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Banach–Saks property in some Banach sequence spaces

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Abstract. It is proved that for any Banach space X property (β) defined by Rolewicz in [22] implies that both X and X^* have the Banach–Saks property. Moreover, in Musielak–Orlicz sequence spaces, criteria for the Banach–Saks property, the near uniform convexity, the uniform Kadec–Klee property and property $(\mathbf{H})$ are given.

1. Introduction. Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ stand for the set of natural numbers, the set of reals and the set of nonnegative reals, respectively. Let $(X, \|\cdot\|)$ be a real Banach space, and X^* be the dual space of X. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X by $\mathrm{conv}(A)$ ($\overline{\mathrm{conv}}(A)$) we denote the convex hull (the closed convex hull) of A. In [2], Clarkson has introduced the concept of uniform convexity.

A norm $\|\cdot\|$ is called *uniformly convex* (written **UC**) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ the inequality $\|x - y\| > \varepsilon$ implies

$$\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta. \tag{1}$$

A Banach space X is said to have the *Banach–Saks property* if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence of its arithmetic means $\left(\frac{1}{n}(z_1 + z_2 + \ldots + z_n)\right)$ is convergent in norm (see [1]).

It is well known that every Banach space X with the Banach–Saks property is reflexive and the converse is not true (see [7]). Kakutani [12] has proved that any uniformly convex Banach space X has the Banach–Saks property. Moreover, he has also proved that if X is a reflexive Banach space

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such that there is $\Theta \in (0,2)$ such that for every sequence (x_n) in $S(X)$ weakly convergent to zero there are $n_1, n_2 \in \mathbb{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$, then X has the Banach–Saks property.

A Banach space X is said to have *property* (**H**) (or the *Kadec–Klee property*) if every weakly convergent sequence on the unit sphere $S(X)$ is convergent in norm (see [11]).

Recall that a sequence (x_n) is said to be an *ε-separated sequence* if for some $\varepsilon > 0$,

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space X is said to have the *uniform Kadec–Klee property* (written **UKK**) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is the weak limit of a norm one ε-separated sequence, then $\|x\| < 1 - \delta$. Every **UKK** Banach space has property (**H**) (see [10]).

A Banach space is said to be *nearly uniformly convex* (written **NUC**) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subset B(X)$ with $\operatorname{sep}(x_n) > \varepsilon$, we have

$$\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset.$$

Huff [10] has proved that X is **NUC** if and only if X is reflexive and **UKK**.

A Banach space X is said to be *nearly uniformly smooth* (**NUS** for short) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each basic sequence (x_n) in $B(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + t\varepsilon$$

for each $t \in [0, \delta]$. Prus [20] has shown that a Banach space X is **NUC** if and only if X^* is **NUS.**

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \operatorname{conv}(\{x\} \cup B(X))$$

(see [5]). A Banach space X has the *drop property* (written (**D**)) if for every closed set C disjoint from $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

In [22], Rolewicz has proved that if the Banach space X has the drop property, then X is reflexive. Montesinos [18] has extended this result showing that X has the drop property if and only if X is a reflexive Banach space with property (**H**).

For any subset C of X we denote by $\alpha(C)$ its *Kuratowski measure of noncompactness*, i.e. the infimum of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ε.

Goebel and Sękowski [8] have extended the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of noncompactness. Namely, they called a norm $\|\cdot\|$ in a Banach space X

Δ-uniformly convex (written $\mathbf{\Delta UC}$) if for any $\varepsilon > 0$ there is $\delta > 0$ such that for each convex set E contained in the closed unit ball $B(X)$ such that $\alpha(E) > \varepsilon$, we have

$$\inf\{\|x\| : x \in E\} < 1 - \delta.$$

It is well known that $\mathbf{\Delta UC}$ coincides with **NUC**.

Rolewicz [22], studying the relationships between **NUC** and the drop property, has defined property (β). A Banach space X is said to have *property* (β) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\alpha(D(x, B(X)) \setminus B(X)) < \varepsilon$$

whenever $1 < \|x\| < 1 + \delta$. It is well known that if a Banach space X has property (β), then its dual space X^* has the normal structure (see [16]). The following result will be very helpful for our considerations (see [15]):

A Banach space X has property (β) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geq \varepsilon$ there is an index k such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

A map $\Phi : \mathbb{R} \to \mathbb{R}_+$ is said to be an *Orlicz function* if Φ vanishes only at 0, and Φ is even, convex, and continuous on the whole $\mathbb{R}_+$ (see [17], [19], [21]).

