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Properties of some integrals related to partial differential equations of order higher than two

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Abstract. We construct fundamental solutions of some partial differential equations of order higher than two and examine properties of these solutions and of some related integrals. The results will be used in our next paper concerning boundary-value problems for these equations.

1. Introduction. Let $x = \chi_p(t)$, $0 < t \le T$, $p = 1, 2$, be equations of two non-intersecting curves on the (x,t) plane.

In the domain

(1) $$\mathbf{S}_T = \{(x,t) : \chi_1(t) < x < \chi_2(t),\ 0 < t \le T\}, \qquad T = \text{const} < \infty,$$

we consider the partial differential equation

(2) $$\mathfrak{L}u \equiv \sum_{i=0}^{n+2} \sum_{j=0}^{m} a_{ij}(x,t) D_x^i D_t^j u - D_x^n D_t^{m+1} u = 0,$$

where $n, m \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $n + m > 0$ (for $n = m = 0$ equation (2) is a parabolic equation of second order, the theory of which is well known), $D_x^i = \partial^i/\partial x^i, D_t^j = \partial^j/\partial t^j$.

We make the following assumptions:

(A.1) There are constants a_0 and a_1 such that

$$0 < a_0 \le a_{n+2,m}(x,t) \le a_1 \quad \text{for } (x,t) \in \overline{\mathbf{S}}_T,$$

where $\overline{\mathbf{S}}_T$ denotes the closure of $\mathbf{S}_T$.

(A.2) The coefficients a_{ij} $(i = 0, 1, \ldots, n+2,\ j = 0, 1, \ldots, m)$ are continuous in $\overline{\mathbf{S}}_T$ and satisfy the Hölder condition with respect to x with exponent α $(0 < \alpha \le 1)$; moreover, $a_{n+2,m}$ satisfies the Hölder condition with respect to t with exponent $\frac{1}{2}\alpha$.

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(A.3) The functions χ_p $(p = 1,2)$ have continuous derivatives up to order $n_* = [(n+1)/2]$ ($[k]$ denotes the largest integer not greater than k) in the interval $[0,T]$ and the highest derivatives satisfy the Hölder condition

$$|\Delta_t[\chi_p^{(n_*)}(t)]| \le \begin{cases} \text{const}\,(\Delta t)^{\alpha/2} & \text{if } n+1 \text{ is even},\\ \text{const}\,(\Delta t)^{(\alpha+1)/2} & \text{if } n+1 \text{ is odd},\end{cases}$$

where $\chi_p^{(0)} = \chi_p$, $\Delta_t[\chi_p(t)] \equiv \chi_p(t+\Delta t) - \chi_p(t)$, $t, t+\Delta t \in [0,T]$, $0 < \alpha \le 1$.

2. Fundamental solutions. Let $n, m, r \in \mathbb{N}_0$, $r \le n$. Consider the operators

$$\mathbf{P}_m[G(x,t;\xi,\tau)] = \begin{cases} \int_0^{t-\tau} \mathbf{P}_{m-1}[G(x,s+\tau;\xi,\tau)]\,ds, & m \in \mathbb{N},\\ G(x,t;\xi,\tau), & m = 0,\end{cases} \tag{3}$$

$$\mathbf{Q}_{n,r}[G(x,t;\xi,\tau)] = \begin{cases} -\int_{x-\xi}^{\infty} \mathbf{Q}_{n-1,r}[G(y+\xi,t;\xi,\tau)]\,dy, & n \in \mathbb{N},\ r = 0,\\ \int_0^{x-\xi} \mathbf{Q}_{n-1,r-1}[G(y+\xi,t;\xi,\tau)]\,dy, & n, r \in \mathbb{N},\\ G(x,t;\xi,\tau), & n = r = 0,\end{cases} \tag{4}$$

where G is a sufficiently regular function such that the expressions on the right-hand side of (3) and (4) make sense for all $(x,t),(\xi,\tau) \in \overline{\mathbf{S}}_T$.

LEMMA 1. *If G has continuous t-derivatives up to order $n_* = [(n+1)/2]$, then*

$$D_t^j \mathbf{P}_m[G(x,t;\xi,\tau)] = \begin{cases} \mathbf{P}_{m-j}[G(x,t;\xi,\tau)] & \text{if } 0 \le j \le m,\\ D_t^{j-m} G(x,t;\xi,\tau) & \text{if } m < j \le m + n_*,\end{cases}$$

where $j = 0,1,\ldots,m+n_*$ and $(x,t),(\xi,\tau) \in \overline{\mathbf{S}}_T$.

