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PM functions, their characteristic intervals and iterative roots

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Abstract. The concept of characteristic interval for piecewise monotone functions is introduced and used in the study of their iterative roots on a closed interval.

I. Introduction. The *iterative root* of order n of a function $F : E \to E$, for a given positive integer n and a given set E, is a function $f : E \to E$ such that

$$f^n = F, \tag{1.1}$$

where f^n denotes the nth iterate of f, i.e., $f^n = f \circ f^{n-1}$ and $f^0 = \mathrm{id}$.

The problem of iterative roots, as an important subject in the theory of functional equations, has been studied deeply in various aspects, for example, for real functions by Bödewadt [2], Fort [4] and Kuczma [7–9], and for complex functions by Kneser [5] and Rice [10], since Babbage [3], Abel [1] and Koenigs [6] initiated that research in the last century. In particular, the research in this field gets very active in Poland and China.

It is well known that a strictly increasing continuous function has continuous iterative roots of any order but a strictly decreasing function has no continuous iterative roots of even order. In particular, for monotone functions we have the following result.

Theorem (Bödewadt [2]). *Let $F : I = [a,b] \to I$ be continuous and strictly increasing. Then for any integer $n \ge 2$ and $A, B \in (a,b)$ with $A < B$, (1.1) has a continuous and strictly increasing solution f on I satisfying $F(a) \le f(A) < f(B) \le F(b)$.*

However, there are few results without monotonicity assumptions.

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In 1993, while visiting Poland, the author had a talk about an interesting method, presented in Chinese by J. Zhang and L. Yang [11], based on introducing the so-called "characteristic interval" for piecewise monotone functions. In this paper this method is presented in detail. In Section II we discuss the properties of this type of functions; Section III is devoted to the notion of characteristic interval and an extension theorem; finally, in Section IV, the results of Section III are applied to give the existence of iterative roots for piecewise monotone functions (abbreviated as PM functions) on $I = [a, b] \subset \mathbb{R}^1$. In Sections II to IV, all considered functions are supposed to be continuous from I into itself.

II. PM functions

Definition 1. An interior point x_0 in I is referred to as a *monotone point* of $F : I \to I$ if F is strictly monotone in a neighborhood of x_0. Otherwise, x_0 is called a *fort* (or a non-monotone point). Furthermore, $F \in C^0(I, I)$ is referred to as a *strictly piecewise monotone function* or *PM function* if F has only finitely many forts in I. Let $N(F)$ denote the number of forts of F, and $\mathrm{PM}(I, I)$ the set of all continuous PM functions from I into itself.

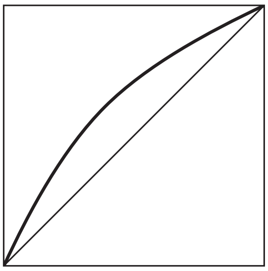

Fig. 1. No fort

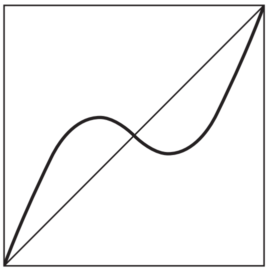

Fig. 2. A PM function

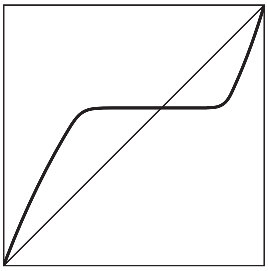

Fig. 3. Infinitely many forts

From Figure 3 we see that a fort may not be an extreme point.

Lemma 2.1 (equivalent definition). *An interior point x_0 in I is a fort of F iff for any $\varepsilon > 0$ there are two points x_1, x_2 in I with $x_1 \neq x_2$, $|x_1 - x_0| < \varepsilon$ and $|x_2 - x_0| < \varepsilon$ such that $F(x_1) = F(x_2)$.*

The simple proof is omitted.

Lemma 2.2. (i) *If $F_1, F_2 \in \mathrm{PM}(I, I)$ then $F_2 \circ F_1 \in \mathrm{PM}(I, I)$.*

(ii) *If $F_2 \circ F_1 \in \mathrm{PM}(I, I)$ then $F_1 \in \mathrm{PM}(I, I)$. Here $\circ$ denotes the composition of functions.*

P r o o f. Let S, S_1, and S_2 denote the sets of forts of $F = F_2 \circ F_1$, F_1, and F_2 respectively, and let $S_3 = \{x \in I \mid F_1(x) \in S_2\}$. Clearly,

$$S = S_1 \cup S_3. \tag{2.1}$$

It follows that the cardinal numbers satisfy

$$\#S \le \#S_1 + \#S_3, \tag{2.2}$$

$$\#S_1 \le \#S. \tag{2.3}$$

Thus (2.3) implies (ii).