A sequence $\Phi = (\Phi_n)$ of Orlicz functions is called a *Musielak–Orlicz function*. By $\Psi = (\Psi_n)$ we denote the complementary function of Φ in the sense of Young, i.e.

$$\Psi_n(v) = \sup\{|v|u - \Phi_n(u) : u \geq 0\}, \qquad n = 1, 2, \ldots$$

Denote by l^0 the space of all real sequences $x = (x(i))$. For a given Musielak–Orlicz function Φ, we define a convex modular $I_\Phi : l^0 \to [0, \infty]$ by the formula

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i)).$$

The *Musielak–Orlicz sequence space l_Φ* is

$$l_\Phi = \{x \in l^0 : I_\Phi(cx) < \infty \text{ for some } c > 0\}.$$

We consider l_Φ equipped with the so-called *Luxemburg norm*

$$\|x\| = \inf\{\varepsilon > 0 : I_\Phi(x/\varepsilon) \leq 1\},$$

under which it is a Banach space (see [3], [19]).

The subspace h_Φ defined by

$$h_\Phi = \{x \in l_\Phi : I_\Phi(cx) < \infty \text{ for every } c > 0\}$$

is called the *subspace of finite* (or *order continuous*) *elements*.

We say an Orlicz function Φ satisfies the δ_2-*condition* ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$, $u_0 > 0$ and a sequence (c_i) of nonnegative numbers such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

It is well known that $h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$ (see [13]).

We say a Musielak–Orlicz function Φ satisfies *condition* $(*)$ if for any $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that $\Phi_i((1+\delta)u) \leq 1$ whenever $\Phi_i(u) \leq 1-\varepsilon$ for $u \in \mathbb{R}$ and all $i \in \mathbb{N}$ (see [14]).

For more details on Musielak–Orlicz spaces we refer to [3] or [19].

2. Auxiliary facts. In order to obtain some new results, we will use the following well-known facts.

LEMMA 1 (see [4]). *If a Musielak–Orlicz function $\Phi = (\Phi_i)$ with all Φ_i finitely valued satisfies condition $(*)$ and $\Phi \in \delta_2$, then for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|x\| < 1 - \delta$ whenever $I_\Phi(x) < 1 - \varepsilon$.*

LEMMA 2 (see [14]). *If a Musielak–Orlicz function $\Phi = (\Phi_i)$ with all Φ_i finitely valued satisfies condition $(*)$ and $\Phi \in \delta_2$, then for every $\varepsilon > 0$ and $c > 0$ there exists $\delta > 0$ such that*

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon$$

whenever $I_\Phi(x) \leq c$ and $I_\Phi(y) < \delta$.

LEMMA 3 (see [6]). *If a Musielak–Orlicz function $\Psi = (\Psi_i) \in \delta_2$, then there exists $\theta \in (0,1)$ and a sequence (h_i) of nonnegative numbers such that $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ and the inequality*

$$\Phi_i\left(\frac{u}{2}\right) \leq \frac{1-\theta}{2}\Phi_i(u)$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$.

3. Results. We start with the following general result.

THEOREM 1. *If a Banach space X has property (β), then both X and X^* have the Banach–Saks property.*

Proof. Assume X has property (β). First, we will prove that X has the Banach–Saks property. Since property (β) implies reflexivity, it is enough to prove that there exists $\Theta \in (0,2)$ such that for each sequence (x_n) in $S(X)$ weakly convergent to zero, there are $n_1, n_2 \in \mathbb{N}$ such that $\|x_{n_1} + x_{n_2}\| < \Theta$.

Since (x_n) is weakly convergent to zero, the set of its elements cannot be compact in $S(X)$. So, there are $\varepsilon_0 > 0$ and a subsequence (z_n) of (x_n)

with $\operatorname{sep}(z_n) \geq \varepsilon_0$. By property ($\beta$) for X, there exists $\delta > 0$ depending on ε_0 only such that for every $z \in S(X)$ there exists $k \in \mathbb{N}$ for which

$$\|z + z_k\| < 2 - \delta$$

(cf. Proposition 1 in [15]). In particular, setting $z = z_1$, a natural number $k(1) \neq 1$ can be found such that

$$\|z_1 + z_{k(1)}\| < \Theta,$$

where $\Theta = 2 - \delta$. This means that X has the Banach–Saks property.

