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On boundary-value problems for partial differential equations of order higher than two

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Abstract. We prove the existence of solutions of some boundary-value problems for partial differential equations of order higher than two. The general idea is similar to that in [1]. We make an essential use of the results of our paper [12].

1. The problem. Let $x = \chi_p(t),\ 0 < t \le T$, $p = 1, 2$, be equations of non-intersecting curves on the (x, t) plane.

In this paper we prove the existence of a solution of the problem

$$(1)\quad \mathfrak{L}u(x,t) \equiv \sum_{i=0}^{n+2}\sum_{j=0}^{m} a_{ij}(x,t)D_x^i D_t^j u(x,t) - D_x^n D_t^{m+1} u(x,t) = f(x,t),$$

where $(x,t) \in \mathbf{S}_T = \{(x,t) : \chi_1(t) < x < \chi_2(t),\ 0 < t \le T\}$, $T = \mathrm{const} < \infty$, $n, m \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $n + m > 0$ (for $n = m = 0$ equation (1) is a parabolic equation of second order, the theory of which is well known), satisfying the initial conditions

$$(2)\qquad D_t^l u(x,0) = 0, \qquad \chi_1(0) \le x \le \chi_2(0),\ l = 0, 1, \ldots, m,$$

and the boundary conditions

$$(3)\qquad \mathbf{B}_l^p u(\chi_p(t), t) \equiv \sum_{k=0}^{r_l^p} b_{kl}^p(t) D_x^k u(\chi_p(t), t) = \mathbf{g}_l^p(t),$$

where $0 < t \le T$, $p = 1, 2$, $l = 1, \ldots, l_0 = [(n+3)/2]$ (denotes the greatest integer function), $0 \le r_1^p < r_2^p < \ldots < r_{l_0}^p \le n+1$, $r_l^p \in \mathbb{N}_0$, $b_{r_l^p, l}^p(t) \ge b_0 = \mathrm{const} > 0$.

We distinguish the following four cases:

1) $r_{l_0}^p < n + 1$, $p = 1$ or $p = 2$, n is odd,

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2) $r_{l_0}^p < n+1$, $p=1$ or $p=2$, n is even,
3) $r_{l_0}^p = n+1$, $p=1$ or $p=2$, n is odd,
4) $r_{l_0}^p = n+1$, $p=1$ or $p=2$, n is even.

We shall exactly analyse cases 1) and 3). The argument in the remaining cases is similar. Note that in cases 1) and 3) we have to put $[(n-1)/2]$ boundary conditions on one of the curves χ_p and $[(n-1)/2]+1$ on the other.

Boundary-value problems in rectangular domains and for particular cases of the operator $\mathcal{L}$ and of the boundary operators $\mathbf{B}_l^p$ have been considered in many papers (see [2], [3], [4], [10] and [15]). In [14] the boundary-value problem for the equation

$$D_x^{n+2}u - D_x^n D_t u = f(x,t,u,\ldots,D_x^{n+1}u)$$

was examined. Paper [13] was devoted to the equation

$$\mathbf{L}(D_x + D_t)^n u(x,t) = f(x,t),$$

where $\mathbf{L} \equiv D_t - a(x,t)D_x^2 + b(x,t)D_x + c(x,t)$. In [5] some boundary-value problems for the equation

$$(D_x^2 - D_t)(aD_x + bD_t + c)u(x,t) = 0$$

were investigated, where a,b,c are constants and $a\cdot b \neq 0$. Moreover, in [11] Cauchy's problem for equation (1) was examined.

Note that particular cases of equation (1) describe the propagation of waves in a compressible viscous medium (see [3], [6], [17]) and some problems of magneto-hydrodynamics (see [8], [9]).

2. Assumptions. We make the following assumptions:

(A.1) There are constants a_0 and a_1 such that

$$0 < a_0 \le a_{n+2,m}(x,t) \le a_1 \quad \text{for } (x,t) \in \overline{\mathbf{S}}_T$$

($\overline{\mathbf{S}}_T$ denotes the closure of $\mathbf{S}_T$).

(A.2) The coefficients a_{ij} ($i=0,1,\ldots,n+2$, $j=0,1,\ldots,m$) are continuous in $\overline{\mathbf{S}}_T$ and satisfy the Hölder condition with respect to x with exponent α ($0<\alpha\le 1$); moreover, $a_{n+2,m}$ satisfies the Hölder condition with respect to t with exponent $\frac{1}{2}\alpha$.

(A.3) The functions χ_p ($p=1,2$) have continuous derivatives up to order $n_* = [(n+1)/2]$ in $[0,T]$ and the highest derivatives satisfy the Hölder condition

$$|\Delta_t[\chi_p^{(n_*)}(t)]| \le \text{const} \begin{cases} (\Delta t)^{\alpha/2} & \text{if } n+1 \text{ is even,} \\ (\Delta t)^{(\alpha+1)/2} & \text{if } n+1 \text{ is odd,} \end{cases}$$

where $\Delta_t[\chi(t)] \equiv \chi(t+\Delta t) - \chi(t)$, $t, t+\Delta t \in [0,T]$, $\alpha \in (0,1]$.

(A.4) The function $f(x,t)$ is defined and continuous for $(x,t)\in \mathbf{S}_T$, and satisfies the inequalities

$$|f(x,t)|\le M_f, \qquad |\Delta_x f(x,t)|\le m_f|\Delta x|^\alpha,$$

where $\Delta_x f(x,t)\equiv f(x+\Delta x,t)-f(x,t)$, $(x,t),(x+\Delta x,t)\in \overline{\mathbf{S}}_T$, M_f, $m_f=\mathrm{const}>0$, $\alpha\in(0,1]$.