On the other hand, $\#S_1 < \infty$ and $\#S_2 < \infty$ imply $\#S_3 < \infty$; otherwise, by $\#S_2 < \infty$, there are infinitely many $x_1 < x_2 < \ldots < x_n < \ldots$ in I such that $F_1(x_i) = F_1(x_j)$, $i \ne j$. By Lemma 2.1 this contradicts the fact that $\#S_1 < \infty$ and implies (i) by (2.2).

COROLLARY 2.3. *If* $f^n \in \mathrm{PM}(I, I)$ *then* $f \in \mathrm{PM}(I, I)$; *and vice versa.*

Furthermore, (2.3) implies for $F \in \mathrm{PM}(I, I)$ that

$$0 = N(F^0) \le N(F) \le N(F^2) \le N(F^3) \le \ldots \le N(F^n) \le \ldots \tag{2.4}$$

Let $H(F)$ denote the smallest positive integer k such that $N(F^k) = N(F^{k+1})$, and let $H(F) = \infty$ when (2.4) is a strictly increasing sequence.

LEMMA 2.4. *Let* $F_1, F_2 \in \mathrm{PM}(I, I)$. *Then* $N(F_2 \circ F_1) = N(F_1)$ *iff* F_2 *is strictly monotone on* $[m, M]$, *the range of* F_1, *where* $m = \min F_1$ *and* $M = \max F_1$.

P r o o f. We use the notations S, S_1, S_2, S_3, F, etc. as in the proof of Lemma 2.2. Note that $[m, M]$ is not a single point set since F_1 as a PM function is not constant. On the one hand, suppose F_2 is strictly monotone on $[m, M]$. For each $x_0 \in S_3$, by the monotonicity of F_2, $F_1(x_0) = m$ or M, that is, x_0 is an extreme point and, of course, a fort of F_1. Thus $S_3 \subset S_1$. From (2.1), $S = S_1$ and $N(F_2 \circ F_1) = N(F_1)$.

On the other hand, for an indirect proof of the necessity we assume that F_2 has a fort x_1 in $[m, M]$. The continuity of F_1 implies that there is a monotone point $x_0 \in (a, b)$ such that $F_1(x_0) = x_1$, i.e., $x_0 \in S_3 \setminus S_1$. Thus $S \setminus S_1 \ne \emptyset$, i.e., $N(F_2 \circ F_1) \ne N(F_1)$. This gives a contradiction.

LEMMA 2.5. *Let* $F \in \mathrm{PM}(I, I)$ *and* $H(F) = k < \infty$. *Then for any integer* $i > 0$, $N(F^k) = N(F^{k+i})$.

P r o o f. Let m_i and M_i denote the minimum and maximum of F^i on I respectively. Since $H(F) = k$ implies

$$N(F^k) = N(F^{k+1}) = N(F \circ F^k), \tag{2.5}$$

by Lemma 2.4, F is strictly monotone on $[m_k, M_k]$. However,

$$m_k \le m_{k+i-1} < M_{k+i-1} \le M_k \quad \text{for } i \ge 1, \tag{2.6}$$

so F is also strictly monotone on $[m_{k+i-1}, M_{k+i-1}]$. By Lemma 2.4,

$$N(F^{k+i-1}) = N(F^{k+i}), \quad i = 1, 2, \ldots \tag{2.7}$$

This completes the proof.

LEMMA 2.6. *If* $H(F) = k$, *then* $H(F^i) = [k/i] + \mathrm{sgn}\{k/i\}$, *for every integer* $i > 0$, *where* $[x]$ *denotes the largest integer not exceeding* x *and* $\{x\} = x - [x]$.

Proof. Let $F_1 = F^i$ and let r denote the right side of the above expression for $H(F^i)$. Since

$$i(r-1) < k \le ir < i(r+1), \tag{2.8}$$

we get

$$N(F_1^{r-1}) = N(F^{i(r-1)}) < N(F^k) = N(F_1^r) = N(F_1^{r+1}). \tag{2.9}$$

This implies $H(F_1) = r$ and completes the proof.

