PM functions, their characteristic intervals
and iterative roots

by Weinian Zhang (Chengdu)

Abstract. The concept of characteristic interval for piecewise monotone functions is introduced and used in the study of their iterative roots on a closed interval.

I. Introduction. The iterative root of order \( n \) of a function \( F : E \to E \), for a given positive integer \( n \) and a given set \( E \), is a function \( f : E \to E \) such that

\[
    f^n = F,
\]

where \( f^n \) denotes the \( n \)th iterate of \( f \), i.e., \( f^n = f \circ f^{n-1} \) and \( f^0 = \text{id} \).

The problem of iterative roots, as an important subject in the theory of functional equations, has been studied deeply in various aspects, for example, for real functions by Bödewadt [2], Fort [4] and Kuczma [7–9], and for complex functions by Kneser [5] and Rice [10], since Babbage [3], Abel [1] and Koenigs [6] initiated that research in the last century. In particular, the research in this field gets very active in Poland and China.

It is well known that a strictly increasing continuous function has continuous iterative roots of any order but a strictly decreasing function has no continuous iterative roots of even order. In particular, for monotone functions we have the following result.

\textbf{Theorem} (Bödewadt [2]). Let \( F : I = [a, b] \to I \) be continuous and strictly increasing. Then for any integer \( n \geq 2 \) and \( A, B \in (a, b) \) with \( A < B \), (1.1) has a continuous and strictly increasing solution \( f \) on \( I \) satisfying \( F(a) \leq f(A) < f(B) \leq F(b) \).

However, there are few results without monotonicity assumptions.

1991 \textit{Mathematics Subject Classification}: 39B12, 39B22.

\textit{Key words and phrases}: iterative root, piecewise monotone function, characteristic interval.
In 1993, while visiting Poland, the author had a talk about an interesting method, presented in Chinese by J. Zhang and L. Yang [11], based on introducing the so-called “characteristic interval” for piecewise monotone functions. In this paper this method is presented in detail. In Section II we discuss the properties of this type of functions; Section III is devoted to the notion of characteristic interval and an extension theorem; finally, in Section IV, the results of Section III are applied to give the existence of iterative roots for piecewise monotone functions (abbreviated as PM functions) on \( I = [a, b] \subset \mathbb{R}^1 \). In Sections II to IV, all considered functions are supposed to be continuous from \( I \) into itself.

II. PM functions

**Definition 1.** An interior point \( x_0 \) in \( I \) is referred to as a **monotone point** of \( F : I \rightarrow I \) if \( F \) is strictly monotone in a neighborhood of \( x_0 \). Otherwise, \( x_0 \) is called a **fort** (or a non-monotone point). Furthermore, \( F \in C^0(I, I) \) is referred to as a **strictly piecewise monotone function** or **PM function** if \( F \) has only finitely many forts in \( I \). Let \( N(F) \) denote the number of forts of \( F \), and \( \text{PM}(I, I) \) the set of all continuous PM functions from \( I \) into itself.

From Figure 3 we see that a fort may not be an extreme point.

**Lemma 2.1** (equivalent definition). An interior point \( x_0 \) in \( I \) is a fort of \( F \) iff for any \( \varepsilon > 0 \) there are two points \( x_1, x_2 \) in \( I \) with \( x_1 \neq x_2, |x_1 - x_0| < \varepsilon \) and \( |x_2 - x_0| < \varepsilon \) such that \( F(x_1) = F(x_2) \).

The simple proof is omitted.

**Lemma 2.2.** (i) If \( F_1, F_2 \in \text{PM}(I, I) \) then \( F_2 \circ F_1 \in \text{PM}(I, I) \).
(ii) If \( F_2 \circ F_1 \in \text{PM}(I, I) \) then \( F_1 \in \text{PM}(I, I) \). Here \( \circ \) denotes the composition of functions.
Characteristic intervals and iterative roots

**Proof.** Let $S, S_1,$ and $S_2$ denote the sets of forts of $F = F_2 \circ F_1,$ $F_1,$ and $F_2$ respectively, and let $S_3 = \{ x \in I \mid F_1(x) \in S_2 \}.$ Clearly,

(2.1) \[ S = S_1 \cup S_3, \]

It follows that the cardinal numbers satisfy

(2.2) \[ \#S \leq \#S_1 + \#S_3, \]

(2.3) \[ \#S_1 \leq \#S. \]

Thus (2.3) implies (ii).

