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On the norm-closure of the class of hypercyclic operators

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Abstract. Let T be a bounded linear operator acting on a complex, separable, infinite-dimensional Hilbert space and let $f : D \to \mathbb{C}$ be an analytic function defined on an open set $D \subseteq \mathbb{C}$ which contains the spectrum of T. If T is the limit of hypercyclic operators and if f is nonconstant on every connected component of D, then $f(T)$ is the limit of hypercyclic operators if and only if $f(\sigma_{\mathrm{W}}(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected, where $\sigma_{\mathrm{W}}(T)$ denotes the Weyl spectrum of T.

1. Terminology and introduction. In this note X always denotes a complex, infinite-dimensional Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. We write $\mathcal{K}(X)$ for the ideal of all compact operators on X. For $T \in \mathcal{L}(X)$ the spectrum of T is denoted by $\sigma(T)$. The reader is referred to [5] for the definitions and properties of Fredholm operators, semi-Fredholm operators and the index $\operatorname{ind}(T)$ of a semi-Fredholm operator T in $\mathcal{L}(X)$. For $T \in \mathcal{L}(X)$ we will use the following notations:

$$\begin{aligned}
\varrho_F(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is Fredholm}\},\\
\varrho_{\text{s-F}}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\},\\
\varrho_{\mathrm{W}}(T) &= \{\lambda \in \varrho_{\mathrm{F}}(T) : \ \operatorname{ind}(\lambda I - T) = 0\},\\
\sigma_0(T) &= \{\lambda \in \sigma(T) : \lambda \text{ is isolated in } \sigma(T), \text{ and } \lambda \in \varrho_{\mathrm{F}}(T)\},\\
\sigma_{\mathrm{F}}(T) &= \mathbb{C} \setminus \varrho_{\mathrm{F}}(T), \qquad \sigma_{\text{s-F}}(T) = \mathbb{C} \setminus \varrho_{\text{s-F}}(T),\\
\sigma_{\mathrm{W}}(T) &= \mathbb{C} \setminus \varrho_{\mathrm{W}}(T) \qquad \text{(Weyl spectrum)},\\
\operatorname{Hol}(T) &= \{f : D(f) \to \mathbb{C} : D(f) \text{ is open}, \sigma(T) \subseteq D(f),\\
&\qquad f \text{ is holomorphic}\},\\
\widetilde{\operatorname{Hol}}(T) &= \{f \in \operatorname{Hol}(T) : f \text{ is nonconstant on every connected}\\
&\qquad \text{component of } D(f)\}.
\end{aligned}$$

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For $f \in \mathrm{Hol}(T)$, the operator $f(T)$ is defined by the well known analytic calculus (see [5]).

If X is separable, then $T \in \mathcal{L}(X)$ is called *hypercyclic* if $\{x, Tx, T^2x, \ldots\}$ is dense in X for some $x \in X$. We denote by $\mathcal{HC}(X)$ the class of all hypercyclic operators in $\mathcal{L}(X)$. The following simple spectral description of the norm-closure $\mathcal{HC}(X)^-$ is due to D. A. Herrero [3], Theorem 2.1:

Theorem 1. *If X is a separable Hilbert space, then $A \in \mathcal{HC}(X)^-$ if and only if A satisfies the conditions*

(1) $\sigma_{\mathrm{W}}(A) \cup \{z \in \mathbb{C} : |z| = 1\}$ *is connected*,
(2) $\sigma_0(A) = \emptyset$, *and*
(3) $\mathrm{ind}(\lambda I - A) \geq 0$ *for all* $\lambda \in \varrho_{\text{s-F}}(A)$.

Furthermore, $\mathcal{HC}(X)^- + \mathcal{K}(X) = \{A \in \mathcal{L}(X) : A$ *satisfies* (1) *and* (3)$\}$.

The main result of the present note reads as follows:

Theorem 2. *Let X be a separable Hilbert space, $T \in \mathcal{HC}(X)^-$ and let $f \in \widetilde{\mathrm{Hol}(T)}$. Then the following assertions are equivalent*:

(1) $f(T) \in \mathcal{HC}(X)^-$.
(2) $f(T) \in \mathcal{HC}(X)^- + \mathcal{K}(X)$.
(3) $f(\sigma_{\mathrm{W}}(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ *is connected.*

As an immediate consequence we have:

Corollary. *Let X, T and f be as in Theorem 2. If $\sigma_{\mathrm{W}}(T)$ is connected and $|f(\lambda_0)| = 1$ for some $\lambda_0 \in \sigma_{\mathrm{W}}(T)$, then $f(T) \in \mathcal{HC}(X)^-$.*

A result closely related to the above corollary can be found in [4], Theorem 2.