LEMMA 2. *If G has continuous x-derivatives up to order $m_* = 2m+1$, then*

$$D_x^i \mathbf{Q}_{n,r}[G(x,t;\xi,\tau)] = \begin{cases} \mathbf{Q}_{n-i,r-i}[G(x,t;\xi,\tau)], & 0 \le i \le r-1,\\ \mathbf{Q}_{n-i,0}[G(x,t;\xi,\tau)], & r \le i \le n,\\ D_x^{i-n} G(x,t;\xi,\tau), & n < i \le n + m_*,\end{cases}$$

where $i = 0,1,\ldots,n+m_*$ and $(x,t),(\xi,\tau) \in \overline{\mathbf{S}}_T$.

The lemmas follow immediately from (3) and (4).

Define

$$\omega^{z,\sigma}(x,t;\xi,\tau) = (t-\tau)^{-1/2} \exp\left[-\frac{(x-\xi)^2}{4a_{n+2,m}(z,\sigma)(t-\tau)}\right], \tag{5}$$

where $(x,t),(\xi,\tau),(z,\sigma) \in \overline{\mathbf{S}}_T$, and

$$w_r(x,t;\xi,\tau) = (\mathbf{P}_m \circ \mathbf{Q}_{n,r})[\omega^{\xi,\tau}(x,t;\xi,\tau)], \qquad r = 0,1,\ldots,n. \tag{6}$$

By Lemmas 1 and 2, the functions $w_r (r = 0,1,\ldots,n)$ are quasi-solutions (see [4], p. 139) of equation (2). Applying Levi's method (see e.g. [4], p.

152) we can construct fundamental solutions of (2) in the form

$$\Lambda_r(x,t;\xi,\tau) = w_r(x,t;\xi,\tau) + \overline{w}_r(x,t;\xi,\tau) \tag{7}$$

$(r = 0,1,\ldots,n)$, where

$$\overline{w}_r(x,t;\xi,\tau) = \int_0^t \int_{\chi_1(\sigma)}^{\chi_2(\sigma)} w_r(x,t;z,\sigma)\Phi_r(z,\sigma;\xi,\tau)\,dz\,d\sigma \tag{8}$$

$(r = 0,1,\ldots,n)$ and Φ_r are solutions of the Volterra equation

$$\Phi_r(x,t;\xi,\tau) = \mathfrak{L}w_r(x,t;\xi,\tau) + \int_\tau^t \int_{\chi_1(\sigma)}^{\chi_2(\sigma)} \mathfrak{L}w_r(x,t;z,\sigma)\Phi_r(z,\sigma;\xi,\tau)\,dz\,d\sigma.$$

It follows immediately from (5), (7) and (8) that

$$|D_x^i D_t^j w_r(x,t;\xi,\tau)| \le C(t-\tau)^{-(n-i+2m-2j-1)/2} \exp\left[-\frac{(x-\xi)^2}{4a_0(t-\tau)}\right], \tag{9}$$

$$|D_x^i D_t^j \overline{w}_r(x,t;\xi,\tau)| \le C(t-\tau)^{-(n-i+2m-2j-1+\alpha)/2} \exp\left[-\frac{(x-\xi)^2}{4a_0(t-\tau)}\right], \tag{10}$$

$$|D_x^i D_t^j \Lambda_r(x,t;\xi,\tau)| \le C(t-\tau)^{-(n-i+2m-2j-1)/2} \exp\left[-\frac{(x-\xi)^2}{4a_0(t-\tau)}\right], \tag{11}$$

where $i,j \in \mathbb{N}_0$, $r \le n$, $(x,t),(\xi,\tau) \in \overline{\mathbf{S}}_T$, $\tau < t$, $0 < \alpha \le 1$, $C = \text{const} > 0$.

