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Partial differential equations in Banach spaces involving nilpotent linear operators

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Abstract. Let E be a Banach space. We consider a Cauchy problem of the type

$$\begin{cases} D_t^k u + \sum_{j=0}^{k-1} \sum_{|\alpha|\le m} A_{j,\alpha}(D_t^j D_x^\alpha u) = f & \text{in } \mathbb{R}^{n+1}, \\ D_t^j u(0,x) = \varphi_j(x) & \text{in } \mathbb{R}^n,\ j=0,\ldots,k-1, \end{cases}$$

where each $A_{j,\alpha}$ is a given continuous linear operator from E into itself. We prove that if the operators $A_{j,\alpha}$ are nilpotent and pairwise commuting, then the problem is well-posed in the space of all functions $u \in C^\infty(\mathbb{R}^{n+1}, E)$ whose derivatives are equi-bounded on each bounded subset of $\mathbb{R}^{n+1}$.

Introduction. Let $k,m,n\in\mathbb{N}$ and let $(E,\|\cdot\|_E)$ be a real or complex Banach space. Following [4], we denote by $V(\mathbb{R}^n,E)$ the space of all functions $u\in C^\infty(\mathbb{R}^n,E)$ such that, for every non-empty bounded set $\Omega\subseteq\mathbb{R}^n$, one has

$$\|u\|_{\Omega,E} := \sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in\Omega} \|D^\alpha u(x)\|_E < \infty,$$

where $D^\alpha u = \partial^{\alpha_1+\ldots+\alpha_n} u/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$, $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

In the present paper, we are interested in the well-posedness in the space $V(\mathbb{R}^n,E)$ of the Cauchy problem

$$(1)\quad \begin{cases} D_t^k u + \sum_{j=0}^{k-1} \sum_{|\alpha|\le m} A_{j,\alpha}(D_t^j D_x^\alpha u) = f & \text{in } \mathbb{R}^{n+1}, \\ D_t^j u(0,x) = \varphi_j(x) & \text{in } \mathbb{R}^n,\ j=0,\ldots,k-1, \end{cases}$$

where each $A_{j,\alpha}$ is a given continuous linear operator from E into itself. We denote by $\mathcal{L}(E)$ the space of all continuous linear operators from E into

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itself, endowed with the usual norm

$$\|A\|_{\mathcal{L}(E)} = \sup_{\|v\|_E \le 1} \|A(v)\|_E.$$

Apparently, the only previous result on this subject is Theorem 1 of [4], where one assumes that

$$\sum_{j=0}^{k-1} \sum_{|\alpha| \le m} \|A_{j,\alpha}\|_{\mathcal{L}(E)} < 1.$$

We wish here to prove another, independent result supposing that the operators $A_{j,\alpha}$ are nilpotent and pairwise commuting. However, a complete characterization of the well-posedness of the problem (1) in the space $V(\mathbb{R}^n, E)$ remains still unknown.

We believe that such a characterization should be quite difficult. To support this, we now discuss a particularly simple case which shows the peculiarity of working in the space $V(\mathbb{R}^n, E)$ rather than in the other spaces usually considered in the theory of linear partial differential equations.

Let σ be a non-negative real number. Denote by $\Gamma^{(\sigma)}(\mathbb{R}^2)$ the (real) Gevrey class in $\mathbb{R}^2$ of index σ. That is to say, $\Gamma^{(\sigma)}(\mathbb{R}^2)$ is the class of all real functions $u \in C^\infty(\mathbb{R}^2)$ such that, for every non-empty bounded set $\Omega \subseteq \mathbb{R}^2$, one has

$$\inf_{\lambda > 0} \sup_{(\alpha,\beta) \in \mathbb{N}_0^2} \sup_{(t,x) \in \Omega} \left(\frac{1}{(\alpha+\beta)!} \right)^\sigma \lambda^{\alpha+\beta} \left| \frac{\partial^{\alpha+\beta} u(t,x)}{\partial t^\alpha \partial x^\beta} \right| < \infty.$$

Recall, in particular, that $\Gamma^{(1)}(\mathbb{R}^2)$ coincides with the class of analytic functions in $\mathbb{R}^2$. Also, observe that $V(\mathbb{R}^2, \mathbb{R}) \subseteq \Gamma^{(0)}(\mathbb{R}^2)$.

