(t) : X \supset D(A(t)) \to X$ is m-accretive, and there exist $\lambda_0 > 0$, a continuous nondecreasing function $l : [0,\infty) \to [0,\infty)$ and a continuous function $h : [0,T] \to X$ such that

$$\|(I+\lambda A(t))^{-1}x - (I+\lambda A(s))^{-1}x\| \leq \lambda\|h(t)-h(s)\|l(\|x\|)$$
$$\text{for all } \lambda \in (0,\lambda_0),\ t,s \in [0,T],\ x \in \overline{D(A(t))}.$$

Assumption (A_1) implies that the set $\overline{D(A(t))}$ is independent of t (see Lemma 3.1 of [6]). Therefore, we will denote this set by $\overline{D}$.

Define

$$C_0 = \{\psi \in C : \psi(0) \in \overline{D}\}.$$

Remark 2.1. Since C_0 is a closed subset of the Banach space C, it is a complete metric space equipped with the metric ϱ_{C_0} given by the formula

$$\varrho_{C_0}(\psi_1,\psi_2) = \|\psi_1 - \psi_2\|_C, \quad \psi_1,\psi_2 \in C_0. \tag{2.1}$$

Let $f : [0,T] \times C \to X$. We will also need the following assumption:

ASSUMPTION (A_2). There exists a constant $L > 0$ such that

$$\|f(s,\psi_1) - f(s,\psi_2)\| \le L\|\psi_1 - \psi_2\|_C \quad \text{for } s \in [0,T],\ \psi_1,\psi_2 \in C,$$

and there exist a continuous nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ and a continuous function $k : [0,T] \to X$ such that

$$\|f(s_1,\psi) - f(s_2,\psi)\| \le \omega(\|\psi\|_C)\|k(s_1) - k(s_2)\| \quad \text{for } s_1, s_2 \in [0,T],\ \psi \in C.$$

3. Auxiliary theorems. Now, we formulate two definitions of weak solutions. The first was given by Evans [6], and the second by Kartsatos and Parrott [8]. Some properties of weak solutions were discussed by Kartsatos in [7].

For a given function $\widetilde{f} : [0,T] \to X$ and $x \in X$, a continuous function $u : [0,T] \to X$ is said to be a *weak solution* of the problem

$$w'(t) + A(t)w(t) = \widetilde{f}(t), \quad t \in [0,T], \quad w(0) = x,$$

if for every $\widetilde{T} \in (0,T]$ there exist a sequence $P^n = \{0 = t_{n0} < t_{n1} < \ldots < t_{nN(n)} = T(n)\}$ $(n \in \mathbb{N})$ of partitions and sequences $\{u_{nj}\}_{j=0,1,\ldots,N(n)}$, $\{\widetilde{f}_{nj}\}_{j=1,\ldots,N(n)}$ $(n \in \mathbb{N})$ of elements in X such that

(i) $\widetilde{T} \le T(n) \le T$ $(n \in \mathbb{N})$ and

$$\lim_{n\to\infty} \max_{j\in\{1,\ldots,N(n)\}} (t_{nj} - t_{n,j-1}) = 0,$$

(ii) $u_{n0} := x$ $(n \in \mathbb{N})$ and

$$\frac{u_{nj} - u_{n,j-1}}{t_{nj} - t_{n,j-1}} + A(t_{nj})u_{nj} = \widetilde{f}_{nj} \quad (j = 1,\ldots,N(n);\ n \in \mathbb{N}),$$

(iii) $\widetilde{f}_n$ is convergent to $\widetilde{f}$ in $L^1(0,T;X)$, where $\widetilde{f}_n(t) := \widetilde{f}_{nj}$ for $t \in (t_{n,j-1}, t_{nj}]$ $(j = 1,\ldots,N(n);\ n \in \mathbb{N})$, and u_n converges uniformly to u on $[0,T]$, where $u_n(t) := u_{nj}$ for $t \in (t_{n,j-1}, t_{nj}]$ $(j = 1,\ldots,N(n);\ n \in \mathbb{N})$.

For given functions $f : [0,T] \times C \to X$ and $\phi \in C_0$, a continuous function $u : [-r,T] \to X$ is said to be a *weak solution* of the problem

$$w'(t) + A(t)w(t) = f(t,w_t), \quad t \in [0,T], \quad w_0 = \phi, \tag{3.1}$$

if $u(t) = \phi(t)$ for $t \in [-r,0]$ and u is a weak solution of the problem

$$w'(t) + A(t)w(t) = f(t,u_t), \quad t \in [0,T], \quad w(0) = \phi(0).$$

Now, we formulate two theorems which are consequences of the results obtained by Kartsatos and Parrott [8].