(A.5) The functions $\mathbf{g}_l^p$, $p=1,2$, $l=1,\ldots,l_0$, are defined and have continuous derivatives $D_t^\nu \mathbf{g}_l^p$ ($\nu=0,1,\ldots,\mathcal{M}=[d_r/2]$, $d_r=n-r_l^p+2m+1$) in $[0,T]$ and satisfy the conditions

$$|\Delta_t[D_t^{\mathcal{M}}\mathbf{g}_l^p(t)]|\le M_g\begin{cases}(\Delta t)^{\alpha/2} & \text{if } d_r \text{ is even,}\\ (\Delta t)^{(\alpha+1)/2} & \text{if } d_r \text{ is odd,}\end{cases}$$

and $D_t^\nu \mathbf{g}_l^p(0)=0$, where $M_g=\mathrm{const}>0, 0<\alpha\le 1$.

(A.6) The functions b_{kl}^p, $p=1,2$, $l=1,\ldots,l_0$, $k=0,1,\ldots,r_l^p$, are defined in $[0,T]$ and have continuous derivatives up to order $\mathcal{M}$.

Remark. Without restricting generality, we can assume $b^p_{r_l^p,l}(t)\ge b_0\equiv 1$.

3. Solution of the problem. In all cases 1)–4) we shall seek a solution of the problem (1)–(3) in the form

$$u(x,t)=\sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \Lambda_{r_q^\sigma}(x,t;\chi_\sigma(\tau),\tau)\varphi_q^\sigma(\tau)\,d\tau+\mathbf{Z}_{\mathbf{S}_T}(x,t), \tag{4}$$

where φ_q^σ are unknown functions, $\Lambda_{r_q^\sigma}$ are the fundamental solutions of (1) constructed in [12] and

$$\mathbf{Z}_{\mathbf{S}_T}(x,t)=\iint_{\mathbf{S}_t}\Lambda_0(x,t;y,\tau)f(y,\tau)\,dy\,d\tau. \tag{5}$$

3.1. Case 1). Observe that the function u given by (4) satisfies equation (1) and initial conditions (2). Boundary conditions (3) lead to the system of equations

$$\mathbf{g}_l^p(t)=\sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \mathbf{B}_l^p\Lambda_{r_q^\sigma}(\chi_p(t),t;\chi_\sigma(\tau),\tau)\varphi_q^\sigma(\tau)\,d\tau+\mathbf{z}_l^p(t), \tag{6}$$

where $\mathbf{z}_l^p(t)=\mathbf{B}_l^p\mathbf{Z}_{\mathbf{S}_T}(\chi_p(t),t)$, $0<t\le T$, $p=1,2$, $l=1,\ldots,l_0$.

By Lemma 3 of [12] we obtain

$$\begin{aligned} &D_x^{r_l^p} w_{r_q^p}(\chi_p(\tau),t;\chi_p(\tau),\tau)\\ &\quad=\begin{cases}0, & 1\le l<q,\\ (-1)^{n-r_l^p}\sqrt{\pi}[\mathfrak{a}(\tau)]^{(n-r_l^p)/2}\Gamma^{-1}(d_r/2)(t-\tau)^{d_r/2-1}, & q\le l\le l_0,\end{cases}\end{aligned} \tag{7}$$

($p=1,2$, $l,q=1,\ldots,l_0$), where $d_r=n-r_l^p+2m+1$ and the functions $w_{r_l^p}$ are defined by formula (6) of [12], and $\mathfrak{a}(\tau)=a_{n+2,m}(\chi_p(\tau),\tau)$.

Using the definition of the operator $\mathbf{I}_\kappa$ ([12], (25)) and (7) we can write

$$(8)\qquad \int_0^t D_x^{r_l^p} w_{r_q^p}(\chi_p(\tau),t;\chi_p(\tau),\tau)\varphi_q^p(\tau)\,d\tau = c_{lq}^p \mathbf{I}_{d_r/2}([\mathbf{a}(t)]^{(n-r_l^p)/2}\varphi_q^p(t))$$

$(p=1,2,\ l,q=1,\ldots,l_0,\ 0<t\le T)$, where

$$(9)\qquad c_{lq}^p = \begin{cases} 0, & 1\le l<q,\\ (-1)^{n-r_l^p}\sqrt{\pi}, & q\le l\le l_0.\end{cases}$$

By (8) and (9) we can rewrite system (6) in the form

$$(10)\qquad \sum_{q=1}^{l_0} c_{lq}^p \mathbf{I}_{d_r/2}([\mathbf{a}(t)]^{(n-r_l^p)/2}\varphi_q^p(t)) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \mathbf{K}_{lq}^{p\sigma}(t,\tau)\varphi_p^\sigma(\tau)\,d\tau + \mathbf{z}_l^p(t) = \mathbf{g}_l^p(t),$$

where

$$(11)\qquad \mathbf{K}_{lq}^{p\sigma}(t,\tau) = \mathbf{B}_l^p \Lambda_{r_q^\sigma}(\chi_p(t),t\chi_p(\tau),\tau) - \begin{cases} 0 & \text{if } \sigma\ne p \text{ or } \sigma=p \text{ and } 1\le l<q,\\ D_x^{r_l^p} w_{r_q^p}(\chi_p(\tau),t;\chi_p(\tau),\tau) & \text{if } \sigma=p \text{ and } q\le l\le l_0,\end{cases}$$

$(p,\sigma=1,2,\ l,q=1,\ldots,l_0,\ 0<t\le T)$.