THEOREM 1. *Let* $F \in \mathrm{PM}(I, I)$ *and* $H(F) > 1$. *Then* F *has no continuous iterative roots of order* n *for* $n > N(F)$.

Proof. Assume $f \in C^0(I, I)$ is an iterative root of order n. By Corollary 2.3, $f \in \mathrm{PM}(I, I)$. Since $H(F) > 1$, i.e., $N(f^{2n}) = N(F^2) > N(F) = N(f^n)$, we see that $H(f) > n$ and $0 = N(f^0) < N(f) < N(f^2) < \ldots < N(f^n)$. This implies $N(f^n) \ge n$, i.e., $N(F) \ge n$, contrary to the assumption.

PROBLEM 1. It is still an open question whether F has an iterative root of order n for all $n \le N(F)$.

This theorem says that F can have continuous iterative roots of infinitely many orders only when $H(F) \le 1$. In what follows, we concentrate on the case where $H(F) \le 1$. We show that on the so-called characteristic interval this case reduces to the monotone case.

III. Characteristic interval. Suppose $H(F) \le 1$. For F non-monotone it follows that $N(F) = N(F^2)$. By Lemma 2.4, F is strictly monotone on $[m, M]$, where $m = \min F$ and $M = \max F$. Obviously, extending appropriately the interval on which F is monotone, one can find two points $a', b' \in I$, $a' < b'$, such that

(i) a' and b' are either forts or endpoints;
(ii) there is no fort inside (a', b');
(iii) $[a', b'] \supset [m, M]$.

DEFINITION 2. The unique interval $[a', b']$ obtained above is referred to as the *characteristic interval* of F.

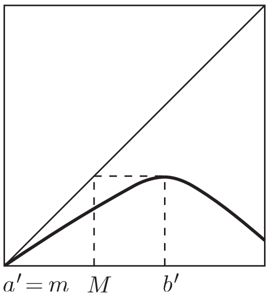

Fig. 4

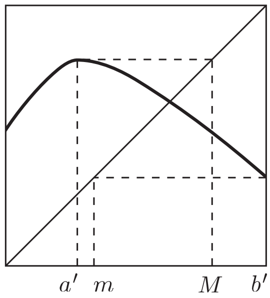

Fig. 5

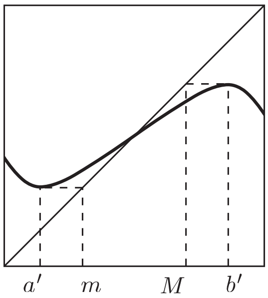

Fig. 6

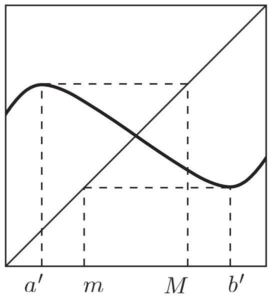

Fig. 7

The above figures illustrate the cases where $H(F) \leq 1$.

THEOREM 2. *Let* $F \in \mathrm{PM}(I, I)$ *and* $H(F) \leq 1$. *Suppose* F *has a continuous iterative root* f *of order* $n > 1$. *Then*

(i) F *is strictly monotone from* $[a', b']$ *into itself*;
(ii) *all periodic points of* F *are inside* $[a', b']$;
(iii) *all periodic points of* f *are inside* $[a', b']$;
(iv) f *is strictly monotone from* $[a', b']$ *into itself*;
(v) $f^n(x) = F(x)$ *for* $x \in [a', b']$;
(vi) *if* $n > N(F) + 1$ *and* $F(x') = a'$ *or* b' *for some* $x' \in I$, *then* $x' \in [a', b']$.

P r o o f. We use the same notations m_i and M_i as in the proof of Lemma 2.5. Obviously, the sequence $\{m_i\}$ is non-decreasing and $\{M_i\}$ is non-increasing. Then (i) follows from the definition of characteristic interval, in particular from $[m_1, M_1] \subset [a', b']$. By Corollary 2.3, f is also strictly monotone on $[a', b']$. To prove (ii), let x_0 be a periodic point of F. Then for some integer $k > 0$, $x_0 = F^k(x_0) \in [m_k, M_k] \subset [m_1, M_1] \subset [a', b']$. Now (iii) follows from (ii) since all periodic points of f are periodic points of F.