On the other hand, $\#S_1 < \infty$ and $\#S_2 < \infty$ imply $\#S_3 < \infty$; otherwise, by $\#S_2 < \infty,$ there are infinitely many $x_1 < x_2 < \ldots < x_n < \ldots$ in $I$ such that $F_1(x_i) = F_1(x_j), i \neq j.$ By Lemma 2.1 this contradicts the fact that $\#S_1 < \infty$ and implies (i) by (2.2).

**Corollary 2.3.** If $f^n \in \text{PM}(I, I)$ then $f \in \text{PM}(I, I);$ and vice versa.

Furthermore, (2.3) implies for $F \in \text{PM}(I, I)$ that

(2.4) \[ 0 = N(F^0) \leq N(F) \leq N(F^2) \leq N(F^3) \leq \ldots \leq N(F^n) \leq \ldots \]

Let $H(F)$ denote the smallest positive integer $k$ such that $N(F^k) = N(F^{k+1}),$ and let $H(F) = \infty$ when (2.4) is a strictly increasing sequence.

**Lemma 2.4.** Let $F_1, F_2 \in \text{PM}(I, I).$ Then $N(F_2 \circ F_1) = N(F_1)$ iff $F_2$ is strictly monotone on $[m, M],$ the range of $F_1,$ where $m = \min F_1$ and $M = \max F_1.$

**Proof.** We use the notations $S, S_1, S_2, S_3, F,$ etc. as in the proof of Lemma 2.2. Note that $[m, M]$ is not a single point set since $F_1$ as a PM function is not constant. On the one hand, suppose $F_2$ is strictly monotone on $[m, M].$ For each $x_0 \in S_3,$ by the monotonicity of $F_2,$ $F_1(x_0) = m$ or $M,$ that is, $x_0$ is an extreme point and, of course, a fort of $F_1.$ Thus $S_3 \subset S_1.$ From (2.1), $S = S_1$ and $N(F_2 \circ F_1) = N(F_1).

On the other hand, for an indirect proof of the necessity we assume that $F_2$ has a fort $x_1$ in $[m, M].$ The continuity of $F_1$ implies that there is a monotone point $x_0 \in (a, b)$ such that $F_1(x_0) = x_1,$ i.e., $x_0 \in S_3 \setminus S_1.$ Thus $S \setminus S_1 \neq \emptyset,$ i.e., $N(F_2 \circ F_1) \neq N(F_1).$ This gives a contradiction.

**Lemma 2.5.** Let $F \in \text{PM}(I, I)$ and $H(F) = k < \infty.$ Then for any integer $i > 0,$ $N(F^k) = N(F^{k+i}).$

**Proof.** Let $m_i$ and $M_i$ denote the minimum and maximum of $F^i$ on $I$ respectively. Since $H(F) = k$ implies

(2.5) \[ N(F^k) = N(F^{k+1}) = N(F \circ F^k), \]

by Lemma 2.4, $F$ is strictly monotone on $[m_k, M_k].$ However,

(2.6) \[ m_k \leq m_{k+i-1} < M_{k+i-1} \leq M_k \quad \text{for} \ i \geq 1, \]
so $F$ is also strictly monotone on $[m_{k+i-1}, M_{k+i-1}]$. By Lemma 2.4,

$$N(F^{k+i-1}) = N(F^{k+i}), \quad i = 1, 2, \ldots$$

(2.7)

This completes the proof.

**Lemma 2.6.** If $H(F) = k$, then $H(F^i) = \lfloor k/i \rfloor + \text{sgn}\{k/i\}$, for every integer $i > 0$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$ and $\{x\} = x - \lfloor x \rfloor$.