(10) is a system of first-kind Volterra equations. Using the method given by Baderko [1] and the properties of the operator $\mathbf{R}_{1/2}$ defined by formula (14) of [12], we reduce this system to a system of second-kind Volterra equations. Applying to both sides of (10) the operator $\mathbf{R}_{1/2}^{d_r}$, where $d_r = n-r_l^p+2m+1$, by Lemma 4 of [12], we obtain

$$(12)\qquad \sum_{q=1}^{l_0} c_{lq}^p[\mathbf{a}(t)]^{(n-r_l^p)/2}\varphi_q^p(t) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\mathbf{R}_{1/2}^{d_r}\Big[\int_0^t \mathbf{K}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau\Big] + \mathbf{R}_{1/2}^{d_r}[\mathbf{z}_l^p(t)] = \mathbf{R}_{1/2}^{d_r}[\mathbf{g}_l^p(t)],\qquad p=1,2,\ l=1,\ldots,l_0, 0<t\le T.$$

By Theorem 1 of [12],

$$(13)\qquad |D_t^\nu \mathbf{K}_{lq}^{p\sigma}(t,\tau)| \le \text{const}\,(t-\tau)^{(d_r-2\nu+\alpha)/2-1}$$

$(\nu=0,1,\ldots,\mathcal{M}=[d_r/2],\ d_r=n-r_l^p+2m+1,\ p,\sigma=1,2,\ l,q=1,\ldots,l_0,$ $0\le\tau<t\le T,\ 0<\alpha\le 1)$.

We consider two cases: (i) d_r is even, (ii) d_r is odd.

In case (i) the function $\mathbf{K}_{lq}^{p\sigma}$ satisfies condition (18) of Lemma 4 of [12] with $N=d_r/2$ and $\varrho=\alpha/2$; hence, in view of formula (19) of [12] we have

$$(14)\qquad \mathbf{R}_{1/2}^{d_r}\Big[\int_0^t \mathbf{K}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau\Big] = \int_0^t D_t^{d_r/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau.$$

In case (ii), $\mathbf{K}_{lq}^{p\sigma}$ satisfies the same condition with $N = (d_r - 1)/2$ and $\varrho = (\alpha+1)/2$; hence, by formula (20) of [12] we get

$$\mathbf{R}_{1/2}^{d_r}\Big[\int_0^t \mathbf{K}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau\Big] = \int_0^t \mathcal{R}_{1/2}[D_t^{d_r/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)]\varphi_q^\sigma(\tau)\,d\tau. \tag{15}$$

By (14) and (15) system (12) can be written in the form

$$\sum_{q=1}^{l_0} c_{lq}^p[\boldsymbol{\alpha}(t)]^{(n-r_l^p)/2}\varphi_q^p(t) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \overline{\mathbf{K}}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau + \overline{\mathbf{z}}_l^p(t) = \overline{\mathbf{g}}_l^p(t) \tag{16}$$

($p = 1,2,\ l = 1,\dots,l_0,\ 0 < t \le T$), where

$$\overline{\mathbf{K}}_{lq}^{p\sigma}(t,\tau) = \begin{cases} D_t^{d_r/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau) & \text{if } d_r \text{ is even},\\ \mathcal{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)] & \text{if } d_r \text{ is odd}, \end{cases} \tag{17}$$

$$\overline{\mathbf{z}}_l^p(t) = \mathbf{R}_{1/2}^{d_r}[\mathbf{z}_l^p(t)], \tag{18}$$

$$\overline{\mathbf{g}}_l^p(t) = \mathbf{R}_{1/2}^{d_r}[\mathbf{g}_l^p(t)]. \tag{19}$$

Now, we estimate the functions $\overline{\mathbf{K}}_{lq}^{p\sigma}$, $\overline{\mathbf{z}}_l^p$ and $\overline{\mathbf{g}}_l^p$. In case (i), by Theorem 1 [12], we have

$$|D_t^{d_r/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)| \le \mathrm{const}\,(t-\tau)^{\alpha/2-1}, \quad 0 \le \tau < t \le T, \tag{20}$$

$$|\varDelta_t D_t^{d_r/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)| \le \mathrm{const}\,(\varDelta t)^{\beta/2}(t-\tau)^{\mu-1}, \tag{21}$$

$0 \le \tau < t \le t + \varDelta t \le T$, $0 < \beta \le \alpha \le 1$, $\mu = \min\{\alpha/2, 1-\alpha/2\}$.

Analogously, in case (ii), we get

$$|D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)| \le \mathrm{const}\,(t-\tau)^{(1+\alpha)/2-1}, \quad 0 \le \tau < t \le T, \tag{22}$$

$$|\varDelta_t D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)| \le \mathrm{const}\,(\varDelta t)^{(1+\alpha)/2}(t-\tau)^{\mu-1}, \tag{23}$$

$0 \le \tau < t \le t + \varDelta t \le T$, $\mu = \min\{\alpha/2, 1-\alpha/2\}$.