Next, we will prove that X^* has the Banach–Saks property. For each sequence (x_n) in $S(X)$ weakly convergent to zero, by the Bessaga–Pełczyński selection principle, there exists a basic subsequence (z_n) of (x_n) (see [7]). Property (β) for X implies that X^* is **NUS** (see [20]), i.e. for any $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that there is $k \in \mathbb{N}$, $k > 1$, such that

$$\|z_1 + tz_k\| < 1 + t\varepsilon$$

for any $t \in [0, \delta]$. In particular, taking $\varepsilon = 1/2$, numbers $\delta_0 \in (0, 1)$ and $k > 1$, $k \in \mathbb{N}$, can be found such that

$$\|z_1 + \delta_0 z_k\| < 1 + \frac{\delta_0}{2}.$$

Hence

$$\|z_1 + z_k\| = \|z_1 + \delta_0 z_k + (1 - \delta_0)z_k\| < 1 + \frac{\delta_0}{2} + (1 - \delta_0) = 2 - \frac{\delta_0}{2},$$

i.e. X^* has the Banach–Saks property.

Theorem 1 cannot be reversed in general. Indeed, note that c_0 as well as its dual l^1 have the Banach–Saks property, but they fail property (β). However, both c_0 and l^1 are not reflexive. It is natural to ask the following

Question. Assume that X is a reflexive Banach space. Does the Banach–Saks property for X and X^* imply property (β) for X?

Now, we will describe some geometric properties in Musielak–Orlicz sequence spaces.

Theorem 2. *If a Musielak–Orlicz function $\Phi = (\Phi_i)$ with all Φ_i finitely valued satisfies condition* ($*$), *then the following statements are equivalent*:

(a) l_Φ *is* **UKK**;
(b) l_Φ *has property* (**H**);
(c) $\Phi \in \delta_2$.

Proof. (a)⇒(b). This holds true for any Banach space (see [10]).

(b)⇒(c). If $\Phi \notin \delta_2$, we can find an element $x = (x(1), x(2), \ldots) \in S(l_\Phi)$ such that $I_\Phi(x) \le 1$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$ (see [13]). Consequently, there is an increasing sequence (n_i) of natural numbers such that

$$\|(0, \ldots, 0, x(n_i + 1), \ldots, x(n_{i+1}), 0, \ldots)\| \ge \frac{1}{2}.$$

Putting

$$x_i = (x(1), \ldots, x(n_i), 0, \ldots, 0, x(n_{i+1} + 1), \ldots), \qquad i = 1, 2, \ldots,$$

we get

(1) $\|x_i\| = 1$, $i = 1, 2, \ldots$;

(2) $x_i \to x$ weakly.

Equalities (1) follow by $I_\Phi(x_i) \le 1$ and $I_\Phi(\lambda x_i) = \infty$ for every $\lambda > 1$ $(i = 1, 2, \ldots)$. We will now prove property (2). For every $y^* \in (l_\Phi)^*$ we have $y^* = y_0^* + y_1^*$ uniquely, where y_0^* and y_1^* are respectively the regular and singular parts of y^*, i.e. y_0^* is determined by a function $y_0 \in l_\Psi$ and $y_1^*(x) = 0$ for any $x \in h_\Phi$ (see [9]). Since $y_0 = (y_0(i)) \in l_\Psi$, there exists $\lambda > 0$ such that $\sum_{i=1}^{\infty} \Psi_i(\lambda y_0(i)) < \infty$. Since $\langle x_i - x, y_1^* \rangle = 0$, we have

$$\begin{aligned}\langle x_i - x, y^* \rangle = \langle x_i - x, y_0^* \rangle &= \sum_{j=n_i+1}^{n_{i+1}} x(j) y_0(j) \\ &\le \frac{1}{\lambda} \sum_{j=n_i+1}^{n_{i+1}} (\Phi_j(x(j)) + \Psi_j(\lambda y_0(j))) \to 0 \quad \text{as } i \to \infty,\end{aligned}$$

which proves (2). We also have

(3) $\|x_i - x\| \ge 1/2$ for all $i \in \mathbb{N}$,

which means that l_Φ does not have property (**H**).