THEOREM 3.1. *Suppose that the operators* $A(t)$, $t \in [0,T]$, *and the function* f *satisfy Assumptions* (A_1) *and* (A_2). *Then for each* $\phi \in C_0$ *there exists* *exactly one weak solution of problem* (3.1). *Moreover*, *if* $\alpha > L$ *is such that*, *for each* $t \in [0,T]$, $A(t) - \alpha I$ *is accretive then*

$$\|u_1(t) - u_2(t)\| \le e^{-(\alpha-L)t}\|\phi_1 - \phi_2\|_C, \quad t \in [0,T],$$

where u_i $(i = 1,2)$ *is the* (*unique*) *weak solution of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,T],$$
$$w_0 = \phi_i \in C_0 \quad (i = 1,2).$$

Theorem 3.2. *Suppose that the operators* $A(t)$, $t \in [0,\infty)$, *and the function* $f : [0,\infty) \times C \to X$ *satisfy Assumptions* (A_1) *and* (A_2) *on the interval* $[0,\infty)$ *in place of* $[0,T]$. *Then for each* $\phi \in C_0$ *there exists exactly one weak solution* u_ϕ *of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,\infty), \quad w_0 = \phi.$$

Moreover, *if* $\alpha > L$ *is such that*, *for each* $t \in [0,\infty)$, $A(t) - \alpha I$ *is accretive then*

$$\|u_1(t) - u_2(t)\| \le e^{-(\alpha-L)t}\|\phi_1 - \phi_2\|_C, \quad t \in [0,\infty),$$

where u_i $(i = 1,2)$ *is the* (*unique*) *weak solution of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,\infty),$$
$$w_0 = \phi_i \in C_0 \quad (i = 1,2).$$

Consequently, u_ϕ *is asymptotically stable.*

4. Result. Let $g : C \to C_0$. We will need the following assumption:

Assumption (A_3). There exist constants $M > 0$ and $\beta \in \mathbb{R}$ such that

$$\|g(w_{\hat{T}}) - g(\widetilde{w}_{\hat{T}})\|_C \le Me^{\beta t_0}\|w - \widetilde{w}\|_{C([t_0,\hat{T}],X)}$$

for all $w, \widetilde{w} \in C([-r,T],X)$ and $\widehat{T} \in [t_0 + r, T]$.

Now, we present two theorems on weak solutions of nonlocal problems.

Theorem 4.1. *Suppose that the operators* $A(t), t \in [0,T]$, *and the functions* $f : [0,T] \times C \to X$ *and* $g : C \to C_0$ *satisfy Assumptions* (A_1)–(A_3). *Moreover*, *suppose that there is* $\alpha > L$ *such that*, *for each* $t \in [0,T]$, *the operator* $A(t) - \alpha I$ *is accretive. Then for each* $T^* \in [t_0 + r, T]$, *where* $t_0 \in (\max\{0, \ln(M)\}/(\alpha - L - \beta), T - r)$, $\ln(M) < (\alpha - L - \beta)(T - r)$ *and* $\beta < \alpha - L$, *there is a unique* $\phi_* \in C_0$ *and exactly one weak solution* $u_* : [-r,T] \to X$ *of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,T], \quad w_0 = \phi_*, \tag{4.1}$$

satisfying the condition

$$(u_*)_0 = g((u_*)_{T^*}) = \phi_*. \tag{4.2}$$

Moreover, for the (unique) weak solution u_ϕ of the problem

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,T], \quad w_0 = \phi, \tag{4.3}$$

where ϕ is an arbitrary function belonging to C_0, the following inequality holds:

$$\|u_\phi(t) - u_*(t)\| \le e^{-(\alpha-L)t}\|\phi - g((u_*)_{T^*})\|_C, \quad t \in [0,T]. \tag{4.4}$$

Proof. By Theorem 3.1, there is exactly one weak solution $u_\phi : [-r,T] \to X$ of problem (4.3), where ϕ is an arbitrary function belonging to C_0. Moreover, by Theorem 3.1, for any two functions $\phi_i \in C_0$ $(i = 1,2)$ the (unique) weak solutions u_{ϕ_i} $(i = 1,2)$ of the problems

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,T],$$
$$w_0 = \phi_i \quad (i = 1,2),$$

respectively, satisfy the inequality

$$\|u_{\phi_1}(t) - u_{\phi_2}(t)\| \le e^{-(\alpha-L)t}\|\phi_1 - \phi_2\|_C, \quad t \in [0,T]. \tag{4.5}$$

Let T^* be an arbitrary number such that $T^* \in [t_0 + r, T]$, where $t_0 \in (\max\{0, \ln(M)\}/(\alpha-L-\beta), T-r, \ln(M) < (\alpha-L-\beta)(T-r)$ and $\beta < \alpha-L$.