The proof of Theorem 2 will be given in Section 3 of this paper. For this proof we need some preliminary results, which we collect in Section 2. Many of these preliminary results can be found in [1], Section 3, in the Hilbert space case.

2. Preliminary results. In this section X will denote an arbitrary complex Banach space.

Proposition 1. *Let $T \in \mathcal{L}(X)$ and $f \in \mathrm{Hol}(T)$.*

(1) $f(\sigma_{\mathrm{F}}(T)) = \sigma_{\mathrm{F}}(f(T))$.
(2) $f(\sigma_{\text{s-F}}(T)) \subseteq \sigma_{\text{s-F}}(f(T))$ (*if f is univalent, we have equality*).
(3) *If $f \in \widetilde{\mathrm{Hol}(T)}$, then $\sigma_0(f(T)) \subseteq f(\sigma_0(T))$.*
(4) *If* $\mathrm{ind}(\lambda I - T) \geq 0$ *for all* $\lambda \in \varrho_{\mathrm{F}}(T)$ *or* $\mathrm{ind}(\lambda I - T) \leq 0$ *for all* $\lambda \in \varrho_{\mathrm{F}}(T)$, *then*
$$\sigma_{\mathrm{W}}(f(T)) = f(\sigma_{\mathrm{W}}(T)).$$

P r o o f. (1) $\sigma_{\mathrm{F}}(T)$ is the spectrum of $T+\mathcal{K}(X)$ in the Banach algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Hence the spectral mapping theorem holds for $\sigma_{\mathrm{F}}(T)$.

(2) See [6], Corollary 1, or [2], Theorem 1.

(3) Let $\mu_0\in\sigma_0(f(T))$; thus μ_0 is an isolated point in $\sigma(f(T))=f(\sigma(T))$ and $\mu_0\in\varrho_{\mathrm{F}}(f(T))$. We have $\mu_0=f(\lambda_0)$ for some $\lambda_0\in\sigma(T)$. By (1), $\lambda_0\in\varrho_{\mathrm{F}}(T)$. Let C denote the connected component of $D(f)$ which contains λ_0. Assume that λ_0 is not isolated in $\sigma(T)$, thus there is a sequence (λ_n) in $C\cap\sigma(T)$ such that $\lambda_n\to\lambda_0$ and $\lambda_n\neq\lambda_0$ for all $n\in\mathbb{N}$. This gives $f(\lambda_n)\to f(\lambda_0)=\mu_0$ $(n\to\infty)$. Since $f(\lambda_n)\in f(\sigma(T))=\sigma(f(T))$ and μ_0 is isolated in $\sigma(f(T))$, we derive $f(\lambda_n)=\mu_0$ for all n. By the uniqueness theorem for analytic functions, it follows that $f(\lambda)=\mu_0$ for all $\lambda\in C$, a contradiction. Thus λ_0 is an isolated point in $\sigma(T)$. Since $\lambda_0\in\varrho_{\mathrm{F}}(T)$, we get $\lambda_0\in\sigma_0(T)$, hence $\mu_0=f(\lambda_0)\in f(\sigma_0(T))$.

(4) follows from [8], Theorem 3.6. ∎

R e m a r k. In general, the spectral mapping theorem for the Weyl spectrum $\sigma_{\mathrm{W}}(T)$ does not hold (see [2], p. 23, or [8], Example 3.3).

Notations. For $T\in\mathcal{L}(X)$, we write $\alpha(T)$ for the dimension of the kernel of T and $\beta(T)$ for the co-dimension of the range of T. Thus, if T is semi-Fredholm,

$$\operatorname{ind}(T)=\alpha(T)-\beta(T)\in\mathbb{Z}\cup\{-\infty,+\infty\}.$$

According to C. Pearcy [7], the next proposition has already appeared in the preprint *Fredholm operators* by P. R. Halmos in 1967. For the convenience of the reader we shall include a proof.