**Proof.** Let $F_1 = F^i$ and let $r$ denote the right side of the above expression for $H(F^i)$. Since

$$i(r - 1) < k \leq ir < i(r + 1),$$

(2.8)

we get

$$N(F_1^{i-1}) = N(F^{i(r-1)}) < N(F^k) = N(F_1^r) = N(F_1^{r+1}).$$

(2.9)

This implies $H(F_1) = r$ and completes the proof.

**Theorem 1.** Let $F \in \text{PM}(I, I)$ and $H(F) > 1$. Then $F$ has no continuous iterative roots of order $n$ for $n > N(F)$.

**Proof.** Assume $f \in C^0(I, I)$ is an iterative root of order $n$. By Corollary 2.3, $f \in \text{PM}(I, I)$. Since $H(F) > 1$, i.e., $N(f^{2n}) = N(F^2) > N(F) = N(f^n)$, we see that $H(f) > n$ and $0 = N(f^0) < N(f) < N(f^2) < \ldots < N(f^n)$. This implies $N(f^n) \geq n$, i.e., $N(F) \geq n$, contrary to the assumption.

**Problem 1.** It is still an open question whether $F$ has an iterative root of order $n$ for all $n \leq N(F)$.

This theorem says that $F$ can have continuous iterative roots of infinitely many orders only when $H(F) \leq 1$. In what follows, we concentrate on the case where $H(F) \leq 1$. We show that on the so-called characteristic interval this case reduces to the monotone case.

**III. Characteristic interval.** Suppose $H(F) \leq 1$. For $F$ non-monotone it follows that $N(F) = N(F^2)$. By Lemma 2.4, $F$ is strictly monotone on $[m, M]$, where $m = \min F$ and $M = \max F$. Obviously, extending appropriately the interval on which $F$ is monotone, one can find two points $a', b' \in I$, $a' < b'$, such that

(i) $a'$ and $b'$ are either forts or endpoints;
(ii) there is no fort inside $(a', b')$;
(iii) $[a', b'] \supset [m, M]$.

**Definition 2.** The unique interval $[a', b']$ obtained above is referred to as the characteristic interval of $F$. 
The above figures illustrate the cases where $H(F) \leq 1$.

**Theorem 2.** Let $F \in \text{PM}(I, I)$ and $H(F) \leq 1$. Suppose $F$ has a continuous iterative root $f$ of order $n > 1$. Then

(i) $F$ is strictly monotone from $[a', b']$ into itself;
(ii) all periodic points of $F$ are inside $[a', b']$;
(iii) all periodic points of $f$ are inside $[a', b']$;
(iv) $f$ is strictly monotone from $[a', b']$ into itself;
(v) $f^n(x) = F(x)$ for $x \in [a', b']$;
(vi) if $n > N(F) + 1$ and $F(x') = a'$ or $b'$ for some $x' \in I$, then $x' \in [a', b']$.

**Proof.** We use the same notations $m_i$ and $M_i$ as in the proof of Lemma 2.5. Obviously, the sequence $\{m_i\}$ is non-decreasing and $\{M_i\}$ is non-increasing. Then (i) follows from the definition of characteristic interval, in particular from $[m_1, M_1] \subset [a', b']$. By Corollary 2.3, $f$ is also strictly monotone on $[a', b']$. To prove (ii), let $x_0$ be a periodic point of $F$. Then for some integer $k > 0$, $x_0 = F^k(x_0) \in [m_k, M_k] \subset [m_1, M_1] \subset [a', b']$. Now (iii) follows from (ii) since all periodic points of $f$ are periodic points of $F$.

Concerning (iv), it suffices to prove that $f(x) \in [a', b']$ for $x \in [a', b']$. In case $f$ is increasing on $[a', b']$, for an indirect proof we assume, without
loss of generality, that \( f(a') < a' \). Since \( f(a) \geq a \), the continuity implies \( f(x_1) = x_1 \) for some \( x_1 \in [a, a'] \), i.e., \( f \) has a periodic point outside \([a', b']\). This contradicts (iii).