Lemma 3. *We have*

$$D_x^i w_r(\chi(\tau),t;\chi(\tau),\tau) = \begin{cases} 0, & 0 \le i < r, \\ (-1)^{n-i}\sqrt{\pi}\,\Gamma^{-1}\left(\frac{n-i+2m+1}{2}\right)[\mathfrak{a}(\tau)]^{(n-i)/2}(t-\tau)^{(n-i+2m-1)/2} & r \le i \le n, \end{cases} \tag{12}$$

$(i,r = 0,1,\ldots,n)$, *where* Γ *is the Euler gamma function*, $\mathfrak{a}(\tau) = a_{n+2,m}(\chi(\tau),\tau)$ *and* χ *denotes* χ_1 *or* χ_2.

Proof. First we consider the case $0 \le i < r$. Applying Lemma 2 and (6) we have

$$D_x^i w_r(\chi(\tau),t;\chi(\tau),\tau) = \mathbf{P}_m(\mathbf{Q}_{n-i,r-i}[\omega^{\chi(\tau),\tau}(\chi(\tau),t;\chi(\tau),\tau)]),$$

hence, by (4) we obtain

$$\mathbf{Q}_{n-i,r-i}[\omega^{\chi(\tau),\tau}(\chi(\tau),t;\chi(\tau),\tau)] = \int_0^{\chi(\tau)-\chi(\tau)} \mathbf{Q}_{n-i-1,r-i-1}[\omega^{\chi(\tau),\tau}(\chi(\tau)+y,t;\chi(\tau),\tau)]\,dy = 0,$$

whence $D_x^i w_r(\chi(\tau),t;\chi(\tau),\tau) = 0$.

For $r \le i \le n$, we make use of the relation

$$D_x^i w_r(\chi(\tau),t;\chi(t),\tau) = \begin{cases} \mathbf{P}_m[\omega^{\chi(\tau),\tau}(\chi(\tau),t;\chi(\tau),\tau)], & i=n, \\ \mathbf{P}_m(\mathbf{Q}_{n-i,0}[\omega^{\chi(\tau),\tau}(\chi(\tau),t;\chi(\tau),\tau)]), & i<n. \end{cases} \tag{13}$$

Let $i<n$. Consider the function

$$\mathbf{Q}^*_{n-i,0}(t,\tau) \equiv \mathbf{Q}_{n-i,0}[\omega^{\chi(\tau),\tau}(\chi(\tau),t;\chi(\tau),\tau)].$$

Changing the order of integration we can write

$$\mathbf{Q}^*_{n-1,0}(t,\tau) = \frac{(-1)^{n-i}}{(n-i-1)!}\int_0^\infty (\vartheta_{n-i})^{n-i-1}(t-\tau)^{-1/2} \times \exp\left[-\frac{(\vartheta_{n-i})^2}{4\mathbf{a}(\tau)(t-\tau)}\right]d\vartheta_{n-i}.$$

Thus, substituting $\eta = \frac{1}{4}(\vartheta_{n-i})^2[\mathbf{a}(\tau)(t-\tau)]^{-1}$ we have

$$\mathbf{Q}^*_{n-1,0}(t,\tau) = \frac{(-1)^{n-i}}{2(n-i-1)!}[4\mathbf{a}(\tau)]^{(n-i)/2}\Gamma^{-1}\left(\frac{n-i}{2}\right)(t-\tau)^{(n-i-1)/2}.$$

By (3) and (13) we finally obtain

$$D_x^i w_r(\chi(\tau),t;\chi(\tau),\tau) = (-1)^{n-i}\sqrt{\pi}[\mathbf{a}(\tau)]^{(n-i)/2} \times \Gamma^{-1}\left(\frac{n-i+2m+1}{2}\right)(t-\tau)^{(n-i+2m-1)/2}.$$

By a similar argument we get (12) in the case $i=n$. Thus, the proof of Lemma 3 is complete.

3. Properties of the operators $\mathfrak{R}_{1/2}, \mathbf{R}_{1/2}, \mathbf{I}_\kappa$. In the present section we consider the operators

$$\mathbf{R}_{1/2}[\varphi(t)] = \frac{1}{\sqrt{\pi}}D_t\Big[\int_0^t (t-s)^{-1/2}\varphi(s)\,ds\Big] \tag{14}$$

and

$$\mathfrak{R}_{1/2}[\Phi(t,\tau)] = \frac{1}{\sqrt{\pi}}D_t\Big[\int_\tau^t (t-s)^{-1/2}\Phi(s,\tau)\,ds\Big], \tag{15}$$

where φ is defined and continuous for $t\in[0,T]$ and Φ is defined and continuous for $(t,\tau)\in[0,T]\times[0,T]$.

