Banach–Saks property in some Banach sequence spaces

by YUNAN CUI (Harbin),HENRYK HUDZIK (Poznań) and RYSZARD PLUCIENNIK (Zielona Góra)

Abstract. It is proved that for any Banach space \( X \) property (\( \beta \)) defined by Rolewicz in [22] implies that both \( X \) and \( X^* \) have the Banach–Saks property. Moreover, in Musielak–Orlicz sequence spaces, criteria for the Banach–Saks property, the near uniform convexity, the uniform Kadec–Klee property and property (H) are given.

1. Introduction. Let \( \mathbb{N} \), \( \mathbb{R} \), and \( \mathbb{R}_+ \) stand for the set of natural numbers, the set of reals and the set of nonnegative reals, respectively. Let \( (X, \|\cdot\|) \) be a real Banach space, and \( X^* \) be the dual space of \( X \). By \( B(X) \) and \( S(X) \) we denote the closed unit ball and the unit sphere of \( X \), respectively. For any subset \( A \) of \( X \) by \( \text{conv}(A) \) (\( \text{conv}(A) \)) we denote the convex hull (the closed convex hull) of \( A \). In [2], Clarkson has introduced the concept of uniform convexity.

A norm \( \|\cdot\| \) is called uniformly convex (written UC) if for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for \( x, y \in S(X) \) the inequality \( \|x - y\| > \varepsilon \) implies

\[
\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta.
\]

A Banach space \( X \) is said to have the Banach–Saks property if every bounded sequence \( (x_n) \) in \( X \) admits a subsequence \( (z_n) \) such that the sequence of its arithmetic means \( \left\{ \frac{1}{n}(z_1 + z_2 + \ldots + z_n) \right\} \) is convergent in norm (see [1]).

It is well known that every Banach space \( X \) with the Banach–Saks property is reflexive and the converse is not true (see [7]). Kakutani [12] has proved that any uniformly convex Banach space \( X \) has the Banach–Saks property. Moreover, he has also proved that if \( X \) is a reflexive Banach space
such that there is $\Theta \in (0, 2)$ such that for every sequence $(x_n)$ in $S(X)$ weakly convergent to zero there are $n_1, n_2 \in \mathbb{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$, then $X$ has the Banach–Saks property.

A Banach space $X$ is said to have property (H) (or the Kadec–Klee property) if every weakly convergent sequence on the unit sphere $S(X)$ is convergent in norm (see [11]).

Recall that a sequence $(x_n)$ is said to be an $\varepsilon$-separated sequence if for some $\varepsilon > 0$,

$$\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$  

A Banach space $X$ is said to have the uniform Kadec–Klee property (written UKK) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x$ is the weak limit of a norm one $\varepsilon$-separated sequence, then $\|x\| < 1 - \delta$. Every UKK Banach space has property (H) (see [10]).

A Banach space is said to be nearly uniformly convex (written NUC for short) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each basic sequence $(x_n)$ in $B(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + t\varepsilon$$

for each $t \in [0, \delta]$. Prus [20] has shown that a Banach space $X$ is NUC if and only if $X^*$ is NUS.

For any $x \not\in B(X)$, the drop determined by $x$ is the set

$$D(x, B(X)) = \text{conv}(\{x\} \cup B(X))$$

(see [5]). A Banach space $X$ has the drop property (written (D)) if for every closed set $C$ disjoint from $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$  

In [22], Rolewicz has proved that if the Banach space $X$ has the drop property, then $X$ is reflexive. Montesinos [18] has extended this result showing that $X$ has the drop property if and only if $X$ is a reflexive Banach space with property (H).

For any subset $C$ of $X$ we denote by $\alpha(C)$ its Kuratowski measure of noncompactness, i.e. the infimum of those $\varepsilon > 0$ for which there is a covering of $C$ by a finite number of sets of diameter less than $\varepsilon$.

Goebel and Sękowski [8] have extended the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of noncompactness. Namely, they called a norm $\|\cdot\|$ in a Banach space $X$
\textit{\(\Delta\)–uniformly convex} (written \(\Delta\text{UC}\)) if for any \(\varepsilon > 0\) there is \(\delta > 0\) such that for each convex set \(E\) contained in the closed unit ball \(B(X)\) such that \(\alpha(E) > \varepsilon\), we have

\[
\inf\{\|x\| : x \in E\} < 1 - \delta.
\]

It is well known that \(\Delta\text{UC}\) coincides with \(\text{NUC}\).