(c)⇒(a). Suppose l_Φ is not **UKK** and $\Phi \in \delta_2$. There exists $\varepsilon_0 > 0$ such that for any $\theta > 0$ there are a sequence (x_n) and an element x in $S(l_\Phi)$ with $\operatorname{sep}(x_n) \ge \varepsilon_0$, $x_n \to x$ weakly and $\|x\| > 1 - \theta$. Since $\operatorname{sep}(x_n) \ge \varepsilon_0$, we can assume without loss of generality that $\|x_n - x\| \ge \varepsilon_0/2$ for every $n \in \mathbb{N}$. Since $\Phi \in \delta_2$ and Φ satisfies condition $(*)$ and x can be assumed to have $\|x\|$ close to 1, we may assume that there is $\eta_0 > 0$ such that $I_\Phi(x_n - x) \ge \eta_0$ and $I_\Phi(x) > 1 - \eta_0/5$. Using again $\Phi \in \delta_2$, there exists $\sigma_0 \in (0, \eta_0/5)$ such that

$$|I_\Phi(x + y) - I_\Phi(x)| < \frac{\eta_0}{5}$$

whenever $I_\Phi(y) < \sigma_0$.

Since $(x_n) \subset S(l_\Phi)$ and $x_n \to x$ weakly, by the lower semicontinuity of the norm with respect to the weak topology, we conclude that there is

$i_0 \in \mathbb{N}$ such that $\sum_{i=i_0+1}^{\infty} \Phi_i(x(i)) < \sigma_0$. By virtue of $x_n \to x$ weakly, which implies that $x_n \to x$ coordinatewise, there exists $n_0 \in \mathbb{N}$ such that

$$\Big|\sum_{i=1}^{i_0} \Phi_i(x_n(i)) - \sum_{i=1}^{i_0} \Phi_i(x(i))\Big| < \frac{\eta_0}{5} \quad \text{and} \quad \sum_{i=1}^{i_0} \Phi_i(x_n(i) - x(i)) < \frac{\eta_0}{5}$$

for $n \ge n_0$. So

$$\begin{aligned} 1 = \sum_{i=1}^{\infty} \Phi_i(x_n(i)) &= \sum_{i=1}^{i_0} \Phi_i(x_n(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) \\ &\ge \sum_{i=1}^{i_0} \Phi_i(x(i)) - \frac{\eta_0}{5} + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)). \end{aligned}$$

Hence

$$\begin{aligned} \eta_0 \le I_\Phi(x_n - x) &= \sum_{i=1}^{\infty} \Phi_i(x_n(i) - x(i)) \\ &= \sum_{i=1}^{i_0} \Phi_i(x_n(i) - x(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i) - x(i)) \\ &< \frac{\eta_0}{5} + \sum_{i=i_0+1}^{\infty} \Phi_i(x_n(i)) + \frac{\eta_0}{5} \le 1 - \sum_{i=1}^{i_0} \Phi_i(x(i)) + \frac{2\eta_0}{5} + \frac{\eta_0}{5} \\ &\le 1 - (1 - \sigma_0) + \frac{3\eta_0}{5} \le 1 - \left(1 - \frac{\eta_0}{5}\right) + \frac{3\eta_0}{5} < \eta_0. \end{aligned}$$

This contradiction proves the implication (c)⇒(a).

COROLLARY 1. *If a Musielak–Orlicz function $\Phi = (\Phi_i)$ with all Φ_i finitely valued satisfies condition* $(*)$, *then the following statements are equivalent*:

(a) l_Φ *is* **NUC**;
(b) l_Φ *has the drop property*;
(c) $\Phi \in \delta_2$ *and* $\Psi \in \delta_2$.

P r o o f. Since **NUC** is equivalent to the conjunction of **UKK** and reflexivity, and the reflexivity of l_Φ is equivalent to the fact that $\Phi \in \delta_2$ and $\Psi \in \delta_2$, by Theorem 2, we get our corollary immediately.

Recall that a *Nakano space* $l^{(p_i)}$ is the Musielak–Orlicz space l_Φ with $\Phi = (\Phi_i)$, where

$$\Phi_i(u) = |u|^{p_i}, \qquad 1 \le p_i < \infty,\ i = 1, 2, \ldots$$

COROLLARY 2. *For any Nakano space $l^{(p_i)}$ the following statements are equivalent*:

(a) $l^{(p_i)}$ *is* **NUC**;

(b) $l^{(p_i)}$ *has the drop property*;

(c) $1 < \liminf_i p_i \le \limsup_i p_i < \infty$.

Proof. This follows immediately by Corollary 1 and the fact that for the Nakano function $\Phi = (\Phi_i)$ with $\Phi_i(u) = |u|^{p_i}$ we have $\Phi \in \delta_2$ if and only if $\limsup_i p_i < \infty$, and its complementary function $\Psi \in \delta_2$ if and only if $\liminf_i p_i > 1$.