On the other hand, in case \( f \) is decreasing on \([a', b']\), \( F \) is strictly monotone on \([f(b'), f(a')]\); otherwise, \( F \) has forts in this interval, and by Lemma 2.2, \( f^{n+1}(x) = F(f(x)) \) and even \( F(F(x)) = f^{2n}(x) \) have forts on \([a', b']\), which implies \( N(F^2) > N(F) \), contrary to \( H(F) \leq 1 \). Furthermore, neither \( a' \) nor \( b' \) is an interior point of \([f(b'), f(a')]\) since \( a' \) and \( b' \) are forts (or endpoints) of \( F \). Thus, in order to prove \([f(b'), f(a')] \subset [a', b'] \) we show that the interior of \([a', b'] \cap [f(b'), f(a')] \) is not empty. Indeed, otherwise \( F(f(x)) = f(F(x)) \) cannot reach the interior of \([a', b']\) for all \( x \in I \). However, \((a', b') \supset (m_1, M_1) \neq \emptyset \), so \( F(f(x)) \) must reach the interior of \([a', b']\) for some \( x \in I \). This contradiction completes the proof of (iv). (iv) yields (v) naturally.

Finally, we prove (vi). Note that \( n > N(F) + 1 > N(F) \) implies \( H(f) < n \); otherwise, \( N(f^n) > N(f^{n-1}) > \ldots > N(f) > N(f^0) = 0 \), which yields a contradiction that \( N(f^n) \geq n > N(F) \). It follows that \( N(f^{n-1}) = N(F) \) and by Lemma 2.5 that \( N(f^{n-1}) = N(f^{n-1} \circ f^{n-1}) \), i.e., \( H(f^{n-1}) \leq 1 \). Thus \([a', b'] \) is also the characteristic interval of \( f^{n-1} \) and \( f^{n-1} \) maps \( I \) into it, since \([\min f^{n-1}, \max f^{n-1}] \subset [m_1, M_1] \) and by Lemma 2.2 (or from (2.1)), \( f^{n-1} \) and \( f^n (= F) \) have common forts. Therefore, the fact that \( F = f \circ f^{n-1} \) reaches \( a' \) (or \( b' \)) on \( I \) implies that \( f \) also reaches \( a' \) (or \( b' \)) on \([a', b']\). In particular, when \( f \) is increasing on \([a', b']\) we can assert that \( f(a') = a' \) (or \( f(b') = b' \)), and then \( F(a') = a' \) (or \( F(b') = b' \)).

Now, we consider the case where \( f \) is decreasing on \([a', b']\). Using the same arguments as above, by the hypothesis that \( n > N(F) + 1 \) we have \( H(f) < n - 1 \) and \( H(f^{n-2}) \leq 1 \), i.e., \( f^{n-2} \) maps \( I \) into \([a', b']\). Thus the fact that \( F = f^2 \circ f^{n-1} \) reaches \( b' \) (or \( a' \)) on \( I \) implies that \( f^2 \) also reaches \( a' \) (or \( b' \)) on \([a', b']\). Since \( f \) is decreasing on \([a', b']\) and \( f(a') \leq b' \), \( f(b') \geq a' \), we see that \( f(a') = b' \) and \( f(b') = a' \), i.e., \( f \) maps \([a', b']\) onto itself. This implies that \( F \) also maps \([a', b']\) onto itself. Of course, \( F \) reaches \( a' \) (or \( b' \)) on \([a', b']\). This completes the proof.

**Theorem 3** (extension). Suppose \( F \in PM(I, I) \) and \( H(F) \leq 1 \). Let \([a', b']\) be the characteristic interval, let \( m \) and \( M \) denote the minimum and maximum of \( F \) on \([a, b]\), and \( m' \) and \( M' \) those on \([a', b']\). If, restricted to \([a', b']\), equation (1.1) has a continuous solution \( f_1 \) which maps \([a', b']\) into itself and maps \([m, M]\) into \([m', M']\), then there exists a continuous function \( f \) from \( I \) into \( I \) such that

(i) \( f(x) = f_1(x) \) for all \( x \in [a', b'] \), and

(ii) \( f \) satisfies (1.1) on the whole interval \( I \).
This theorem says that the problem of iterative roots can be reduced to that for monotone functions on a subinterval.