Rolewicz [22], studying the relationships between \(\text{NUC}\) and the drop property, has defined property \((\beta)\). A Banach space \(X\) is said to have property \((\beta)\) if for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
\alpha(D(x, B(X) \setminus B(X))) < \varepsilon
\]

whenever \(1 < \|x\| < 1 + \delta\). It is well known that if a Banach space \(X\) has property \((\beta)\), then its dual space \(X^*\) has the normal structure (see [16]). The following result will be very helpful for our considerations (see [15]):

A Banach space \(X\) has property \((\beta)\) if and only if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for each element \(x \in B(X)\) and each sequence \((x_n)\) in \(B(X)\) with

\[
\text{sep}(x_n) \geq \varepsilon
\]

there is an index \(k\) such that

\[
\left\|\frac{x + x_k}{2}\right\| \leq 1 - \delta.
\]

A map \(\Phi : \mathbb{R} \to \mathbb{R}_+\) is said to be an \textit{Orlicz function} if \(\Phi\) vanishes only at 0, and \(\Phi\) is even, convex, and continuous on the whole \(\mathbb{R}_+\) (see [17], [19], [21]).

A sequence \(\Phi = (\Phi_n)\) of Orlicz functions is called a \textit{Musielak–Orlicz function}. By \(\Psi = (\Psi_n)\) we denote the complementary function of \(\Phi\) in the sense of Young, i.e.

\[
\Psi_n(v) = \sup\{\|u - \Phi_n(u) : u \geq 0\}, \quad n = 1, 2, \ldots
\]

Denote by \(l^0\) the space of all real sequences \(x = (x(i))\). For a given Musielak–Orlicz function \(\Phi\), we define a convex modular \(I_\Phi : l^0 \to [0, \infty]\) by the formula

\[
I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i)).
\]

The Musielak–Orlicz sequence space \(l_\Phi\) is

\[
l_\Phi = \{x \in l^0 : I_\Phi(cx) < \infty\text{ for some }c > 0\}.
\]

We consider \(l_\Phi\) equipped with the so-called \textit{Luxemburg norm}

\[
\|x\| = \inf\{\varepsilon > 0 : I_\Phi(x/\varepsilon) \leq 1\},
\]

under which it is a Banach space (see [3], [19]).

The subspace \(h_\Phi\) defined by

\[
h_\Phi = \{x \in l_\Phi : I_\Phi(cx) < \infty\text{ for every }c > 0\}
\]

is called the \textit{subspace of finite (or order continuous) elements}.\]
We say an Orlicz function \( \Phi \) satisfies the \( \delta_2 \)-condition \((\Phi \in \delta_2 \text{ for short})\) if there exist constants \( k \geq 2, \ u_0 > 0 \) and a sequence \((c_i)\) of nonnegative numbers such that \(\sum_{i=1}^{\infty} c_i < \infty\) and the inequality
\[
\Phi_i(2u) \leq k\Phi_i(u) + c_i
\]
holds for every \( i \in \mathbb{N} \) and \( u \in \mathbb{R} \) satisfying \( |u| \leq u_0 \).

It is well known that \( h_\Phi = l_\Phi \) if and only if \( \Phi \in \delta_2 \) (see [13]).

We say a Musielak–Orlicz function \( \Phi \) satisfies condition \((\ast)\) if for any \( \varepsilon \in (0, 1) \) there exists \( \delta > 0 \) such that \( \Phi_i((1+\delta)u) \leq 1 \) whenever \( \Phi_i(u) \leq 1 - \varepsilon \) for \( u \in \mathbb{R} \) and all \( i \in \mathbb{N} \) (see [14]).

For more details on Musielak–Orlicz spaces we refer to [3] or [19].

2. Auxiliary facts.
In order to obtain some new results, we will use the following well-known facts.

**Lemma 1** (see [4]). If a Musielak–Orlicz function \( \Phi = (\Phi_i) \) with all \( \Phi_i \) finitely valued satisfies condition \((\ast)\) and \( \Phi \in \delta_2 \), then for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \|x\| < 1 - \delta \) whenever \( I_\Phi(x) < 1 - \varepsilon \).