From (22) and (23) it follows that the functions $D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}$ satisfy the assumptions of Lemma 6 of [12], and therefore

$$|\mathcal{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)]| \le \mathrm{const}\,(t-\tau)^{\alpha/2-1}, \quad 0 \le \tau < t \le T, \tag{24}$$

$$|\varDelta_t\mathcal{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{K}_{lq}^{p\sigma}(t,\tau)]| \le \mathrm{const}\,(\varDelta t)^{\beta/2}(t-\tau)^{\mu-1}, \tag{25}$$

$0 \le \tau < t \le t + \varDelta t \le T$, $0 < \beta \le \alpha \le 1$, $\mu = \min\{\alpha/2, 1-\alpha/2\}$.

Combining (20), (21), (24) and (25), we arrive at

$$|\overline{\mathbf{K}}_{lq}^{p\sigma}(t,\tau)| \le \mathrm{const}\,(t-\tau)^{\alpha/2-1}, \quad 0 \le \tau < t \le T, \tag{26}$$

(27) $|\Delta_t \overline{\mathbf{K}}^{p\sigma}_{lq}(t,\tau)| \le \text{const}\,(\Delta t)^{\beta/2}(t-\tau)^{\mu-1}, \quad 0 \le \tau < t \le t+\Delta t \le T,$

$p,\sigma = 1,2,\ l,q = 1,\dots,l_0,\ 0 < \beta \le \alpha \le 1,\ \mu = \min\{\alpha/2, 1-\alpha/2\}$.

Now, we examine the function $\overline{\mathbf{g}}^p_l$ given by (19). If d_r is even, by (A.5) the function $\overline{\mathbf{g}}^p_l$ satisfies the assumptions of Lemma 5 of [12] with $N = d_r/2$, and so

$$\overline{\mathbf{g}}^p_l(t) = D_t^{d_r/2}\mathbf{g}^p_l(t), \quad 0 \le \tau < t \le T.$$

If d_r is odd, by (A.5), $\overline{\mathbf{g}}^p_l$ satisfies the assumptions of that lemma with $N = (d_r-1)/2$, and thus

$$\overline{\mathbf{g}}^p_l(t) = \mathbf{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{g}^p_l(t)], \quad 0 \le \tau < t \le T.$$

Hence

$$\overline{\mathbf{g}}^p_l(t) = \begin{cases} D_t^{d_r/2}\mathbf{g}^p_l(t) & \text{if } d_r \text{ is even},\\ \mathbf{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{g}^p_l(t)] & \text{if } d_r \text{ is odd},\end{cases} \tag{28}$$

$(d_r = n - r^p_l + 2m + 1,\ p = 1,2,\ l = 1,\dots,l_0,\ 0 < t \le T)$.

From (28) and (A.5) in case (i) we obtain

$$|\Delta_t \overline{\mathbf{g}}^p_l(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0 \le t < t+\Delta t \le T, \quad \overline{\mathbf{g}}^p_l(0) = 0. \tag{29}$$

In case (ii) we have

$$|\Delta_t D^{(d_r-1)/2}\mathbf{g}^p_l(t)| \le \text{const}\,(\Delta t)^{(1+\alpha)/2}, \quad 0 \le t < t+\Delta t \le T,$$
$$D_t^{(d_r-1)/2}\mathbf{g}^p_l(0) = 0,$$

hence, by Lemma 2 of [16], we also get (29).

It remains to investigate the function $\overline{\mathbf{z}}^p_l$ given by (18). Using (5) and Lemma 5 of [12], we obtain

$$\overline{\mathbf{z}}^p_l(t) = \begin{cases} D_t^{d_r/2}\mathbf{z}^p_l(t) & \text{if } d_r \text{ is even},\\ \mathbf{R}_{1/2}[D_t^{(d_r-1)/2}\mathbf{z}^p_l(t)] & \text{if } d_r \text{ is odd},\end{cases}$$

$(d_r = n - r^p_l + 2m + 1,\ p = 1,2,\ l = 1,\dots,l_0,\ 0 < t \le T)$; hence, by Lemma 8 of [12], we find

$$|\Delta_t \overline{\mathbf{z}}^p_l(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0 \le t < t+\Delta t \le T, \quad \overline{\mathbf{z}}^p_l(0) = 0. \tag{30}$$

Now, we return to system (16). Multiplying both sides by $[\mathbf{a}(t)]^{-(n-r^p_l)/2}$ we obtain

$$\sum_{q=1}^{l_0} c^p_{lq}\varphi^p_q(t) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \overline{\overline{\mathbf{K}}}^{p\sigma}_{lq}(t,\tau)\varphi^\sigma_q(\tau)\,d\tau + \overline{\overline{\mathbf{z}}}^p_l(t) = \overline{\overline{\mathbf{g}}}^p_l(t) \tag{31}$$

$(p = 1,2,\ l = 1,\dots,l_0,\ 0 < t \le T)$, where

$$\overline{\overline{\mathbf{K}}}^{p\sigma}_{lq}(t,\tau) = [\mathbf{a}(t)]^{-(n-r^p_l)/2}\overline{\mathbf{K}}^{p\sigma}_{lq}(t,\tau), \quad \overline{\overline{\mathbf{z}}}^p_l(t) = [\mathbf{a}(t)]^{-(n-r^p_l)/2}\overline{\mathbf{z}}^p_l(t),$$
$$\overline{\overline{\mathbf{g}}}^p_l(t) = [\mathbf{a}(t)]^{-(n-r^p_l)/2}\overline{\mathbf{g}}^p_l(t), \qquad \mathbf{a}(t) = a_{n+2,m}(\chi_p(t),t).$$

Using assumptions (A.1), (A.2) it can be proved that the functions $\overline{\overline{\mathbf{K}}}{}^{p\sigma}_{lq}$, $\overline{\overline{\mathbf{z}}}{}^{p}_{l}$ and $\overline{\overline{\mathbf{g}}}{}^{p}_{l}$ satisfy the estimates (26), (27), (29) and (30) respectively.