**Proof.** Let $F_1$ be the restriction of $F$ to $[a', b']$. By Theorem 2, its inverse

$$F_1^{-1} : [m', M'] \to [a', b']$$

is continuous. Let

$$f = F_1^{-1} \circ f_1 \circ F$$

on $I$. Because $F(x) \in [m, M] \subset [a', b']$ for $x \in I$ and $f_1(y) \in [m', M']$ for $y \in [m, M]$, the definition in (3.1) is reasonable and $f : [a, b] \to [a', b']$ is continuous. Obviously, for $x \in I$,

$$f^n(x) = (F_1^{-1} \circ f_1^n \circ F)(x) = (F_1^{-1} \circ F_1 \circ F)(x) = F(x).$$

This completes the proof.

**IV. Existence of iterative roots**

**Theorem 4.** Let $F \in \text{PM}(I, I)$ and $H(F) \leq 1$. Suppose

(a) $F$ is increasing on its characteristic interval $[a', b']$, and

(b) $F(x)$ on $I$ cannot reach $a'$ and $b'$ unless $F(a') = a'$ or $F(b') = b'$.

Then for any integer $n > 1$, $F$ has a continuous iterative root of order $n$. Moreover, these conditions are necessary for $n > N(F) + 1$.

**Proof.** By Bödewadt’s theorem stated in Section I, $F$ on $[a', b']$ has a continuous iterative root $f_1$ of order $n$, which satisfies $m' = F(a') \leq f_1(m) < f_1(M) \leq F(b') = M'$. By Theorem 3, equation (1.1) has a continuous solution $f$ on the whole interval $I$, which is an extension of $f_1$. In particular, for $n > N(F) + 1$, by Theorem 2(vi), the condition (b) is necessary. Furthermore, it is well known that a strictly decreasing function has no continuous iterative roots of even order, so (a) is also necessary.

**Problem 2.** Does $F$ have iterative roots of order $n$ for $n \leq N(F) + 1$ when $H(F) \leq 1$ and $F(x') = a'$ (or $b'$) for some $x' \in I$ but $x' \notin [a', b']$?

**Theorem 5.** Suppose $F \in \text{PM}(I, I)$, $H(F) \leq 1$, and $F$ is decreasing on its characteristic interval $[a', b']$. If either $F(a') = b'$ and $F(b') = a'$, or $a' < F(x) < b'$ on $I$, then for any odd $n > 0$, $F$ has an iterative root of order $n$, and for even $n$, (1.1) has no continuous solutions.

**Proposition.** Suppose $F : [a', b'] \to [a', b']$ is continuous and decreasing, and either $F(a') = b'$ and $F(b') = a'$, or $a' < F(x) < b'$ on $[a', b']$. Then for $n \geq 1$,

$$f^{2n+1} = F$$
has a decreasing $C^0$ solution $f$ on $[a', b']$ such that
\begin{equation}
F(b') \leq f(M) < f(m) \leq F(a'),
\end{equation}
where $m = \min F$ and $M = \max F$.

**Proof.** Since (4.2) is trivial when $F(a') = b'$ and $F(b') = a'$, we only prove the proposition under the condition that $a' < F(x) < b'$ on $[a', b']$. Note that $F$ has a unique fixed point $x_0$ in $(a', b')$ and
\begin{equation}
(4.3)
\quad a' \leq m < x_0 < M \leq b'.
\end{equation}

![Fig. 8](image)