**Lemma 2** (see [14]). If a Musielak–Orlicz function \( \Phi = (\Phi_i) \) with all \( \Phi_i \) finitely valued satisfies condition \((\ast)\) and \( \Phi \in \delta_2 \), then for every \( \varepsilon > 0 \) and \( c > 0 \) there exists \( \delta > 0 \) such that
\[
|I_\Phi(x + y) - I_\Phi(x)| < \varepsilon
\]
whenever \( I_\Phi(x) \leq c \) and \( I_\Phi(y) < \delta \).

**Lemma 3** (see [6]). If a Musielak–Orlicz function \( \Psi = (\Psi_i) \in \delta_2 \), then there exists \( \theta \in (0, 1) \) and a sequence \((h_i)\) of nonnegative numbers such that \( \sum_{i=1}^{\infty} \Phi_i(h_i) < \infty \) and the inequality
\[
\Phi_i\left(\frac{u}{2}\right) \leq \frac{1-\theta}{2} \Phi_i(u)
\]
holds for every \( i \in \mathbb{N} \) and \( u \in \mathbb{R} \) with \( \Phi_i(h_i) \leq \Phi_i(u) \leq 1 \).

3. Results.
We start with the following general result.

**Theorem 1.** If a Banach space \( X \) has property \((\beta)\), then both \( X \) and \( X^* \) have the Banach–Saks property.

**Proof.** Assume \( X \) has property \((\beta)\). First, we will prove that \( X \) has the Banach–Saks property. Since property \((\beta)\) implies reflexivity, it is enough to prove that there exists \( \Theta \in (0, 2) \) such that for each sequence \( (x_n) \) in \( S(X) \) weakly convergent to zero, there are \( n_1, n_2 \in \mathbb{N} \) such that \( \|x_{n_1} + x_{n_2}\| < \Theta \).

Since \( (x_n) \) is weakly convergent to zero, the set of its elements cannot be compact in \( S(X) \). So, there are \( \varepsilon_0 > 0 \) and a subsequence \((z_n)\) of \((x_n)\)
Banach–Saks property

with sep($z_n$) $\geq \varepsilon_0$. By property ($\beta$) for $X$, there exists $\delta > 0$ depending on $\varepsilon_0$ only such that for every $z \in S(X)$ there exists $k \in \mathbb{N}$ for which

$$\|z + z_k\| < 2 - \delta$$

(cf. Proposition 1 in [15]). In particular, setting $z = z_1$, a natural number $k(1) \neq 1$ can be found such that

$$\|z_1 + z_{k(1)}\| < \Theta,$$

where $\Theta = 2 - \delta$. This means that $X$ has the Banach–Saks property.

Next, we will prove that $X^*$ has the Banach–Saks property. For each sequence $(x_n)$ in $S(X)$ weakly convergent to zero, by the Bessaga–Pełczyński selection principle, there exists a basic subsequence $(z_n)$ of $(x_n)$ (see [7]).

Property ($\beta$) for $X$ implies that $X^*$ is NUS (see [20]), i.e. for any $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that there is $k \in \mathbb{N}$, $k > 1$, such that

$$\|z_1 + tz_k\| < 1 + t\varepsilon$$

for any $t \in [0, \delta]$. In particular, taking $\varepsilon = 1/2$, numbers $\delta_0 \in (0, 1)$ and $k > 1$, $k \in \mathbb{N}$, can be found such that

$$\|z_1 + \delta_0 z_k\| < 1 + \frac{\delta_0}{2}.$$ 

Hence

$$\|z_1 + z_k\| = \|z_1 + \delta_0 z_k + (1 - \delta_0) z_k\| < 1 + \frac{\delta_0}{2} + (1 - \delta_0) = 2 - \frac{\delta_0}{2},$$

i.e. $X^*$ has the Banach–Saks property.