Now, we treat system (31) as an algebraic system with respect to the functions φ^p_q, $p = 1, 2$, $q = 1, \ldots, l_0$. Its determinant is of the form

$$\mathbf{W} = \begin{vmatrix} c^p_{11} & 0 & 0 & \ldots & 0 \\ c^p_{21} & c^p_{22} & 0 & \ldots & 0 \\ c^p_{31} & c^p_{32} & c^p_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c^p_{l_0,1} & c^p_{l_0,2} & c^p_{l_0,3} & \ldots & c^p_{l_0,l_0} \end{vmatrix}.$$

Hence, in view of (9), we have

$$\mathbf{W} = c^p_{11}c^p_{22}\ldots c^p_{l_0,l_0} = (-1)^{nl_0-(r^p_1+r^p_2+\ldots+r^p_{l_0})}(\sqrt{\pi})^{l_0} \neq 0$$

on one of the curves χ_p (see §1) and

$$\mathbf{W} = c^p_{11}c^p_{22}\ldots c^p_{l_*-1,l_*-1}c^p_{l_*+1,l_*+1}\ldots c^p_{l_0,l_0} \neq 0$$

on the other. Cramer's formulae yield

$$\varphi^p_q(t) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \widetilde{\mathbf{K}}{}^{p\sigma}_{lq}(t,\tau)\varphi^\sigma_q(\tau)\,d\tau + \widetilde{\mathbf{z}}{}^p_l(t) = \widetilde{\mathbf{g}}{}^p_l(t), \tag{32}$$

where

$$\widetilde{\mathbf{K}}{}^{p\sigma}_{lq}(t,\tau) = \sum_{v=1}^{l_0} A^p_{lv}\overline{\overline{\mathbf{K}}}{}^{p\sigma}_{vq}(t,\tau), \qquad \widetilde{\mathbf{z}}{}^p_l(t) = \sum_{v=1}^{l_0} A^p_{lv}\overline{\overline{\mathbf{z}}}{}^p_v(t),$$

$$\widetilde{\mathbf{g}}{}^p_l(t) = \sum_{v=1}^{l_0} A^p_{lv}\overline{\overline{\mathbf{g}}}{}^p_v(t), \qquad A^p_{lv} = C^p_{lv}/\mathbf{W},$$

$p = 1, 2$, $l = 1, \ldots, l_0$, $0 < t \le T$ (C^p_{lv} denotes the algebraic complement of c^p_{lv} in $\mathbf{W}$).

It is easy to see that $\widetilde{\mathbf{K}}{}^{p\sigma}_{lq}$, $\widetilde{\mathbf{z}}{}^p_l$ and $\widetilde{\mathbf{g}}{}^p_l$ satisfy the same estimates as $\overline{\overline{\mathbf{K}}}{}^{p\sigma}_{lq}$, $\overline{\overline{\mathbf{z}}}{}^p_l$ and $\overline{\overline{\mathbf{g}}}{}^p_l$ respectively. Thus, (32) is a system of second-type Volterra integral equations with weak singularities and hence it has a solution of the form

$$\varphi^p_l(t) = \widetilde{\mathbf{g}}{}^p_l(t) - \widetilde{\mathbf{z}}{}^p_l(t) + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0}\int_0^t \mathcal{K}^{p\sigma}_{lq}(t,\tau)[\widetilde{\mathbf{g}}{}^\sigma_q(\tau) - \widetilde{\mathbf{z}}{}^\sigma_q(\tau)]\,d\tau, \tag{33}$$

where $\mathcal{K}^{p\sigma}_{lq}$ denote the resolvent kernels of the $\widetilde{\mathbf{K}}{}^{p\sigma}_{lq}$, $p, \sigma = 1, 2$, $l, q = 1, \ldots, l_0$. Moreover, the estimates (26), (27), (29) and (30) imply

$$|\Delta_t\varphi^p_l(t)| \le \mathrm{const}\,(\Delta t)^{\beta/2}, \qquad \varphi^p_l(0) = 0 \tag{34}$$

($p = 1, 2$, $l = 1, \ldots, l_0$, $0 \le t < t + \Delta t \le T$, $0 < \beta \le \alpha \le 1$).

3.2. Case 3). Without losing generality we may assume that on both the curves χ_p, l_0-1 conditions are posed given by the operators $\mathbf{B}_l^p$, $p=1,2$, $l=1,\ldots,l_0-1$, with $0\le r_1^p<r_2^p<\ldots<r_{l_0-1}^p<n+1$, and moreover, one more condition given by $\mathbf{B}_{l_0}^p$ with $r_{l_0}^1=n+1$ is posed on χ_1.

