On continuous solutions of a functional equation

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Abstract. This paper discusses continuous solutions of the functional equation
\( \varphi[f(x)] = g(x, \varphi(x)) \) in topological spaces.

Let us consider the equation
\[ (1) \quad \varphi[f(x)] = g(x, \varphi(x)) \]
with \( \varphi : X \to Y \) as unknown function.

In order to obtain a solution of equation (1), it is enough to extend a function defined on a set which for every \( x \) contains exactly one element of the form \( f^k(x) \), where \( k = 0, \pm 1, \pm 2, \ldots \) and \( f^k(x) \) denotes the \( k \)th iterate of the function \( f \) (cf. [3] and [4]). In the case when \( X \) is an open interval and \( Y \) is a Banach space, it is well known under what conditions these extensions are continuous (cf. [5]). Paper [6] by M. Sablik brings theorems which answer the above question for \( X \) and \( Y \) contained in some Banach spaces ([6, Th. 2.1, Th. 2.2]). In the case when \( X \) and \( Y \) are locally convex vector spaces the continuity of similar extensions was examined by W. Smajdor in [7] but for the Schröder equation (i.e. \( \varphi[f(x)] = s\varphi(x), \ 0 < |s| < 1 \)). We are going to adopt the method given in that paper to the more general situation.

We shall employ Baron’s Extension Theorem proved in [1] (cf. also [2]). This theorem concerns extending solutions of functional equations from a neighbourhood of a distinguished point (Lemma 7).

We shall deal with the following hypotheses:

(i) \( X \) is a Hausdorff topological space; \( \xi \) is a given (and fixed) point of \( X \); \( Y \) is a topological space.

(ii) The function \( f \) maps \( X \) into \( X \) in such a manner that
\[ (2) \quad f \text{ is homeomorphism of } X \text{ onto } f(X); \]
\[ (3) \quad \xi \in \text{int } f(X); \]

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\( \lim_{n \to \infty} f^n(x) = \xi \) for every \( x \in X \);

(5) each neighbourhood \( U \) of the point \( \xi \) contains a neighbourhood \( W \) of \( \xi \) such that \( \text{cl}\, f(W) \subset W \subset U \).

(iii) The function \( g : X \times Y \to Y \) is continuous; for every \( x \in X \setminus \{\xi\} \) the function \( g(x, \cdot) \) is a bijection and the function \( h : (X \setminus \{\xi\}) \times Y \to Y \) defined by
\[
h(x, y) = g(x, \cdot)^{-1}(y)
\]
is continuous.

Evidently

(6) \( f(\xi) = \xi \).

According to (3) and (5) we can find a neighbourhood \( W \) of \( \xi \) such that \( W \subset \text{int}\, f(X) \) and \( \text{cl}\, f(W) \subset W \). Obviously \( f^2(W) \subset f(W) \), thus \( \text{cl}\, f^2(W) \subset \text{cl}\, f(W) \subset W \subset f(X) \). By (2) we have
\[
\text{cl}\, f^2(W) = \text{cl}\, f^2(W) \cap f(X) = f(\text{cl}\, f(W)) \subset f(W).
\]
Putting \( V_0 := f(W) \) we obtain an open set with the following properties:

(7) \( \xi \in V_0, \ \text{cl}\, V_0 \subset \text{int}\, f(X) \),

(8) \( \text{cl}\, f(V_0) \subset V_0 \).

Moreover, by induction we have

(9) \( f^k(V_0) \) is open, \( k = 0, 1, 2, \ldots \),

(10) \( \text{cl}\, f^{k+1}(V_0) \subset f^k(V_0), \ \text{for} \ k = 0, 1, 2, \ldots \).

Fix an open set \( V_0 \) satisfying (7) and (8) and put

(11) \( A_0 := \text{cl}\, V_0 \setminus \text{cl}\, f(V_0) \),

(12) \( C_0 := \text{cl}\, V_0 \setminus V_0 \).

We have the following

**Lemma 1.**

(13) \( A_0 = C_0 \cup \text{int}\, A_0 \),

(14) \( \text{cl}\, A_0 \subset A_0 \cup f(C_0) \).

**Proof of (13).** Recalling (11) and (12) we have \( A_0 \subset C_0 \cup (A_0 \setminus C_0) \subset C_0 \cup (V_0 \setminus \text{cl}\, f(V_0)) \subset C_0 \cup \text{int}\, A_0 \). The converse inclusion follows immediately from (11), (12) and (8).