Theorem 1 cannot be reversed in general. Indeed, note that $c_0$ as well as its dual $l_1$ have the Banach–Saks property, but they fail property ($\beta$). However, both $c_0$ and $l_1$ are not reflexive. It is natural to ask the following

**Question.** Assume that $X$ is a reflexive Banach space. Does the Banach–Saks property for $X$ and $X^*$ imply property ($\beta$) for $X$?

Now, we will describe some geometric properties in Musielak–Orlicz sequence spaces.

**Theorem 2.** If a Musielak–Orlicz function $\Phi = (\Phi_i)$ with all $\Phi_i$ finitely valued satisfies condition $(\ast)$, then the following statements are equivalent:

(a) $l_\Phi$ is UKK;
(b) $l_\Phi$ has property (H);
(c) $\Phi \in \delta_2$. 
Proof. (a)⇒(b). This holds true for any Banach space (see [10]).

(b)⇒(c). If Φ ∉ δ2, we can find an element x = (x(1), x(2), ...) ∈ S(IΦ) such that IΦ(x) ≤ 1 and IΦ(λx) = ∞ for any λ > 1 (see [13]). Consequently, there is an increasing sequence (n_i) of natural numbers such that

\[ \|(0, \ldots, 0, x(n_i + 1), \ldots, x(n_{i+1}), 0, \ldots)\| \geq \frac{1}{2}. \]

Putting

\[ x_i = (x(1), \ldots, x(n_i), 0, \ldots, 0, x(n_{i+1}), \ldots), \quad i = 1, 2, \ldots, \]

we get

1. \( \|x_i\| = 1, \ i = 1, 2, \ldots; \)
2. \( x_i \to x \) weakly.

Equalities (1) follow by \( I_\Phi(x_i) \leq 1 \) and \( I_\Phi(\lambda x_i) = \infty \) for every \( \lambda > 1 \) (i = 1, 2, ...). We will now prove property (2). For every \( y^* \in (I_\Phi)^* \) we have \( y^* = y^*_0 + y^*_1 \) uniquely, where \( y^*_0 \) and \( y^*_1 \) are respectively the regular and singular parts of \( y^* \), i.e. \( y^*_0 \) is determined by a function \( y_0 \in I_\Phi \) and \( y^*_1(x) = 0 \) for any \( x \in h_\Phi \) (see [9]). Since \( y_0 = (y_0(i)) \in l_\Phi \), there exists \( \lambda > 0 \) such that \( \sum_{i=1}^\infty \|\gamma_j(\lambda y_0(i))\| < \infty \). Since \( \langle x_i - x, y^*_1 \rangle = 0 \), we have

\[ \langle x_i - x, y^* \rangle = \langle x_i - x, y^*_0 \rangle = \sum_{j=n_{i+1}}^{n_i+1} x(j)y_0(j) \]

\[ \leq \frac{1}{\lambda} \sum_{j=n_{i+1}}^{n_i+1} (\Phi_j(x(j)) + \Psi_j(\lambda y_0(j))) \to 0 \quad \text{as} \quad i \to \infty, \]

which proves (2). We also have

3. \( \|x_i - x\| \geq 1/2 \) for all \( i \in \mathbb{N} \),

which means that \( I_\Phi \) does not have property (H).

(c)⇒(a). Suppose \( I_\Phi \) is not UKK and \( \Phi \in \delta_2 \). There exists \( \varepsilon_0 > 0 \) such that for any \( \theta > 0 \) there are a sequence \( (x_n) \) and an element \( x \) in \( S(I_\Phi) \) with \( \text{sep}(x_n) \geq \varepsilon_0 \), \( x_n \to x \) weakly and \( \|x\| > 1 - \theta \). Since \( \text{sep}(x_n) \geq \varepsilon_0 \), we can assume without loss of generality that \( \|x_n - x\| \geq \varepsilon_0/2 \) for every \( n \in \mathbb{N} \). Since \( \Phi \in \delta_2 \) and \( \Phi \) satisfies condition (*) and \( x \) can be assumed to have \( \|x\| \) close to 1, we may assume that there is \( \eta_0 > 0 \) such that \( I_\Phi(x_n - x) \geq \eta_0 \) and \( I_\Phi(x) > 1 - \eta_0/5 \). Using again \( \Phi \in \delta_2 \), there exists \( \sigma_0 \in (0, \eta_0/5) \) such that

\[ |I_\Phi(x + y) - I_\Phi(x)| < \frac{\eta_0}{5} \]

whenever \( I_\Phi(y) < \sigma_0 \).