Proposition 2. *If T and S are semi-Fredholm operators with $\alpha(T)$ and $\alpha(S)$ finite* [*resp. $\beta(T)$ and $\beta(S)$ finite*], *then TS is a semi-Fredholm operator with $\alpha(TS)<\infty$* [*resp. $\beta(TS)<\infty$*] *and*

$$\operatorname{ind}(TS)=\operatorname{ind}(T)+\operatorname{ind}(S).$$

P r o o f. It suffices to consider the case where $\alpha(T),\alpha(S)<\infty$.

C a s e 1: *T and S are Fredholm operators*. Then it is well known that TS is Fredholm and $\operatorname{ind}(TS)=\operatorname{ind}(T)+\operatorname{ind}(S)$ (see [5], §71).

C a s e 2: *T or S is not Fredholm*. Thus $\beta(T)=\infty$ or $\beta(S)=\infty$. Use [5], §82, Aufgaben 2, 4, to get: TS is semi-Fredholm, $\alpha(TS)<\infty$, $\beta(TS)=\infty$. Hence $\operatorname{ind}(TS)=-\infty=\operatorname{ind}(T)+\operatorname{ind}(S)$. ∎

Proposition 3. *Let $T\in\mathcal{L}(X)$ satisfy*

$$\sigma_0(T)=\emptyset \quad \textit{and} \quad \operatorname{ind}(\lambda I-T)\geq 0 \quad \textit{for all } \lambda\in\varrho_{\text{s-F}}(T).$$

If $f\in\widetilde{\mathrm{Hol}(T)}$ then we have

(1) $\sigma_0(f(T)) = \emptyset$,

(2) $\operatorname{ind}(\mu I - f(T)) \geq 0$ *for all* $\mu \in \varrho_{\text{s-F}}(f(T))$.

Proof. (1) follows from Proposition 1(3).

(2) Take $\mu_0 \in \varrho_{\text{s-F}}(f(T))$ and put $g(\lambda) = \mu_0 - f(\lambda)$. If g has no zeroes in $\sigma(T)$, then $g(T) = \mu_0 I - f(T)$ is invertible in $\mathcal{L}(X)$, thus $\operatorname{ind}(\mu_0 I - f(T)) = 0$. If g has zeroes in $\sigma(T)$, then g has only a finite number of zeroes in $\sigma(T)$, since $f \in \widetilde{\operatorname{Hol}(T)}$. Let $\lambda_1, \ldots, \lambda_k$ be those zeroes and $\nu_1, \ldots, \nu_k$ their respective orders. Then we have

$$g(\lambda) = h(\lambda) \prod_{j=1}^{k} (\lambda_j - \lambda)^{\nu_j}$$

with $h \in \operatorname{Hol}(T)$ and $h(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. Therefore $h(T)$ is invertible and

$$g(T) = h(T) \prod_{j=1}^{k} (\lambda_j I - T)^{\nu_j}.$$

Since $0 \in \varrho_{\text{s-F}}(g(T))$, we get, by Proposition 1(2),

$$\lambda_1, \ldots, \lambda_k \in \varrho_{\text{s-F}}(T).$$

Since $\operatorname{ind}(\lambda_j I - T) \geq 0$, we have

$$\beta(\lambda_j I - T) < \infty \quad \text{for } j = 1, \ldots, k.$$

Thus by Proposition 2 (recall that $\beta(h(T)) = 0 < \infty$),

$$\begin{aligned} \operatorname{ind}(\mu_0 I - f(T)) &= \operatorname{ind}(g(T)) \\ &= \underbrace{\operatorname{ind}(h(T))}_{=0} + \sum_{j=1}^{k} \nu_j \underbrace{\operatorname{ind}(\lambda_j I - T)}_{\geq 0} \geq 0. \ \blacksquare \end{aligned}$$

Remark. The description of the index in [1], Theorem 3.7, sheds more light on claim (4) of Proposition 1 and on claim (2) of Proposition 3 in the Hilbert space case.

3. Proof of Theorem 2. (1)⇒(2). Clear.

(3)⇒(1). Use Proposition 1(4), Proposition 3 and Theorem 1.

(2)⇒(3). By Theorem 1, $\sigma_W(f(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected. Use again Proposition 1(4) to derive (3). $\blacksquare$

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