Concerning (iv), it suffices to prove that $f(x) \in [a', b']$ for $x \in [a', b']$. In case f is increasing on $[a', b']$, for an indirect proof we assume, without loss of generality, that $f(a') < a'$. Since $f(a) \geq a$, the continuity implies $f(x_1) = x_1$ for some $x_1 \in [a, a')$, i.e., f has a periodic point outside $[a', b']$. This contradicts (iii).

On the other hand, in case f is decreasing on $[a', b']$, F is strictly monotone on $[f(b'), f(a')]$; otherwise, F has forts in this interval, and by Lemma 2.2, $f^{n+1}(x) = F(f(x))$ and even $F(F(x)) = f^{2n}(x)$ have forts on $[a', b']$, which implies $N(F^2) > N(F)$, contrary to $H(F) \leq 1$. Furthermore, neither a' nor b' is an interior point of $[f(b'), f(a')]$ since a' and b' are forts (or endpoints) of F. Thus, in order to prove $[f(b'), f(a')] \subset [a', b']$ we show that the interior of $[a', b'] \cap [f(b'), f(a')]$ is not empty. Indeed, otherwise $F(f(x)) = f(F(x))$ cannot reach the interior of $[a', b']$ for all $x \in I$. However, $(a', b') \supset (m_1, M_1) \neq \emptyset$, so $F(f(x))$ must reach the interior of $[a', b']$ for some $x \in I$. This contradiction completes the proof of (iv). (iv) yields (v) naturally.

Finally, we prove (vi). Note that $n > N(F) + 1 > N(F)$ implies $H(f) < n$; otherwise, $N(f^n) > N(f^{n-1}) > \ldots > N(f) > N(f^0) = 0$, which yields a contradiction that $N(f^n) \geq n > N(F)$. It follows that $N(f^{n-1}) = N(F)$ and by Lemma 2.5 that $N(f^{n-1}) = N(f^{n-1} \circ f^{n-1})$, i.e., $H(f^{n-1}) \leq 1$. Thus $[a', b']$ is also the characteristic interval of f^{n-1} and f^{n-1} maps I into it, since $[\min f^{n-1}, \max f^{n-1}] \supset [m_1, M_1]$ and by Lemma 2.2 (or from (2.1)), f^{n-1} and f^n $(= F)$ have common forts. Therefore, the fact that $F = f \circ f^{n-1}$ reaches a' (or b') on I implies that f also reaches a' (or b') on $[a', b']$. In particular, when f is increasing on $[a', b']$ we can assert that $f(a') = a'$ (or $f(b') = b'$), and then $F(a') = a'$ (or $F(b') = b'$).

Now, we consider the case where f is decreasing on $[a', b']$. Using the same arguments as above, by the hypothesis that $n > N(F) + 1$ we have $H(f) < n - 1$ and $H(f^{n-2}) \leq 1$, i.e., f^{n-2} maps I into $[a', b']$. Thus the fact that $F = f^2 \circ f^{n-1}$ reaches a' (or b') on I implies that f^2 also reaches a' (or b') on $[a', b']$. Since f is decreasing on $[a', b']$ and $f(a') \leq b'$, $f(b') \geq a'$, we see that $f(a') = b'$ and $f(b') = a'$, i.e., f maps $[a', b']$ onto itself. This implies that F also maps $[a', b']$ onto itself. Of course, F reaches a' (or b') on $[a', b']$. This completes the proof.

THEOREM 3 (extension). *Suppose* $F \in \mathrm{PM}(I, I)$ *and* $H(F) \leq 1$. *Let* $[a', b']$ *be the characteristic interval, let* m *and* M *denote the minimum and maximum of* F *on* $[a, b]$, *and* m' *and* M' *those on* $[a', b']$. *If, restricted to* $[a', b']$, *equation* (1.1) *has a continuous solution* f_1 *which maps* $[a', b']$ *into itself and maps* $[m, M]$ *into* $[m', M']$, *then there exists a continuous function* f *from* I *into* I *such that*

(i) $f(x) = f_1(x)$ *for all* $x \in [a', b']$, *and*

(ii) f *satisfies* (1.1) *on the whole interval* I.

This theorem says that the problem of iterative roots can be reduced to that for monotone functions on a subinterval.