Next, define a mapping $F_{T^*} : C_0 \to C_0$ by the formula

$$F_{T^*}(\phi) = g((u_\phi)_{T^*}), \quad \phi \in C_0. \tag{4.6}$$

Observe that, from Remark 2.1, from (2.1) and (4.6), from Assumption (A_3), from (4.5) and from the fact that $T^* \in [t_0 + r, T]$ and $t_0 > \max\{0, \ln(M)\}/(\alpha - L - \beta)$,

$$\begin{aligned}
\varrho_{C_0}(F_{T^*}(\phi_1), F_{T^*}(\phi_2)) &= \|F_{T^*}(\phi_1) - F_{T^*}(\phi_2)\|_C = \|g((u_{\phi_1})_{T^*}) - g((u_{\phi_2})_{T^*})\|_C \\
&\le Me^{\beta t_0}\|u_{\phi_1} - u_{\phi_2}\|_{C([t_0,T^*],X)} = Me^{\beta t_0} \sup_{t\in[t_0,T^*]} \|u_{\phi_1}(t) - u_{\phi_2}(t)\| \\
&\le Me^{\beta t_0} \sup_{t\in[t_0,T^*]} e^{-(\alpha-L)t}\|\phi_1 - \phi_2\|_C \\
&\le Me^{(-\alpha+\beta+L)t_0}\|\phi_1 - \phi_2\|_C < \varrho_{C_0}(\phi_1, \phi_2) \quad \text{for } \phi_1, \phi_2 \in C_0.
\end{aligned}$$

Hence, by the Banach contraction theorem F_{T^*} has a unique fixed point $\phi_* \in C_0$. Moreover, by Theorem 3.1, there exists exactly one weak solution $u_* : [-r,T] \to X$ of problem (4.1). Obviously, condition (4.2) holds.

Finally, Theorem 3.1 implies that

$$\|u_\phi(t) - u_*(t)\| \le e^{-(\alpha-L)t}\|\phi - \phi_*\|_C, \quad t \in [0,T],$$

where u_ϕ is the unique weak solution of problem (4.3).

From the above inequality and from (4.2), we have (4.4).

The proof of Theorem 4.1 is complete.

As a consequence of Theorem 3.2 and of an argument similar to the argument from the proof of Theorem 4.1, we obtain the following theorem:

THEOREM 4.2. *Suppose that the operators* $A(t)$, $t \in [0,\infty)$, *and the functions* $f:[0,\infty)\times C \to X$ *and* $g: C \to C_0$ *satisfy Assumptions* (A_1)–(A_3) *on the interval* $[0,\infty)$ *in place of* $[0,T]$. *Moreover, suppose that there is* $\alpha > L$ *such that, for each* $t \in [0,\infty)$, *the operator* $A(t) - \alpha I$ *is accretive. Then for each* $T^* > t_0 + r$, *where* $t_0 > \max\{0, \ln(M)\}/(\alpha - L - \beta)$ *and* $\beta < \alpha - L$, *there is a unique* $\phi_* \in C_0$ *and exactly one weak solution* $u_*: [-r,\infty) \to X$ *of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,\infty), \quad w_0 = \phi_*,$$

satisfying the condition

$$(u_*)_0 = g((u_*)_{T^*}) = \phi_*.$$

Moreover, for the (*unique*) *weak solution* u_ϕ *of the problem*

$$w'(t) + A(t)w(t) = f(t, w_t), \quad t \in [0,\infty), \quad w_0 = \phi,$$

where ϕ *is an arbitrary function belonging to* C_0, *the following inequality holds*:

$$\|u_\phi(t) - u_*(t)\| \le e^{-(\alpha-L)t}\|\phi - g((u_*)_{T^*})\|_C, \quad t \in [0,\infty).$$

Consequently, u_* *is asymptotically stable.*

Remark 4.1. Let g be a function defined by the formula

$$g(\psi) = Me^{\beta t_0}\psi \quad \text{for } \psi \in C, \tag{4.7}$$

where $M > 0$, $\beta < \alpha - L$, $\alpha > L$, $\ln(M) < (\alpha - L - \beta)(T - r)$ (L is the constant from Assumption (A_3)) and $t_0 \in (\max\{0, \ln(M)\}/(\alpha - L - \beta), T - r)$.

If the following condition holds:

$$\psi \in C \ \Rightarrow\ Me^{\beta t_0}\psi(0) \in \overline{D}$$

then $g: C \to C_0$.