**Proof of (14).** Let \( x \in \text{cl}\, A_0 \setminus A_0 \). Then from the definition of \( A_0 \) we infer that \( x \in \text{cl}\, f(V_0) \). Since, by (9) and (11), \( f(V_0) \) is an open set disjoint from \( A_0 \), it follows that \( x \notin f(V_0) \). Applying (8), (7) and (2) we get
\[
x \in \text{cl}\, f(V_0) \setminus f(V_0) = \text{cl}\, f(V_0) \cap f(X) \setminus f(V_0) = f(\text{cl}\, V_0) \setminus f(V_0) = f(\text{cl}\, V_0 \setminus V_0) = f(C_0),
\]
which was to be proved.
Put
\begin{align}
A_k &= f^k(A_0), \quad k = 0, 1, 2, \ldots, \\
C_k &= f^k(C_0), \quad k = 0, 1, 2, \ldots
\end{align}

By continuity of $f^k$, $k = 0, 1, 2, \ldots$, from (15), (11), (10) and (7) we have
\begin{align}
\text{cl} A_k &\subset \text{cl} f^k(A_0) \subset \text{cl} f^k(\text{cl}\ V_0) \subset \text{cl} \text{cl} f^k(V_0) \subset \text{cl} f^k(V_0) \\
& \subset \text{cl} V_0 \subset \text{int} f(X) \subset f(X).
\end{align}

Using the above inclusions and induction we can derive from Lemma 1 the next one:

**Lemma 2.**
\begin{align}
A_k &= C_k \cup \text{int} A_k, \quad k = 0, 1, 2, \ldots, \\
\text{cl} A_k &\subset A_k \cup C_{k+1}, \quad k = 0, 1, 2, \ldots
\end{align}

We have

**Lemma 3.**
\begin{align}
A_k \cap A_l &= \emptyset \quad \text{for } k \neq l, \quad k, l = 0, 1, 2, \ldots
\end{align}

**Proof.** Fix $l, k \in \{0, 1, 2, \ldots\}$, $l \neq k$. Let $l \geq k + 1$. Then, by (2) and (10) we get $A_l \subset f^l(\text{cl}\ V_0) = \text{cl} f^l(V_0) \subset \text{cl} f^{k+l}(V_0) = f^k(\text{cl} f(V_0))$. Now, (20) follows from the fact that $A_k \cap f^k(\text{cl} f(V_0)) = \emptyset$.

Put
\begin{align}
P := \bigcap_{k=0}^{\infty} f^k(V_0).
\end{align}

**Lemma 4.**
\begin{align}
P &\text{ is closed;} \\
\xi &\in P; \\
f(P) &= P; \\
f(V_0 \setminus P) &\subset V_0 \setminus f(P); \\
P \neq X &\implies \xi \notin \text{int} P; \\
X \setminus P &= \bigcup_{k=0}^{\infty} [f^{-k}(V_0) \setminus P].
\end{align}

**Proof.** It follows from (10) that $\bigcap_{n=0}^{\infty} f^n(V_0) = \bigcap_{n=0}^{\infty} \text{cl} f^n(V_0)$ thus (22) is true. (23) follows from (6) and (7), and (24) results from (10). Since $f(V_0 \setminus P) = f(V_0) \setminus f(P)$, (25) follows from (8) and (24).

To prove (26) let $x \in X \setminus P$. Then, by (24), $f^k(x) \in X \setminus P$, $k = 0, 1, 2, \ldots$ and $\xi = \lim_{k \to \infty} f^k(x) \in X \setminus \text{int} P$.

Finally, (27) follows from (4) and (7).
Lemma 5.
\[ \text{cl} V_0 \setminus P = \bigcup_{k=0}^{\infty} A_k. \]

Proof. Fix \( k \in \{0, 1, 2, \ldots \} \) and \( x \in A_k \). Then \( x \in \text{cl} V_0 \) by (17). Using the definition of \( A_k \) we infer that \( x \notin f^k(\text{cl} f(V_0)) \). This implies that \( x \notin f^{k+1}(V_0) \) and, consequently, \( x \notin P \). Now, fix \( x \in \text{cl} V_0 \setminus P \). Take the smallest non-negative \( k \) such that \( x \notin f^k(V_0) \). If \( k = 0 \), then \( x \in \text{cl} V_0 \setminus V_0 \subset A_0 \). If \( k > 0 \), then either \( x \in \text{cl} f^k(V_0) \) or not. In the first case, recalling (15), we have \( x \in \text{cl} f^k(V_0) \setminus f^k(V_0) \subset A_k \). In the other case we have \( x \in \text{cl} f^{k-1}(V_0) \setminus cl f^k(V_0) = A_{k-1} \). This implies that \( x \in \bigcup_{k=0}^{\infty} A_k \).