Proof. Let F_1 be the restriction of F to $[a', b']$. By Theorem 2, its inverse $F_1^{-1} : [m', M'] \to [a', b']$ is continuous. Let

$$f = F_1^{-1} \circ f_1 \circ F \tag{3.1}$$

on I. Because $F(x) \in [m, M] \subset [a', b']$ for $x \in I$ and $f_1(y) \in [m', M']$ for $y \in [m, M]$, the definition in (3.1) is reasonable and $f : [a, b] \to [a', b']$ is continuous. Obviously, for $x \in I$,

$$f^n(x) = (F_1^{-1} \circ f_1^n \circ F)(x) = (F_1^{-1} \circ F_1 \circ F)(x) = F(x). \tag{3.2}$$

This completes the proof.

IV. Existence of iterative roots

Theorem 4. *Let $F \in \mathrm{PM}(I, I)$ and $H(F) \le 1$. Suppose*

(a) *F is increasing on its characteristic interval $[a', b']$, and*
(b) *$F(x)$ on I cannot reach a' and b' unless $F(a') = a'$ or $F(b') = b'$.*

Then for any integer $n > 1$, F has a continuous iterative root of order n. Moreover, these conditions are necessary for $n > N(F) + 1$.

Proof. By Bödewadt's theorem stated in Section I, F on $[a', b']$ has a continuous iterative root f_1 of order n, which satisfies $m' = F(a') \le f_1(m) < f_1(M) \le F(b') = M'$. By Theorem 3, equation (1.1) has a continuous solution f on the whole interval I, which is an extension of f_1. In particular, for $n > N(F) + 1$, by Theorem 2(vi), the condition (b) is necessary. Furthermore, it is well known that a strictly decreasing function has no continuous iterative roots of even order, so (a) is also necessary.

Problem 2. Does F have iterative roots of order n for $n \le N(F) + 1$ when $H(F) \le 1$ and $F(x') = a'$ (or b') for some $x' \in I$ but $x' \notin [a', b']$?

Theorem 5. *Suppose $F \in \mathrm{PM}(I, I)$, $H(F) \le 1$, and F is decreasing on its characteristic interval $[a', b']$. If either $F(a') = b'$ and $F(b') = a'$, or $a' < F(x) < b'$ on I, then for any odd $n > 0$, F has an iterative root of order n, and for even n, (1.1) has no continuous solutions.*

Proposition. *Suppose $F : [a', b'] \to [a', b']$ is continuous and decreasing, and either $F(a') = b'$ and $F(b') = a'$, or $a' < F(x) < b'$ on $[a', b']$. Then for $n \ge 1$,*

$$f^{2n+1} = F \tag{4.1}$$

has a decreasing C^0 *solution* f *on* $[a', b']$ *such that*

$$F(b') \le f(M) < f(m) \le F(a'), \tag{4.2}$$

where $m = \min F$ *and* $M = \max F$.

P r o o f. Since (4.2) is trivial when $F(a') = b'$ and $F(b') = a'$, we only prove the proposition under the condition that $a' < F(x) < b'$ on $[a', b']$. Note that F has a unique fixed point x_0 in (a', b') and

$$a' \le m < x_0 < M \le b'. \tag{4.3}$$

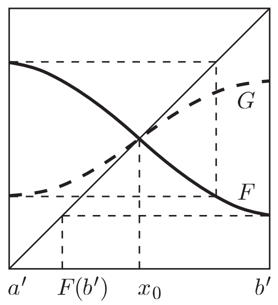

Fig. 8

Clearly, $G = F^2$ is C^0 and increasing on $[a', b']$ and certainly on $[a', x_0]$. By Bödewadt's theorem (see Section I), the equation

$$g^2 = G \tag{4.4}$$

has an increasing C^0 solution g on $[a', x_0]$ such that

$$G(a') = g(F(b')). \tag{4.5}$$

Here (4.5) is guaranteed by the fact that $a' < F(b') < x_0$. Furthermore, the monotonicity implies that $G(x) > x$ and then $g(x) > x$ on $[a', x_0]$, and that $G(x_0) = x_0$ and $g(x_0) = x_0$, so g maps the subinterval $[F(b'), x_0]$ into itself. By Bödewadt's theorem, the equation

$$h^{2n+1} = g \tag{4.6}$$

also has an increasing C^0 solution h on $[F(b'), x_0]$ such that

$$g(F(b')) \le h(\beta) \tag{4.7}$$

for $\beta := \min\{g(m), F(M)\}$. Here (4.7) is guaranteed by the fact that $\beta > \min\{g(a'), F(b')\} = F(b')$, since (4.5) associated with (4.4) implies that

$$g(a') = F(b'). \tag{4.8}$$

In particular, because g is continuous and increasing on $[a', x_0]$ and $[a', x_0] \supset [F(b'), x_0]$, using Bödewadt's inductive construction of iterative roots, one can extend the solution h of (4.6) to the whole interval $[a', x_0]$, i.e., h is increasing and continuous on $[a', x_0]$ and (4.6) still holds. Let

$$h_1(x) = \begin{cases} h(x), & x \in [a', x_0], \\ F^{-1} \circ h \circ F(x), & x \in (x_0, b']. \end{cases} \tag{4.9}$$