The operators $\mathbf{R}_{1/2}$ and $\mathfrak{R}_{1/2}$ were introduced by Baderko [1].

Moreover, we define

$$\mathbf{R}^k_{1/2}[\varphi(t)] = \mathbf{R}_{1/2}[\mathbf{R}^{k-1}_{1/2}[\varphi(t)]], \quad k\in\mathbb{N}, \quad \mathbf{R}^0_{1/2}[\varphi(t)] = \varphi(t) \tag{16}$$

and

$$(17)\quad \mathfrak{R}^k_{1/2}[\Phi(t,\tau)] = \mathfrak{R}_{1/2}[\mathfrak{R}^{k-1}_{1/2}[\Phi(t,\tau)]], \quad k \in \mathbb{N}, \quad \mathfrak{R}^0_{1/2}[\Phi(t,\tau)] = \Phi(t,\tau).$$

LEMMA 4. *Let $N \in \mathbb{N}_0$. If a function Ψ has continuous derivatives $D^j_t\Psi$, $j = 0,1,\ldots,N$, and*

$$(18)\quad |D^j_t\Psi(t,\tau)| \le \mathrm{const}\,(t-\tau)^{N-j+\varrho-1}, \quad 0 \le \tau < t \le T,\ 0 < \varrho < 1,$$

and a function φ is continuous in $[0,T]$, then

$$(19)\quad \mathbf{R}^{2N}_{1/2}\Big[\int_0^t \Psi(t,\tau)\varphi(\tau)\,d\tau\Big] = \int_0^t D^N_t\Psi(t,\tau)\varphi(\tau)\,d\tau, \quad 0 < \varrho \le 1/2,$$

$$(20)\quad \mathbf{R}^{2N+1}_{1/2}\Big[\int_0^t \Psi(t,\tau)\varphi(\tau)\,d\tau\Big] = \int_0^t \mathfrak{R}_{1/2}[D^N_t\Psi(t,\tau)]\varphi(\tau)\,d\tau, \quad 1/2 < \varrho < 1.$$

LEMMA 5. *Let $N \in \mathbb{N}_0$. If a function ψ is defined in $[0,T]$ and has continuous derivatives $D^j_t\psi$, $j = 0,1,\ldots,N$, and*

$$D^j_t\psi(0) = 0, \quad j = 0,1,\ldots,N,$$

then

$$\mathbf{R}^{2N}_{1/2}[\psi(t)] = D^N_t\psi(t), \quad \mathbf{R}^{2N+1}_{1/2}[\psi(t)] = \mathfrak{R}_{1/2}[D^N_t\psi(t)], \quad 0 < t \le T.$$

We omit the inductive proofs of Lemmas 4 and 5.

LEMMA 6. *If Φ satisfies the conditions*

$$(21)\quad |\Phi(t,\tau)| \le \mathrm{const}\,(t-\tau)^{(1+\alpha)/2-1}, \quad 0 \le \tau < t \le T,$$

$$(22)\quad |\Delta_t\Phi(t,\tau)| \le \mathrm{const}\,(\Delta t)^{(1+\alpha)/2}(t-\tau)^{\mu-1}, \quad 0 \le \tau < t \le t+\Delta t \le T,$$

where $\mu = \min\{\alpha/2, 1-\alpha/2\}$, then

$$(23)\quad |\mathfrak{R}_{1/2}[\Phi(t,\tau)]| \le \mathrm{const}\,(t-\tau)^{\alpha/2-1}, \quad 0 \le \tau < t \le T,$$

$$(24)\quad |\Delta_t\mathfrak{R}_{1/2}[\Phi(t,\tau)]| \le \mathrm{const}\,(\Delta t)^{\beta/2}(t-\tau)^{\mu-1}, \quad 0 \le \tau < t \le t+\Delta t \le T,$$

where $0 < \beta \le \alpha \le 1$.

The proof of Lemma 6 is similar to that of Lemma 3 in [1].