Since \( (x_n) \subset S(I_\Phi) \) and \( x_n \to x \) weakly, by the lower semicontinuity of the norm with respect to the weak topology, we conclude that there is
\[ i_0 \in \mathbb{N} \text{ such that } \sum_{i=i_0+1}^{\infty} \Phi_i(x(i)) < \sigma_0. \]

By virtue of \( x_n \to x \) weakly, which implies that \( x_n \to x \) coordinatewise, there exists \( n_0 \in \mathbb{N} \) such that

\[
\left| \sum_{i=1}^{i_0} \Phi_i(x_n(i)) - \sum_{i=1}^{i_0} \Phi_i(x(i)) \right| < \frac{\eta_0}{5} \quad \text{and} \quad \sum_{i=1}^{i_0} \Phi_i(x_n(i)) - x(i) < \frac{\eta_0}{5}
\]

for \( n \geq n_0 \). So

\[
1 = \sum_{i=1}^{\infty} \Phi_i(x_n(i)) = \sum_{i=1}^{i_0} \Phi_i(x_n(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i))
\]

\[
\geq \sum_{i=1}^{i_0} \Phi_i(x(i)) - \frac{\eta_0}{5} + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)).
\]

Hence

\[
\eta_0 \leq I_{\Phi}(x_n - x) = \sum_{i=1}^{\infty} \Phi_i(x_n(i) - x(i))
\]

\[
= \sum_{i=1}^{i_0} \Phi_i(x_n(i) - x(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i) - x(i))
\]

\[
< \frac{\eta_0}{5} + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) + \frac{\eta_0}{5} \leq 1 - \sum_{i=1}^{i_0} \Phi_i(x(i)) + \frac{2\eta_0}{5} + \frac{\eta_0}{5}
\]

\[
\leq 1 - (1 - \sigma_0) + \frac{3\eta_0}{5} \leq 1 - \left( 1 - \frac{\eta_0}{5} \right) + \frac{3\eta_0}{5} < \eta_0.
\]

This contradiction proves the implication \((c) \Rightarrow (a)\).

**Corollary 1.** If a Musielak–Orlicz function \( \Phi = (\Phi_i) \) with all \( \Phi_i \) finitely valued satisfies condition \((*)\), then the following statements are equivalent:

(a) \( l_\Phi \) is NUC;
(b) \( l_\Phi \) has the drop property;
(c) \( \Phi \in \delta_2 \) and \( \Psi \in \delta_2 \).

**Proof.** Since NUC is equivalent to the conjunction of UKK and reflexivity, and the reflexivity of \( l_\Phi \) is equivalent to the fact that \( \Phi \in \delta_2 \) and \( \Psi \in \delta_2 \), by Theorem 2, we get our corollary immediately.

Recall that a Nakano space \( l^{(p_i)} \) is the Musielak–Orlicz space \( l_\Phi \) with \( \Phi = (\Phi_i) \), where

\[
\Phi_i(u) = |u|^{p_i}, \quad 1 \leq p_i < \infty, \quad i = 1, 2, \ldots
\]

**Corollary 2.** For any Nakano space \( l^{(p_i)} \) the following statements are equivalent:
(a) $l^{(p_i)}$ is NUC;
(b) $l^{(p_i)}$ has the drop property;
(c) $1 < \liminf_i p_i \leq \limsup_i p_i < \infty$.

**Proof.** This follows immediately by Corollary 1 and the fact that for the Nakano function $\Phi = (\Phi_i)$ with $\Phi_i(u) = |u|^{p_i}$ we have $\Phi \in \delta_2$ if and only if $\limsup_i p_i < \infty$, and its complementary function $\Psi \in \delta_2$ if and only if $\liminf_i p_i > 1$.

**Corollary 3.** Let $l^{(p_i)}$ be a Nakano space. Then the following statements are equivalent:

(a) $l^{(p_i)}$ is UKK;
(b) $l^{(p_i)}$ has property $(H)$;
(c) $\limsup_i p_i < \infty$.