Clearly, $G = F^2$ is $C^0$ and increasing on $[a', b']$ and certainly on $[a', x_0]$. By Bödewadt’s theorem (see Section I), the equation
\begin{equation}
(4.4)
\quad g^2 = G
\end{equation}
has an increasing $C^0$ solution $g$ on $[a', x_0]$ such that
\begin{equation}
(4.5)
\quad G(a') = g(F(b')).
\end{equation}
Here (4.5) is guaranteed by the fact that $a' < F(b') < x_0$. Furthermore, the monotonicity implies that $G(x) > x$ and then $g(x) > x$ on $[a', x_0]$, and that $G(x_0) = x_0$ and $g(x_0) = x_0$, so $g$ maps the subinterval $[F(b'), x_0]$ into itself. By Bödewadt’s theorem, the equation
\begin{equation}
(4.6)
\quad h^{2n+1} = g
\end{equation}
also has an increasing $C^0$ solution $h$ on $[F(b'), x_0]$ such that
\begin{equation}
(4.7)
\quad g(F(b')) \leq h(\beta)
\end{equation}
for $\beta := \min\{g(m), F(M)\}$. Here (4.7) is guaranteed by the fact that $\beta > \min\{g(a'), F(b')\} = F(b')$, since (4.5) associated with (4.4) implies that
\begin{equation}
(4.8)
\quad g(a') = F(b').
\end{equation}
In particular, because $g$ is continuous and increasing on $[a', x_0]$ and $[a', x_0] \supset [F(b'), x_0]$, using Bödewadt’s inductive construction of iterative roots, one
can extend the solution $h$ of (4.6) to the whole interval $[a', x_0]$, i.e., $h$ is increasing and continuous on $[a', x_0]$ and (4.6) still holds. Let

$$
(4.9) \quad h_1(x) = \begin{cases} 
    h(x), & x \in [a', x_0], \\
    F^{-1} \circ h \circ F(x), & x \in (x_0, b'].
\end{cases}
$$

Clearly, $h_1$ is increasing and $C^0$ on $[a', b']$ and

$$
(4.10) \quad h_1^{2n+1} = g, \quad h_1 \circ F = F \circ h_1.
$$

Let

$$
(4.11) \quad f(x) = h_1^{-2n} \circ F(x), \quad x \in [a', b'].
$$

Obviously, $f$ is decreasing and $C^0$ on $[a', b']$, and

$$
(4.12) \quad f^{2n+1} = h_1^{-2n(2n+1)} \circ F^{2n+1} \quad \text{(by (4.10))}
= g^{-2n} \circ F^{2n} \circ F = F,
$$

that is, $f$ is a solution of (4.1) on $[a', b']$.

Moreover, on $[x_0, b']$,

$$
(4.13) \quad f(x) = h^{-2n} \circ F;
$$
on $[a', x_0]$, the range of $F$ is contained by $[x_0, b']$ and then

$$
(4.14) \quad f(x) = h_1^{-2n} \circ F = (F^{-1} \circ h \circ F)^{-2n} \circ F \\
= (F^{-1} \circ h^{-1} \circ F)^{2n} \circ F = F^{-1} \circ h^{-2n} \circ F^2 \\
= F^{-1} \circ h^{-2n} \circ G = F^{-1} \circ h^{2n+2}.
$$

Thus the inequalities in (4.2), that is,

$$
(4.15) \quad f(m) \leq F(a'), \quad f(M) \geq F(b'),
$$

are equivalent to

$$
(4.16) \quad h^{2n+2}(m) \geq G(a'), \quad h^{-2n} \circ F(M) \geq F(b'),
$$

and to

$$
(4.17) \quad h(g(m)) \geq G(a'), \quad h(F(M)) \geq g(F(b')) = G(a') \quad \text{(by (4.5))}.
$$

Obviously (4.16) and (4.17) hold by (4.7) and by the monotonicity of $h$. This completes the proof.

**Proof of Theorem 5.** For odd $n$, the result follows easily from the above Proposition and Theorem 3. For even $n$, the result is obvious since a strictly decreasing function has no iterative roots of even order, as stated in Section I.

By Theorem 3 (extension theorem), many known results for monotone functions can be generalized to PM functions under the hypothesis that $H(F) \leq 1$. 


Acknowledgements. The author thanks the referee for his helpful comments.

References


Centre for Math. Sciences
CICA, Academia Sinica
Chengdu 610041, P.R. China

Reçu par la Rédaction le 14.11.1994
Révisé le 11.9.1995