Now, we rewrite formula (4) in a form more suitable for further considerations:

$$(35)\qquad u(x,t)=\int_0^t \Lambda_{n+1}(x,t;\chi_1(\tau),\tau)\varphi_{l_0}^1(\tau)\,d\tau + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t \Lambda_{r_q^\sigma}(x,t;\chi_\sigma(\tau),\tau)\varphi_q^\sigma\,d\tau+\mathbf{Z}_{\mathbf{S}_T}(x,t),$$

where the functions $\Lambda_{r_q^\sigma}$ for $\sigma=1,2$, $q=1,\ldots,l_0-1$ are defined by formula (7) of [12] and

$$(36)\qquad \Lambda_{n+1}(x,t;y,\tau)=\Lambda_{r_*^1}(x,t;y,\tau)$$

$((x,t),(y,\tau)\in\overline{\mathbf{S}}_T)$, where r_*^1 is a positive integer with $0\le r_*^1\le n$, $r_*^1\ne r_l^1$ for $l=0,1,\ldots,l_0-1$.

Applying to both sides of (35) the operator $\mathbf{B}_{l_0}^1$ given by (3), we get

$$(37)\qquad \mathbf{B}_{l_0}^1u(x,t)=\int_0^t \mathbf{B}_{l_0}^1\Lambda_{r_*^1}(x,t;\chi_1(\tau),\tau)\varphi_{l_0}^1(\tau)\,d\tau + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t \mathbf{B}_{l_0}^1\Lambda_{r_q^\sigma}(x,t;\chi_1(\tau),\tau)\varphi_q^\sigma(\tau)\,d\tau+\mathbf{B}_{l_0}^1\mathbf{Z}_{\mathbf{S}_T}(x,t).$$

By (5) and Lemma 2 of [12] we can write

$$\mathbf{B}_{l_0}^1\Lambda_{r_*^1}(x,t;\chi_1(\tau),\tau)=\mathbf{P}_m[D_x\omega^{\chi_1(\tau),\tau}(x,t;\chi_1(\tau),\tau)]+\mathbf{B}_{l_0}^1\overline{w}_{r_*^1}(x,t;\chi_1(\tau),\tau)$$

$((x,t)\in\overline{\mathbf{S}}_T)$. Consider the integral

$$\mathbf{J}_m(x,t)=\int_0^t \mathbf{P}_m[D_x\omega^{\chi_1(\tau),\tau}(x,t;\chi_1(\tau),\tau)]\varphi_{l_0}^1(\tau)\,d\tau\qquad(m\in\mathbb{N}_0).$$

We investigate its behaviour as $x\to\chi_1(t)$, $(x,t)\in\mathbf{S}_T$. For $m=0$ we have

$$\mathbf{J}_0(x,t)=\int_0^t D_x\omega^{\chi_1(\tau),\tau}(x,t;\chi_1(\tau),\tau)\varphi_{l_0}^1(\tau)\,d\tau.$$

This is a heat potential of second kind which has the following property ([7], p. 1085):

$$(38)\qquad \lim_{x\to\chi_1(t)}\mathbf{J}_0(x,t)=-\sqrt{\frac{\pi}{\mathfrak{a}(t)}}\varphi_{l_0}^1(t)+\mathbf{J}_0(\chi_1(t),t),\qquad (x,t)\in\mathbf{S}_T,$$

where $\mathfrak{a}(t)=a_{n+2,0}(\chi_1(t),t)$.

For $m > 0$ the integral $\mathbf{J}_m$ can be written in the form

$$\mathbf{J}_m(x,t) = \int_0^t \left[\int_\tau^t \frac{(t-\zeta_m)^{m-1}}{(m-1)!} D_x \omega^{\chi_1(\tau),\tau}(x,\zeta_m;\chi_1(\tau),\tau) d\zeta_m \right] \varphi^1_{l_0}(\tau)\, d\tau.$$

It follows that

$$\mathbf{J}_m(x,t) = \int_0^t \frac{(t-\zeta_m)^{m-1}}{(m-1)!} \mathbf{J}_0(x,\zeta_m)\, d\zeta_m,$$

and hence, by (38), we obtain

$$\lim_{x\to\chi_1(t)} \mathbf{J}_m(x,t) = -\int_0^t \frac{(t-\zeta_m)^{m-1}}{(m-1)!} \sqrt{\frac{\pi}{\mathfrak{a}(t)}} \varphi^1_{l_0}(\zeta_m)\, d\zeta_m + \mathbf{J}_m(\chi_1(t),t) \tag{39}$$

$((x,t) \in \mathbf{S}_T, m \in \mathbb{N})$.

Making use of the definition of the operator $\mathbf{I}_\kappa$ (see (25) in [12]), formulae (38) and (39) can be written in the form

$$\lim_{x\to\chi_1(t)} \mathbf{J}_m(x,t) = -\mathbf{I}_m \left[\sqrt{\frac{\pi}{\mathfrak{a}(t)}} \varphi^1_{l_0}(t) \right] + \mathbf{J}_m(\chi_1(t),t) \tag{40}$$

$((x,t) \in \mathbf{S}_T, m \in \mathbb{N}_0)$, where $\mathfrak{a}(t) = a_{n+2,m}(\chi_1(t),t)$.