Lemma 6. For every \( x \in X \setminus P \) the set \( A_0 \) contains exactly one element of the orbit \( C(x) := \{ f^k(x) : k = 0, \pm 1, \pm 2, \ldots \text{ and } f^k(x) \text{ is defined} \} \).

Proof. First we prove the uniqueness. Suppose that for some \( x \in X \setminus P \), \( x_0 \) and \( y_0 \) are two different elements of \( A_0 \cap C(x) \). Then there exists \( k > 0 \) such that \( y_0 = f^k(x_0) \) (otherwise we interchange \( x_0 \) and \( y_0 \)). Since \( x_0 \in \text{cl} V_0 \) we infer that \( y_0 \in f^k(\text{cl} V_0) = \text{cl} f^k(V_0) \subset \text{cl} f(V_0) \), which is impossible.

To prove the existence suppose that \( A_0 \cap C(x) = \emptyset \) for some \( x \in X \setminus P \). In view of (4) there exists an integer \( n \geq 0 \) such that \( f^n(x) \in V_0 \). Defining \( r := f^n(x) \) we have \( r \in V_0 \cap C(x) \). Since \( A_0 \cap C(x) = \emptyset \) we obtain \( r \in \text{cl} f(V_0) \), i.e. \( r \in f(X) \) in view of (8) and (7). This implies that \( f^{-1}(r) \) is defined. We have
\[ f^{-1}(r) \in f^{-1}(\text{cl} f(V_0)) \subset f^{-1}(\text{cl} f(V_0) \cap f(X)) = f^{-1}(f(\text{cl} V_0)) = \text{cl} V_0. \]
Hence \( f^{-1}(r) \in \text{cl} V_0 \cap C(x) \), which again implies that \( f^{-1}(r) \in \text{cl} f(V_0) \subset V_0 \subset f(X) \). By induction we can prove that \( f^{-i}(r) \) is defined for every integer \( i \geq 0 \) and \( f^{-i}(r) \in V_0 \). This together with the equation \( r = f^i[f^{-i}(r)], \ i = 0, 1, 2, \ldots \), implies that \( r \in P \). This yields \( x \in P \), which is impossible. Thus \( A_0 \cap C(x) = \emptyset \).

Lemma 7 (K. Baron). Let \( X \) and \( Y \) be topological spaces, \( U \subset X \) an open set, \( h : X \times Y \to Y \) and \( f : X \to X \) continuous functions. If \( f(U) \subset U \) and for every \( x \in X \) there exists a positive integer \( k \) such that \( f^k(x) \in U \), then for every solution \( \varphi_0 : U \to Y \) of the functional equation
\[ \varphi(x) = h(x, \varphi[f(x)]) \]
there exists exactly one solution \( \varphi : X \to Y \) of this equation such that \( \varphi(x) = \varphi_0(x) \), \( x \in U \). If \( \varphi_0 \) is continuous then so is \( \varphi \).

Theorem. Let hypotheses (i)–(iii) be satisfied. Let \( V_0 \) be an open set satisfying (7) and (8) and let the sets \( P, A_0, C_1 \) be defined by (21), (11) and (16). Then for every continuous function \( \psi : A_0 \cup C_1 \to Y \) such that
\[ \psi(x) = g(f^{-1}(x), \psi[f^{-1}(x)]) \quad \text{for } x \in C_1 \]
there exists exactly one solution \( \varphi : X \setminus P \to Y \) of equation (1) such that
\[
\varphi|_{A_0 \cup C_1} = \psi.
\]