Now, let ψ be a function defined for all $t \in [0,T]$ and satisfying the Hölder condition with exponent $\alpha_\psi \in (0,1]$. Consider the operator $\mathbf{I}_\kappa$ given by the formula

$$(25)\quad \mathbf{I}_\kappa[\psi(t)] = \Gamma^{-1}(\kappa)\int_0^t (t-\tau)^{\kappa-1}\psi(\tau)\,d\tau, \quad \kappa > 0.$$

The operator $\mathbf{I}_\kappa$ was introduced in [1] where it was proved that

$$\mathbf{R}_{1/2}[\mathbf{I}_\kappa[\psi(t)]] = \begin{cases} \mathbf{I}_{\kappa-1/2}[\psi(t)] & \text{if } \kappa > 1/2, \\ \psi(t) & \text{if } \kappa = 1/2. \end{cases}$$

One may prove the following

Lemma 7. *Let $k \in \mathbb{N}$ and $\kappa \in [k/2, \infty)$. Then*

$$\mathbf{R}^k_{1/2}[\mathbf{I}_\kappa[\psi(t)]] = \begin{cases} \mathbf{I}_{\kappa-k/2}[\psi(t)] & \text{if } \kappa > k/2, \\ \psi(t) & \text{if } \kappa = k/2. \end{cases}$$

4. Properties of the functions $\mathbf{K}^{p\sigma}_{lq}$. Consider the functions

$$\mathbf{K}^{p\sigma}_{lq}(t,\tau) = \mathbf{B}^p_l \Lambda_{r^\sigma_q}(\chi_p(t), t; \chi_\sigma(\tau), \tau) - \begin{cases} 0, & \sigma \neq p \text{ or } \sigma = p,\ 1 \le l < q, \\ D_x^{r^p_l} w_{r^p_l}(\chi_p(\tau), t; \chi_p(\tau), \tau), & \sigma = p,\ q \le l \le l_0, \end{cases} \tag{26}$$

where $0 \le r^p_1 < r^p_2 < \ldots < r^p_{l_0} \le n$, $r^p_l \in \mathbb{N}_0$, $p, \sigma = 1, 2$, $l, q = 1, 2, \ldots, l_0$, $l_0 = [(n+3)/2]$,

$$\mathbf{B}^p_l \equiv D_x^{r^p_l} + \sum_{k=0}^{r^p_l - 1} b^p_{kl}(t) D^k_x \tag{27}$$

and b^p_{kl} has continuous derivatives up to order $\mathcal{M} = [d_r/2]$, $d_r = n - r^p_l + 2m + 1$.

Theorem 1. *For $\nu = 0, 1, \ldots, \mathcal{M}$, we have*

$$|D^\nu_t \mathbf{K}^{p\sigma}_{lq}| \le \text{const}\,(t-\tau)^{(d_r - 2\nu + \alpha)/2 - 1}, \quad 0 \le \tau < t \le T, \tag{28}$$

$$|\Delta_t D^{\mathcal{M}}_t \mathbf{K}^{p\sigma}_{lq}| \le \text{const} \begin{cases} (\Delta t)^{\alpha/2}(t-\tau)^{\mu-1} & \textit{if } d_r \textit{ is even}, \\ (\Delta t)^{(\alpha+1)/2}(t-\tau)^{\mu-1} & \textit{if } d_r \textit{ odd}, \end{cases} \tag{29}$$

$(0 \le \tau < t \le t + \Delta t \le T)$, *where* $\mu = \min\{\alpha/2, 1 - \alpha/2\}$, $0 < \alpha \le 1$.

Proof. We consider in detail the case $\sigma = p$. The case $\sigma \neq p$ can be investigated in a similar way. The νth derivative of $\mathbf{K}^{pp}_{lq}$ is given by the formula (see [3], p. 33)

$$\begin{aligned} D^\nu_t \mathbf{K}^{pp}_{lq} &= D^\nu_t D_x^{r^p_l} w_{r^p_q}(\chi_p(t), t; \chi_p(\tau), \tau) \\ &+ \sum_{j=1}^{\nu} \sum_{i_1 + 2i_2 + \ldots + \nu i_\nu = \nu} \sum_{i_1 + i_2 + \ldots + i_\nu = j} \frac{\nu!}{i_1! i_2! \ldots i_\nu!} \\ &\times D^{\nu-j}_t D_x^{r^p_l + j} w_{r^p_q}(\chi_p(t), t; \chi_p(\tau), \tau) \left[\frac{\chi'_p(t)}{1!}\right]^{i_1} \left[\frac{\chi''_p(t)}{2!}\right]^{i_2} \ldots \left[\frac{\chi^{(\nu)}_p(t)}{\nu!}\right]^{i_\nu} \end{aligned} \tag{30}$$

$$+ D_t^\nu D_x^{r_l^p} \overline{w}_{r_q^p}(\chi_p(t), t; \chi_p(\tau), \tau) + D_t^\nu \Big[\sum_{k=0}^{r_l^p - 1} b_{kl}^p(t) D_x^k \Lambda_{r_q^p}(\chi_p(t), t; \chi_p(\tau), \tau)\Big]$$

$(\nu = 0, 1, \ldots, \mathcal{M})$.