Passing to the limit $x \to \chi_1(t)$ in (37), we have

$$\begin{aligned} \mathbf{g}^1_{l_0}(t) = {} & -\mathbf{I}_m \left[\sqrt{\frac{\pi}{\mathfrak{a}(t)}} \varphi^1_{l_0}(t) \right] + \int_0^t \mathbf{K}^{11}_{l_0 l_0}(t,\tau) \varphi^1_{l_0}(\tau)\, d\tau \\ & + \sum_{\sigma=1}^{2} \sum_{q=1}^{l_0-1} \int_0^t \mathbf{K}^{1\sigma}_{l_0 q}(t,\tau) \varphi^\sigma_q(\tau)\, d\tau + \mathbf{z}^1_{l_0}(t), \end{aligned} \tag{41}$$

where $\mathbf{K}^{11}_{l_0 l_0}(t,\tau) = \mathbf{B}^1_{l_0} \Lambda_{r^1_*}(\chi_1(t),t;\chi_1(\tau),\tau)$, $\mathbf{K}^{1\sigma}_{l_0 q}(t,\tau) = \mathbf{B}^1_{l_0} \Lambda_{r^\sigma_q}(\chi_1(t),t;$ $\chi_\sigma(\tau),\tau)$, $\sigma = 1,2$, $q = 1,\dots,l_0-1$, $0 < t \le T$, the operators $\mathbf{B}^1_{l_0}$ are defined by formula (34) of [12] and the functions $\mathbf{z}^1_{l_0}$ are given by relation (42) of [12].

Applying $\mathbf{R}^{2m}_{1/2}$ to both sides of (41), by Lemmas 4 and 5 of [12], we obtain

$$\begin{aligned} & -\sqrt{\frac{\pi}{\mathfrak{a}(t)}} \varphi^1_{l_0}(t) + \int_0^t \overline{\mathbf{K}}^{11}_{l_0 l_0}(t,\tau) \varphi^1_{l_0}(\tau)\, d\tau \\ & \quad + \sum_{\sigma=1}^{2} \sum_{q=1}^{l_0-1} \int_0^t \overline{\mathbf{K}}^{1\sigma}_{l_0 q}(t,\tau) \varphi^\sigma_q(\tau)\, d\tau + \overline{\mathbf{z}}^1_{l_0}(t) = \overline{\mathbf{g}}^1_{l_0}(t), \qquad 0 < t \le T, \end{aligned} \tag{42}$$

where $\overline{\mathbf{K}}^{11}_{l_0 l_0}(t,\tau) = D^m_t \mathbf{K}^{11}_{l_0 l_0}(t,\tau)$, $\overline{\mathbf{K}}^{1\sigma}_{l_0 q}(t,\tau) = D^m_t \mathbf{K}^{1\sigma}_{l_0 q}(t,\tau)$, $\overline{\mathbf{z}}^1_{l_0}(t) = D^m_t \mathbf{z}^1_{l_0}(t)$, $\overline{\mathbf{g}}^1_{l_0}(t) = D^m_t \mathbf{g}^1_{l_0}(t)$, $\sigma = 1,2$, $q = 1,\dots,l_0-1$.

Using Theorem 2 of [12] we find the estimates

$$(43)\qquad |\overline{\mathbf{K}}^{11}_{l_0l_0}(t,\tau)| \le \text{const}\,(t-\tau)^{\alpha/2-1}, \qquad 0\le\tau<t\le T,$$

$$(44)\qquad |\overline{\mathbf{K}}^{1\sigma}_{l_0q}(t,\tau)| \le \text{const}\,(t-\tau)^{\alpha/2-1}, \qquad 0\le\tau<t\le T,$$

$$(45)\qquad |\Delta_t\overline{\mathbf{K}}^{11}_{l_0l_0}(t,\tau)| \le \text{const}\,(\Delta t)^{\beta/2}(t-\tau)^{\mu-1}, \qquad 0\le\tau<t\le t+\Delta t\le T,$$

$$(46)\qquad |\Delta_t\overline{\mathbf{K}}^{1\sigma}_{l_0q}(t,\tau)| \le \text{const}\,(\Delta t)^{\beta/2}(t-\tau)^{\mu-1}, \qquad 0\le\tau<t\le t+\Delta t\le T,$$

where $\sigma=1,2$, $q=1,\ldots,l_0-1$, $0<\beta\le\alpha\le1$, $\mu=\min\{\alpha/2,1-\alpha/2\}$.

Similarly, using Lemma 9 of [12], we have

$$(47)\qquad |\Delta_t\overline{\mathbf{z}}^1_{l_0}(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0\le t<t+\Delta t\le T, \quad \overline{\mathbf{z}}^1_{l_0}(0)=0,$$

moreover, in view of assumption (A.5), we get

$$(48)\qquad |\Delta_t\overline{\mathbf{g}}^1_{l_0}(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0\le t<t+\Delta t\le T, \quad \overline{\mathbf{g}}^1_{l_0}(0)=0.$$

Observe that equation (42) can be written in the form

$$(49)\qquad \varphi^1_{l_0}(t)+\int_0^t \widetilde{\mathbf{K}}^{11}_{l_0l_0}(t,\tau)\varphi^1_{l_0}(\tau)\,d\tau + \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t \widetilde{\mathbf{K}}^{1\sigma}_{l_0q}(t,\tau)\varphi^\sigma_q(\tau)\,d\tau + \widetilde{\mathbf{z}}^1_{l_0}(t) = \widetilde{\mathbf{g}}^1_{l_0}(t),$$

where $\widetilde{\mathbf{K}}^{11}_{l_0l_0}(t,\tau) = -\sqrt{\mathfrak{a}(t)/\pi}\cdot\overline{\mathbf{K}}^{11}_{l_0l_0}(t,\tau)$, $\widetilde{\mathbf{K}}^{1\sigma}_{l_0q}(t,\tau) = -\sqrt{\mathfrak{a}(t)/\pi}\cdot\overline{\mathbf{K}}^{1\sigma}_{l_0q}(t,\tau)$, $\widetilde{\mathbf{z}}^1_{l_0}(t) = -\sqrt{\mathfrak{a}(t)/\pi}\cdot\overline{\mathbf{z}}^1_{l_0}(t)$, $\widetilde{\mathbf{g}}^1_{l_0}(t) = -\sqrt{\mathfrak{a}(t)/\pi}\cdot\overline{\mathbf{g}}^1_{l_0}(t)$, $\sigma=1,2$, $q=1,\ldots,l_0-1$, $0<t\le T$.