Observe that

$$\begin{aligned}\|g(w_{\hat T}) - g(\widetilde{w}_{\hat T})\|_C &= Me^{\beta t_0}\|w_{\hat T} - \widetilde{w}_{\hat T}\|_C = Me^{\beta t_0}\sup_{t\in[-r,0]}\|w_{\hat T}(t) - \widetilde{w}_{\hat T}(t)\| \\ &= Me^{\beta t_0}\sup_{t\in[-r,0]}\|w(t+\widehat{T}) - \widetilde{w}(t+\widehat{T})\| \\ &\le Me^{\beta t_0}\|w - \widetilde{w}\|_{C([t_0,\hat T],X)}\end{aligned}$$

for all $w, \widetilde{w} \in C([-r,T],X)$ and $\widehat{T} \in [t_0 + r, T]$.

Consequently, g satisfies Assumption (A_3) and Theorem 4.1 can be applied if the other assumptions are satisfied. In particular, for each $T^* \in [t_0 + r, T]$ the nonlocal condition (4.2) is of the form

$$u_*(t) = Me^{\beta t_0}u_*(t + T^*) \quad \text{for } t \in [-r, 0]. \tag{4.8}$$

It is easy to see that if the interval $[0,T]$ is replaced by $[0,\infty)$ in (4.7) then g satisfies Assumption (A_3) on $[0,\infty)$ and Theorem 4.2 can be applied if M, β and t_0 satisfy the suitable assumptions of Theorem 4.2. Moreover, the nonlocal condition (4.2) is of the form (4.8).

Remark 4.2. Let g be a function defined by the formula

$$(g(\psi))(t) = \frac{Me^{\beta t_0}}{r} \int_{-r}^{t} \psi(\tau)\, d\tau \quad \text{for } \psi \in C,\ t \in [-r, 0], \tag{4.9}$$

where $M > 0$, $\beta < \alpha - L$, $\alpha > L$, $\ln(M) < (\alpha - L - \beta)(T - r)$ (L is the constant from Assumption (A_3)) and $t_0 \in (\max\{0, \ln(M)\}/(\alpha - L - \beta), T - r)$.

If the following condition holds:

$$\psi \in C \Rightarrow \frac{Me^{\beta t_0}}{r} \int_{-r}^{0} \psi(\tau)\, d\tau \in \overline{D}$$

then $g : C \to C_0$.

Observe that

$$\begin{aligned}
\|g(w_{\hat{T}}) - g(\widetilde{w}_{\hat{T}})\|_C &= \sup_{t \in [-r,0]} \|(g(w_{\hat{T}}))(t) - (g(\widetilde{w}_{\hat{T}}))(t)\| \\
&= \frac{Me^{\beta t_0}}{r} \sup_{t \in [-r,0]} \Big\| \int_{-r}^{t} [w_{\hat{T}}(\tau) - \widetilde{w}_{\hat{T}}(\tau)]\, d\tau \Big\| \\
&= \frac{Me^{\beta t_0}}{r} \sup_{t \in [-r,0]} \Big\| \int_{-r}^{t} [w(\tau + \hat{T}) - \widetilde{w}(\tau + \hat{T})]\, d\tau \Big\| \\
&\le Me^{\beta t_0} \sup_{t \in [-r,0]} \sup_{\tau \in [-r,t]} \|w(\tau + \widehat{T}) - \widetilde{w}(\tau + \widehat{T})\| \\
&\le Me^{\beta t_0} \sup_{\tau \in [-r,0]} \|w(\tau + \widehat{T}) - \widetilde{w}(\tau + \widehat{T})\| \\
&\le Me^{\beta t_0} \|w - \widetilde{w}\|_{C([t_0, \hat{T}], X)}
\end{aligned}$$

for all $w, \widetilde{w} \in C([-r, T], X)$ and $\widehat{T} \in [t_0 + r, T]$.

Consequently, g satisfies Assumption (A_3) and Theorem 4.1 can be applied if the other assumptions are satisfied. In particular, for each $T^* \in [t_0 + r, T]$ the nonlocal condition (4.2) is of the form

$$u_*(t) = \frac{Me^{\beta t_0}}{r} \int_{-r}^{t} u_*(\tau + T^*)\, d\tau \quad \text{for } t \in [-r, 0]. \tag{4.10}$$

It is easy to see that if the interval $[0, T]$ is replaced by $[0, \infty)$ in (4.9) then g satisfies Assumption (A_3) on $[0, \infty)$ and Theorem 4.2 can be applied if M, β and t_0 satisfy the suitable assumptions of Theorem 4.2. Moreover, the nonlocal condition (4.2) is of the form (4.10).

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