**Proof.** This follows immediately by Theorem 2 and the fact that the Nakano function $\Phi = (\Phi_i)$ with $\Phi_i(u) = |u|^{p_i}$ satisfies the $\delta_2$-condition if and only if condition (c) is satisfied.

**Theorem 3.** If a Musielak–Orlicz function $\Phi = (\Phi_i)$, with all $\Phi_i$ finitely valued and satisfying $\Phi_i(u)/u \to 0$ as $u \to 0$, satisfies condition $(\ast)$, then $l_{\Phi}$ has the Banach–Saks property if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.

**Proof.** Since the Banach–Saks property implies reflexivity and the reflexivity of $l_{\Phi}$ is equivalent to $\Phi \in \delta_2$ and $\Psi \in \delta_2$, we only need to prove sufficiency. By $\Psi \in \delta_2$, there exists $\Theta \in (0, 1)$ and a sequence $(h_i)$ of positive numbers such that $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ and

$$\Phi_i \left( \frac{u}{2} \right) \leq (1 - \Theta) \frac{\Phi_i(u)}{2}$$

for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$ (see Lemma 3).

By $\Phi \in \delta_2$ and condition $(\ast)$ for $\Phi$, for any $\varepsilon \in (0, \Theta/16)$, there exists a $\delta \in (0, \Theta)$ such that

$$|I_{\Phi}(y + z) - I_{\Phi}(y)| < \frac{\varepsilon}{2}$$

whenever $I_{\Phi}(y) \leq 1$, $I_{\Phi}(z) \leq \delta$ (see Lemma 2).

For each sequence $(x_n)$ of $S(l_{\Phi})$ with $x_n \to 0$ weakly, we have $x_n \to 0$ coordinatewise, so there are $i_0$ and $n_0 \in \mathbb{N}$ such that $\sum_{i=i_0}^{\infty} \Phi_i(x_1(i)) < \delta$, $\sum_{i=i_0}^{\infty} \Phi_i(h_i) < \delta/16$ and $\sum_{i=1}^{n_0} \Phi_i(x_n(i)) < \delta$ for $n > n_0$. Hence $\sum_{i=i_0}^{\infty} \Phi_i(x_n(i)) \geq 1/2$ for $n > n_0$ and
\[ I_{\Phi} \left( \frac{x_1 + x_n}{2} \right) = \sum_{i=1}^{i_0} \Phi_i \left( \frac{x_n(i) + x_1(i)}{2} \right) + \sum_{i=i_0+1}^{\infty} \Phi_i \left( \frac{x_n(i) + x_1(i)}{2} \right) \]

\[ \leq \sum_{i=1}^{i_0} \frac{\Phi_i(x_1(i))}{2} + \frac{1 - \Theta}{2} \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(h_i) + \epsilon \]

\[ = \frac{1}{2} \left\{ \sum_{i=1}^{i_0} \Phi_i(x_1(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) \right\} \]

\[ + \sum_{i=i_0+1}^{\infty} \Phi_i(h_i) + \epsilon - \frac{\Theta}{2} \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) \]

\[ \leq 2 \cdot \frac{1}{2} + \frac{\Theta}{16} + \frac{\Theta}{4} = 1 - \frac{\Theta}{8}, \]

for \( n > n_0 \).

In view of Lemma 1, there exists \( \delta > 0 \) independent of \( x_1 \) and \( x_n \) such that

\[ \|x_1 + x_n\| < 2 - \delta \quad \text{for } n > n_0. \]

The proof of Theorem 3 is finished.

**Corollary 4.** The Nakano sequence space \( l^{(p_i)} \) has the Banach–Saks property if and only if \( 1 < \lim \inf p_i \leq \lim \sup p_i < \infty \).

**References**


Department of Mathematics
Harbin University of Science and Technology
Xuefu Road 52
150080 Harbin, China

Faculty of Mathematics
and Computer Science
Adam Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
E-mail: hudzik@math.amu.edu.pl

Institute of Mathematics
T. Kotarbiński Pedagogical University
Pl. Słowiański 9
65-069 Zielona Góra, Poland
E-mail: rplucien@wsp.zgora.pl

Reçu par la Rédaction le 15.4.1996
Revisé le 25.6.1996