From assumptions (A.1) and (A.2) it follows that $\overline{\overline{\mathbf{K}}}^{11}_{l_0l_0}$, $\overline{\overline{\mathbf{K}}}^{1\sigma}_{l_0q}$, $\overline{\overline{\mathbf{z}}}^1_{l_0}$ and $\overline{\overline{\mathbf{g}}}^1_{l_0}$ satisfy inequalities (43)–(48) respectively. This means that if we treat the functions φ^σ_q, $\sigma=1,2$, $q=1,\ldots,l_0-1$, as parameters, then (49) is a second-kind Volterra equation with respect to $\varphi^1_{l_0}$. Because the singularity of the kernel of this equation is weak one can solve it.

Imposing on the function u, given by formula (35), the remaining boundary conditions (3) given by the operators $\mathbf{B}^p_1$, $\mathbf{B}^p_2$, …, $\mathbf{B}^p_{l_0-1}$ with $0\le r^p_1 < r^p_2<\ldots<r^p_{l_0-1}<n+1$ ($p=1,2$, $l_0=[(n+3)/2]$), we obtain the following system of integral equations:

$$(50)\qquad \sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t \mathbf{B}^p_l \Lambda_{r^\sigma_q}(\chi_p(t),t;\chi_p(\tau),\tau)\varphi^\sigma_q(\tau)\,d\tau + \int_0^t \mathbf{B}^p_l\Lambda_{r^1_*}(\chi_p(t),t;\chi_1(\tau),\tau)\varphi^1_{l_0}(\tau)\,d\tau + \mathbf{z}^p_l(t) = \mathbf{g}^p_l(t),$$

$p=1,2$, $l=1,\ldots,l_0-1$, $0<t\le T$.

System (50) is a system of first-kind Volterra integral equations with $2(l_0-1)$ equations and $2(l_0-1)$ unknown functions φ_q^σ, $\sigma=1,2$, $q=1,\ldots,l_0-1$. Now, we apply to system (50) the method presented in subsection 3.1 to obtain

$$(51)\quad \varphi_l^p(t)+\sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t \widetilde{\mathsf{K}}_{lq}^{p\sigma}(t,\tau)\varphi_q^\sigma(\tau)\,d\tau = \int_0^t \widetilde{\mathsf{K}}_{l_0l_0}^{11}(t,\tau)\varphi_{l_0}^1(\tau)\,d\tau-\widetilde{\mathsf{g}}_l^p(t)-\widetilde{\mathsf{z}}_l^p(t),$$

$p=1,2$, $l=1,\ldots,l_0-1$, $0<t\le T$.

The functions $\widetilde{\mathsf{K}}_{lq}^{p\sigma}$, $\widetilde{\mathsf{g}}_l^p$ and $\widetilde{\mathsf{z}}_l^p$ satisfy inequalities (26), (27), (29) and (30), respectively, thus (51) is a system of second-kind Volterra integral equations with weak singularities.

Finally, we are able to find a solution of system (49), (51) in the form

$$(52)\quad \varphi_l^p(t)=\overline{\mathsf{g}}_l^p(t)-\overline{\mathsf{z}}_l^p(t) +\sum_{\sigma=1}^{2}\sum_{q=1}^{l_0-1}\int_0^t [\mathcal{K}_{lq}^{p\sigma}(t,\tau)-\mathcal{K}_{l_0l_0}^{11}(t,\tau)][\overline{\mathsf{g}}_q^\sigma(\tau)-\overline{\mathsf{z}}_q^\sigma(\tau)]\,d\tau$$

($l=1,\ldots,l_0$ for $p=1$, $l=1,\ldots,l_0-1$, for $p=2$), where $\mathcal{K}_{lq}^{p\sigma}$ and $\mathcal{K}_{l_0l_0}^{11}$ are the resolvent kernels of $\widetilde{\mathsf{K}}_{lq}^{p\sigma}$ and $\widetilde{\mathsf{K}}_{l_0l_0}^{11}$, respectively. Furthermore, by (26)–(27), (29)–(30) and (43)–(48) we obtain

$$(53)\qquad |\Delta_t\varphi_l^p(t)|\le \mathrm{const}\,(\Delta t)^{\beta/2},\quad 0\le t<t+\Delta t\le T,\quad \varphi_l^p(0)=0$$

($p=1,2$, $l=1,\ldots,l_0-1$), where $0<\beta\le\alpha\le 1$.

As a result of the foregoing considerations we can formulate the following theorem:

THEOREM 1. *If assumptions* (A.1)–(A.6) *are satisfied then there exists a solution* u *of the problem* (1)–(3). *It is given by relation* (4), *where the functions* φ_q^σ *are defined by formula* (33) *in case* 1); *by a formula similar to* (33) *in case* 2) *and then they satisfy inequality* (34); *by formula* (52) *in case* 3); *and by a formula similar to* (52) *in case* 4) *and then they satisfy inequality* (53).

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