We denote the summands on the right-hand side of (30) by $K_1(t,\tau)$, $K_2(t,\tau)$, $K_3(t,\tau)$ and $K_4(t,\tau)$, respectively.

We only prove (29). The proof of (28) is similar, but easier.

Let d_r be even. We consider two cases: (i) $0 \le r_l^p < r_q^p \le n$, (ii) $0 \le r_q^p \le r_l^p \le n$.

In case (i) by Lemmas 1 and 3 we get

$$K_1(t,\tau) = -\int_0^{\chi_p(t)-\chi_p(\tau)} \mathbf{Q}_{N,R}[D_t^{(n-r_l^p+1)/2} \omega^{\chi_p(\tau),\tau}(\xi_1 + \chi_p(\tau), t; \chi_p(\tau), \tau)] d\xi_1,$$

where $\mathbf{Q}_{N,R} = \mathbf{Q}_{n-r_l^p-1, r_q^p - r_l^p - 1}$, hence

$$\begin{aligned}
&\Delta_t K_1(t,\tau)\\
&= -\int_{\chi_p(t)-\chi_p(\tau)}^{\chi_p(t+\Delta t)-\chi_p(\tau)} \mathbf{Q}_{N,R}[D_t^{(n-r_l^p+1)/2} \omega^{\chi_p(\tau),\tau}(\xi_1 + \chi_p(\tau), t+\Delta t; \chi_p(\tau), \tau)] d\xi_1\\
&\quad + \int_0^{\chi_p(t)-\chi_p(\tau)} \mathbf{Q}_{N,R}[D_t^{(n-r_l^p+1)/2} \omega^{\chi_p(\tau),\tau}(\xi_1 + \chi_p(\tau), t; \chi_p(\tau), \tau)\\
&\quad - D_t^{(n-r_l^p+1)/2} \omega^{\chi_p(\tau),\tau}(\xi_1 + \chi_p(\tau), t; \chi_p(\tau), \tau)]\, d\xi_1\\
&\equiv \Delta_t K_{11}(t,\tau) + \Delta_t K_{12}(t,\tau).
\end{aligned}$$

Applying the estimate (2.11) of [2] and (4), we can write

$$\begin{aligned}
&|\mathbf{Q}_{N,R}[D_t^{(n-r_l^p+1)/2} \omega^{\chi_p(\tau),\tau}(\xi_1 + \chi_p(\tau), t+\Delta t; \chi_p(\tau), \tau)]|\\
&\qquad \le \text{const}\,(t+\Delta t - \tau)^{(r_l^p - r_q^p - 1)/2} \int_0^{\xi_1} \ldots \int_0^{\xi_{r_l^p - r_q^p - 1}} d\xi_{r_l^p - r_q^p - 1} \ldots d\xi_1,
\end{aligned}$$

and hence we obtain

$$|\Delta_t K_{11}(t,\tau)| \le \text{const}\,(t+\Delta t - \tau)^{(r_l^p - r_q^p - 1)/2} \int_{\chi_p(t)-\chi_p(\tau)}^{\chi_p(t+\Delta t)-\chi_p(\tau)} \xi_1^{r_l^p - r_q^p - 1} d\xi_1.$$

Since in this case $r_l^p < r_q^p$ and $r_l^p, r_q^p \in \mathbb{N}$, we have the estimate

$$|\Delta_t K_{11}(t,\tau)| \le \text{const}\,(\Delta t)^{\alpha/2}(t-\tau)^{\mu-1}, \quad \mu \le 1 - \alpha/2.$$