Clearly, h_1 is increasing and C^0 on $[a', b']$ and

$$h_1^{2n+1} = g, \qquad h_1 \circ F = F \circ h_1. \tag{4.10}$$

Let

$$f(x) = h_1^{-2n} \circ F(x), \qquad x \in [a', b']. \tag{4.11}$$

Obviously, f is decreasing and C^0 on $[a', b']$, and

$$\begin{aligned} f^{2n+1} &= h_1^{-2n(2n+1)} \circ F^{2n+1} \qquad \text{(by (4.10))} \\ &= g^{-2n} \circ F^{2n} \circ F = F, \end{aligned} \tag{4.12}$$

that is, f is a solution of (4.1) on $[a', b']$.

Moreover, on $[x_0, b']$,

$$f(x) = h^{-2n} \circ F; \tag{4.13}$$

on $[a', x_0]$, the range of F is contained by $[x_0, b']$ and then

$$\begin{aligned} f(x) &= h_1^{-2n} \circ F = (F^{-1} \circ h \circ F)^{-2n} \circ F \\ &= (F^{-1} \circ h^{-1} \circ F)^{2n} \circ F = F^{-1} \circ h^{-2n} \circ F^2 \\ &= F^{-1} \circ h^{-2n} \circ G = F^{-1} \circ h^{2n+2}. \end{aligned} \tag{4.14}$$

Thus the inequalities in (4.2), that is,

$$f(m) \le F(a'), \qquad f(M) \ge F(b'),$$

are equivalent to

$$h^{2n+2}(m) \ge G(a'), \qquad h^{-2n} \circ F(M) \ge F(b'), \tag{4.15}$$

and to

$$h(g(m)) \ge G(a'), \tag{4.16}$$

$$h(F(M)) \ge g(F(b')) = G(a') \qquad \text{(by (4.5))}. \tag{4.17}$$

Obviously (4.16) and (4.17) hold by (4.7) and by the monotonicity of h. This completes the proof.

Proof of Theorem 5. For odd n, the result follows easily from the above Proposition and Theorem 3. For even n, the result is obvious since a strictly decreasing function has no iterative roots of even order, as stated in Section I.

By Theorem 3 (extension theorem), many known results for monotone functions can be generalized to PM functions under the hypothesis that $H(F) \le 1$.

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References

[1] N. H. Abel, *Oeuvres Complètes*, t. II, Christiania, 1881, 36–39.
[2] U. T. Bödewadt, *Zur Iteration reeller Funktionen*, Math. Z. 49 (1944), 497–516.
[3] J. M. Dubbey, *The Mathematical Work of Charles Babbage*, Cambridge Univ. Press, 1978.
[4] M. K. Fort Jr., *The embedding of homeomorphisms in flows*, Proc. Amer. Math. Soc. 6 (1955), 960–967.
[5] H. Kneser, *Reelle analytische Lösungen der Gleichung* $\varphi(\varphi(x)) = e^x$ *und verwandter Funktionalgleichungen*, J. Reine Angew. Math. 187 (1950), 56–67.
[6] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Ann. Ecole Norm. Sup. (3) 1 (1884), Suppl., 3–41.
[7] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat. 46, PWN, Warszawa, 1968.
[8] —, *Fractional iteration of differentiable functions*, Ann. Polon. Math. 22 (1969/70), 217–227.
[9] M. Kuczma and A. Smajdor, *Fractional iteration in the class of convex functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 717–720.
[10] R. E. Rice, B. Schweizer and A. Sklar, *When is* $f(f(z)) = az^2 + bz + c$?, Amer. Math. Monthly 87 (1980), 252–263.
[11] J. Zhang and L. Yang, *Discussion on iterative roots of piecewise monotone functions*, Acta Math. Sinica 26 (1983), 398–412 (in Chinese).

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