**Proof.** In view of Lemma 6 the Theorem from [3] (cf. also [4, Theorem 1.1]) may be applied. It follows from that theorem and Lemma 5 that the function \( \Phi : \text{cl} V_0 \setminus P \to Y \) defined by
\[
\Phi(x) = \psi_n(x), \quad x \in A_n, \quad n \geq 0,
\]
where the functions \( \psi_n : A_n \to Y \) are given by
\[
\psi_0 = \psi|_{A_0}, \quad \psi_{n+1}(x) = g(f^{-1}(x), \psi_n[f^{-1}(x)]),
\]
is a unique solution of equation (1) on \( \text{cl} V_0 \setminus P \) such that
\[
\Phi|_{A_0} = \psi_0.
\]

We are going to prove that \( \Phi \) is continuous on \( \text{cl} V_0 \setminus P \). By definition of \( \Phi \) and Lemma 3 it follows that \( \Phi \) is continuous on \( \bigcup_{k=0}^{\infty} \text{int} A_k \). We shall show that it is also continuous on \( C_1 \). First observe that
\[
\Phi(x) = \psi(x) \quad \text{for } x \in A_0 \cup C_1.
\]
Indeed, if \( x \in C_1 \) then \( f^{-1}(x) \in C_0 \subset A_0 \) and by (30), (31) and (28) we have
\[
\Phi(x) = \psi_1(x) = g(f^{-1}(x), \psi_0[f^{-1}(x)]) = g(f^{-1}(x), \psi[f^{-1}(x)]) = \psi(x).
\]
Next, fix an \( x_0 \in C_1 \) and a neighbourhood \( U \) of \( \Phi(x_0) \). From the continuity of \( \psi \) on \( A_0 \cup C_1 \) and (33) there exists a neighbourhood \( V^1_{x_0} \) of \( x_0 \) such that
\[
\Phi(V^1_{x_0} \cap (A_0 \cup C_1)) \subset U.
\]
By the continuity of \( g(\cdot, \psi(\cdot)) \) on \( A_0 \cup C_1 \) and since \( f^{-1}(x_0) \in A_0 \) and \( g(f^{-1}(x_0), \psi[f^{-1}(x_0)]) = \Phi(x_0) \) we can find a neighbourhood \( W \) of \( f^{-1}(x_0) \) such that
\[
g(\cdot, \psi(\cdot))[W \cap (A_0 \cup C_1)] \subset U.
\]
Putting \( V^2_{x_0} := f(W) \cap V_0 \) we obtain a neighbourhood of \( x_0 \) such that
\[
\Phi(V^2_{x_0} \cap (A_1 \cup C_2)) \subset U.
\]
Indeed, for \( x \in V^2_{x_0} \cap (A_1 \cup C_2) \) we have \( f^{-1}(x) \in W \cap (A_0 \cup C_1) \) and by (33) and (35), \( \Phi(x) = g(f^{-1}(x), \Phi[f^{-1}(x)]) = g(f^{-1}(x), \psi[f^{-1}(x)]) \in U \). Now \( x_0 \in C_1 \) implies that \( x_0 \notin f(V_0) \) and by (10), \( x_0 \notin \text{cl} f^2(V_0) \). Hence \( V^2_{x_0} := V^1_{x_0} \cap V^2_{x_0} \setminus \text{cl} f^2(V_0) \) is an open neighbourhood of \( x_0 \). Moreover, since \( V^2_{x_0} \subset \text{cl} V_0 \setminus f^2(V_0) \subset A_0 \cup C_1 \) by (34) and (36) we get \( \Phi(V^2_{x_0}) \subset U \). This proves the continuity of \( \Phi \) at points of \( C_1 \). Hence the continuity of \( \Phi \) on \( C_k, \ k = 0, 1, 2, \ldots \) may be obtained by induction. From (18) and Lemma 5 we see that \( \Phi \) is continuous on \( \text{cl} V_0 \setminus P \).
Hypothesis (iii) implies that $\Phi_{|V_0 \setminus P}$ is a solution of the equation
\begin{equation}
(37) \quad \Phi(x) = h(x, \Phi[f(x)])
\end{equation}
on $V_0 \setminus P$. Observe that by (22) the set $V_0 \setminus P$ is open in $X \setminus P$ and that for every $x \in X \setminus P$ there exists $k \in \{0, 1, 2, \ldots\}$ such that $f^k(x) \in V_0 \setminus P$ (by (4) and (7)). Thus from Lemma 7 it follows that there exists exactly one solution $\varphi: X \setminus P \to Y$ of (37). It is easy to verify that the function $\varphi$ satisfies equation (1) and condition (29).

References