COROLLARY 3. *Let $l^{(p_i)}$ be a Nakano space. Then the following statements are equivalent*:

(a) $l^{(p_i)}$ *is* **UKK**;

(b) $l^{(p_i)}$ *has property* (**H**);

(c) $\limsup_i p_i < \infty$.

Proof. This follows immediately by Theorem 2 and the fact that the Nakano function $\Phi = (\Phi_i)$ with $\Phi_i(u) = |u|^{p_i}$ satisfies the δ_2-condition if and only if condition (c) is satisfied.

THEOREM 3. *If a Musielak–Orlicz function $\Phi = (\Phi_i)$, with all Φ_i finitely valued and satisfying $\Phi_i(u)/u \to 0$ as $u \to 0$, satisfies condition $(*)$, then l_Φ has the Banach–Saks property if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.*

Proof. Since the Banach–Saks property implies reflexivity and the reflexivity of l_Φ is equivalent to $\Phi \in \delta_2$ and $\Psi \in \delta_2$, we only need to prove sufficiency. By $\Psi \in \delta_2$, there exists $\Theta \in (0,1)$ and a sequence (h_i) of positive numbers such that $\sum_{i=1}^\infty \Phi_i(h_i) < \infty$ and

$$\Phi_i\left(\frac{u}{2}\right) \le (1-\Theta)\frac{\Phi_i(u)}{2}$$

for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\Phi_i(h_i) \le \Phi_i(u) \le 1$ (see Lemma 3).

By $\Phi \in \delta_2$ and condition $(*)$ for Φ, for any $\varepsilon \in (0, \Theta/16)$, there exists a $\delta \in (0, \Theta)$ such that

$$|I_\Phi(y+z) - I_\Phi(y)| < \frac{\varepsilon}{2}$$

whenever $I_\Phi(y) \le 1$, $I_\Phi(z) \le \delta$ (see Lemma 2).

For each sequence (x_n) of $S(l_\Phi)$ with $x_n \to 0$ weakly, we have $x_n \to 0$ coordinatewise, so there are i_0 and $n_0 \in \mathbb{N}$ such that $\sum_{i=i_0+1}^\infty \Phi_i(x_1(i)) < \delta$, $\sum_{i=i_0+1}^\infty \Phi_i(h_i) < \delta/16$ and $\sum_{i=1}^{i_0} \Phi_i(x_n(i)) < \delta$ for $n > n_0$. Hence $\sum_{i=i_0+1}^\infty \Phi_i(x_n(i)) \ge 1/2$ for $n > n_0$ and

$$
\begin{aligned}
I_\Phi\Big(\frac{x_1+x_n}{2}\Big) &= \sum_{i=1}^{i_0}\Phi_i\Big(\frac{x_n(i)+x_1(i)}{2}\Big) + \sum_{i=i_0+1}^{\infty}\Phi_i\Big(\frac{x_n(i)+x_1(i)}{2}\Big)\\
&\le \sum_{i=1}^{i_0}\frac{\Phi_i(x_1(i))}{2} + \sum_{i=i_0+1}^{\infty}\Phi_i\Big(\frac{x_n(i)}{2}\Big)+\varepsilon\\
&\le \sum_{i=1}^{i_0}\frac{\Phi_i(x_1(i))}{2} + \frac{1-\Theta}{2}\sum_{i=i_0+1}^{\infty}\Phi_i(x_n(i)) + \sum_{i=i_0+1}^{\infty}\Phi_i(h_i)+\varepsilon\\
&= \frac{1}{2}\Big\{\sum_{i=1}^{i_0}\Phi_i(x_1(i)) + \sum_{i=i_0+1}^{\infty}\Phi_i(x_n(i))\Big\}\\
&\quad + \sum_{i=i_0+1}^{\infty}\Phi_i(h_i)+\varepsilon - \frac{\Theta}{2}\sum_{i=i_0+1}^{\infty}\Phi_i(x_n(i))\\
&\le 2\cdot\frac{1}{2}+\frac{\Theta}{16}+\frac{\Theta}{16}-\frac{\Theta}{4} = 1-\frac{\Theta}{8},
\end{aligned}
$$

for $n > n_0$.

In view of Lemma 1, there exists $\delta > 0$ independent of x_1 and x_n such that

$$\|x_1 + x_n\| < 2 - \delta \quad \text{for } n > n_0.$$

The proof of Theorem 3 is finished.

COROLLARY 4. *The Nakano sequence space $l^{(p_i)}$ has the Banach–Saks property if and only if* $1 < \liminf_i p_i \le \limsup_i p_i < \infty$.

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