The estimation of $\Delta_t K_{12}$ is based on the inequality

$$|\Delta_t D_t^{(n-r_l^p+1)/2}\omega^{\chi_p(\tau),\tau}(\xi_1+\chi_p(\tau),t;\chi_p(\tau),\tau)| \\ \le \mathrm{const}\,(\Delta t)^{\alpha/2}(t-\tau)^{(n-r_l^p+2-\alpha)/2}\exp\left[-\frac{\xi_1^2}{4a_0(t-\tau)}\right].$$

As a consequence we get

$$|\Delta_t K_{12}(t,\tau)| \le \mathrm{const}\,(\Delta t)^{\alpha/2}(t-\tau)^{\mu-1}, \quad \mu \le 1-\alpha/2.$$

Combining the results obtained above we have (in case (i))

$$|\Delta_t K_1(t,\tau)| \le \mathrm{const}\,(\Delta t)^{\alpha/2}(t-\tau)^{\mu-1}, \quad \mu \le 1-\alpha/2. \tag{31}$$

In case (ii), by Lemmas 1 and 2, we can write

$$K_1(t,\tau) \\ = -\int\limits_0^{\chi_p(t)-\chi_p(\tau)} \mathbf{Q}_{n-r_l^p-1,0}[D_t^{(n-r_l^p+1)/2}\omega^{\chi_p(\tau),\tau}(\xi_1+\chi_p(t),t;\chi_p(\tau),\tau)]\,d\xi_1$$

and hence, proceeding analogously to case (i), we also get the estimate (31).

Now, we estimate the expression $\Delta_t K_2$ appearing in (30). It suffices to consider $\Delta_t\widetilde{K}_2$, where

$$\widetilde{K}_2(t,\tau) = \chi(t)\overline{K}_2(t,\tau) \tag{32}$$

with

$$\chi(t) = \left[\frac{\chi_p'(t)}{1!}\right]^{i_1}\left[\frac{\chi_p''(t)}{2!}\right]^{i_2}\cdots\left[\frac{\chi_p^{(\nu)}(t)}{\nu!}\right]^{i_\nu},$$

$$\overline{K}_2(t,\tau) = D_t^{\nu-j}D_x^{r_l^p+j}w_{r_q^p}(\chi_p(t),t;\chi_p(\tau),\tau).$$

Clearly,

$$\Delta_t\widetilde{K}_2(t,\tau) = \overline{K}_2(t+\Delta t,\tau)\Delta_t\chi(t) + \chi(t)\Delta_t\overline{K}_2(t,\tau). \tag{33}$$

It follows from inequality (9) and assumption (A.3) that

$$|\overline{K}_2(t+\Delta t,\tau)\Delta_t\chi(t)| \le \mathrm{const}\,(\Delta t)^{\alpha/2}(t-\tau)^{\mu-1}, \quad \mu \le \alpha/2.$$

The second expression appearing in (33) can be estimated in a similar manner by applying assumption (A.3). As a consequence we arive at an estimate analogous to (31).

Estimating $K_3(t,\tau)$ and $K_4(t,\tau)$ in (30) does not cause any additional difficulties. It is based on the inequalities (10) and (11), and also leads to inequalities analogous to (31).

We have proved (29) in the case when d_r is even. For d_r odd it can be proved in a similar way. Thus, the proof of Theorem 1 is complete.

Next, we consider the functions

(34) $$\mathbf{K}^{p\sigma}_{l_0 q}(t,\tau) = \mathbf{B}^p_{l_0} \Lambda_{r^\sigma_q}(\chi_p(t), t; \chi_\sigma(\tau), \tau),$$

where $p, \sigma = 1, 2$, $q = 1, \ldots, l_0$, $l_0 = [(n+3)/2]$, $0 \le \tau < t \le T$ and the operators $\mathbf{B}^p_{l_0}$ are given by the formula

(35) $$\mathbf{B}^p_{l_0} \equiv D^{n+1}_x + \sum_{k=0}^{n} b^p_{k,l_0}(t) D^k_x$$

where b^p_{k,l_0} has continuous derivatives up to order m.

THEOREM 9. *For $\nu = 0, 1, \ldots, m$, we have*

(36) $$|D^\nu_t \mathbf{K}^{p\sigma}_{l_0 q}(t,\tau)| \le \text{const}\,(t-\tau)^{(2m-2\nu+\alpha)/2-1}, \quad 0 \le \tau < t \le T,$$

(37) $$|\Delta_t D^m_t \mathbf{K}^{p\sigma}_{l_0 q}(t,\tau)| \le \text{const}\,(\Delta t)^{\beta/2}(t-\tau)^{\mu-1}, \quad 0 \le \tau < t \le t + \Delta t \le T,$$

where $\mu = \min\{\alpha/2, 1-\alpha/2\}$, $0 < \beta \le \alpha \le 1$.

The proof of Theorem 2 is similar to that of Theorem 1.

5. Properties of the functions z^p_l. Let $f(y,\tau)$ be defined and continuous for $(y,\tau) \in \mathbf{S}_T$. We consider the functions

(38) $$z^p_l(t) = \iint_{\mathbf{S}_t} \mathbf{B}^p_l \Lambda_0(\chi_p(t), t; y, \tau) f(y,\tau)\, dy\, d\tau,$$

where $p = 1, 2$, $l = 1, \ldots, l_0$, $l_0 = [(n+3)/2]$, $\mathbf{B}^p_l$ is given by (27) and

$$\mathbf{S}_t = \{(y,\tau) : \chi_1(\tau) \le y \le \chi_2(\tau),\ 0 < \tau < t\}.$$

LEMMA 8. *For $\nu = 0, 1, \ldots, \mathcal{M} = [d_r/2]$, $d_r = n - r^p_l + 2m + 1$, we have*

(39) $$|\Delta_t D^{\mathcal{M}}_t z^p_l(t)| \le \text{const} \begin{cases} (\Delta t)^{\alpha/2} & \text{if } d_r \text{ is even}, \\ (\Delta t)^{(\alpha+1)/2} & \text{if } d_r \text{ is odd}, \end{cases}$$

$(0 \le t < t + \Delta t \le T,\ 0 < \alpha \le 1)$,

(40) $$D^\nu_t z^p_l(0) = 0, \quad \nu = 0, 1, \ldots, \mathcal{M}.$$

Proof. We consider two cases: (i) d_r is even, (ii) d_r is odd.

In case (i) we use the decomposition

(41) $$\begin{aligned} D^\nu_t z^p_l(t) &= \iint_{\mathbf{S}_t} D^\nu_t D^{r^p_l}_x \Lambda_0(\chi_p(t), t; y, \tau) f(y,\tau)\, dy\, d\tau \\ &\quad + \iint_{\mathbf{S}_t} D^\nu_t \Big[\sum_{k=0}^{r^p_l - 1} b^p_{kl}(t) D^k_x \Lambda_0(\chi_p(t), t; y, \tau) \Big] f(y,\tau)\, dy\, d\tau \\ &= \mathbf{z}_1(t) + \mathbf{z}_2(t). \end{aligned}$$

By (11) and (A.3) we have

$$|D_t^{\mathcal{M}} D_x^{r_l^p} \Lambda_0(\chi_p(t), t; y, \tau)| \le \text{const}\,(t-\tau)^{-1} \exp\left[-\frac{(\chi_p(t)-y)^2}{4a_0(t-\tau)}\right],$$

and hence,

$$|\Delta_t \mathbf{z}_1(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0 \le t < t+\Delta t \le T.$$

By a similar argument we get the same estimate for $\mathbf{z}_2(t)$.

The proof of (39) in case (ii) is analogous. Furthermore, (40) follows immediately from (41) and (11). Thus, the lemma is proved.

Finally, we consider the functions

$$\boldsymbol{z}_{l_0}^p(t) = \iint_{\mathbf{S}_t} \mathbf{B}_{l_0}^p \Lambda_0(\chi_p(t), t; y, \tau) f(y, \tau)\, dy\, d\tau, \tag{42}$$

where $p = 1, 2$, $l_0 = [(n+3)/2]$ and $\mathbf{B}_{l_0}^p$ is defined by (35).

Lemma 9. *For $\nu = 0, 1, \ldots, m$, we have*

$$|\Delta_t D_t^m \boldsymbol{z}_{l_0}^p(t)| \le \text{const}\,(\Delta t)^{\alpha/2}, \quad 0 \le t < t+\Delta t \le T,\ 0 < \alpha \le 1, \tag{43}$$

$$D_t^\nu \boldsymbol{z}_{l_0}^p(0) = 0, \quad \nu = 0, 1, \ldots, m. \tag{44}$$

The proof is similar to that of Lemma 8.

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