Given a real number a, consider now the differential operator $P_a : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$P_a(u) = \frac{\partial^k u}{\partial t^k} + a \frac{\partial^m u}{\partial x^m}.$$

Then, according to the classical work of Malgrange [3], we have

$$P_a(C^\infty(\mathbb{R}^2)) = C^\infty(\mathbb{R}^2) \quad \text{for every } a \in \mathbb{R}.$$

Analogously, we have

$$P_a(\Gamma^{(\sigma)}(\mathbb{R}^2)) = \Gamma^{(\sigma)}(\mathbb{R}^2) \quad \text{for every } a \in \mathbb{R} \text{ and } \sigma \in [0,1[\cup ([1,\infty[\cap \mathbb{Q}).$$

Precisely, this follows from Theorems 9.4 and 9.6 of [5] (see also p. 408 and pp. 467–468) when $\sigma \in [0,1[$, and from Theorem 4.1 of [1] when $\sigma \in [1,\infty[\cap \mathbb{Q}$ (in the case $\sigma = 1$ the result was previously proved in [2]).

Now, we come to the space $V(\mathbb{R}^2, \mathbb{R})$ (for short $V(\mathbb{R}^2)$). On the basis of Theorem 4 of [4], we have

$$P_a(V(\mathbb{R}^2)) = V(\mathbb{R}^2) \quad \text{if and only if} \quad a \ne \pm 1.$$

1. The result. Our result is the following.

THEOREM 1. *Let $k,n,m \in \mathbb{N}$, and let $\{A_{j,\alpha}\}_{j=0,\ldots,k-1,\,\alpha\in\mathbb{N}_0^n,\,|\alpha|\le m}$ be a family of pairwise commuting elements of $\mathcal{L}(E)$ such that for some $\overline{q} \in \mathbb{N}$ one has*

$$\|A_{j,\alpha}^{\bar q}\|_{\mathcal{L}(E)} = 0 \quad \text{for each } j = 0,1,\ldots,k-1 \text{ and } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le m.$$

Then for each $f \in V(\mathbb{R}^{n+1},E)$ and each $\varphi_0,\varphi_1,\ldots,\varphi_{k-1} \in V(\mathbb{R}^n,E)$ there exists a unique function $u \in V(\mathbb{R}^{n+1},E)$ such that for each $t\in\mathbb{R}$ and each $x\in\mathbb{R}^n$ one has

$$\begin{cases} D_t^k u(t,x) + \sum_{j=0}^{k-1}\sum_{|\alpha|\le m} A_{j,\alpha}(D_t^j D_x^\alpha u(t,x)) = f(t,x), \\ D_t^j u(0,x) = \varphi_j(x), \quad j=0,\ldots,k-1. \end{cases} \tag{2}$$

Moreover, if p is the cardinality of the set $\{A_{j,\alpha} : A_{j,\alpha} \ne 0,\ j=0,\ldots,k-1,\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le m\}$ and $\bar s := k^2p\overline{q}$, then for each bounded set $\Omega \subseteq \mathbb{R}^n$, and each $r\ge 0$ and $\lambda > 0$, if one puts

$$\sigma := \max\Big\{\lambda, \sum_{j=0}^{k-1}\sum_{|\alpha|\le m} \lambda^{j-k+1}\|A_{j,\alpha}\|_{\mathcal{L}(E)}\Big\},$$

one has the following inequality:

$$\begin{aligned} \max_{0\le i\le k-1} \lambda^{-i}\|D_t^i u\|_{[-r,r]\times\Omega,E} \\ \le \min\Big\{&\big(\max_{0\le j\le \bar s-1}\sigma^j\big)\max_{0\le i\le k-1}(\lambda^{-i}\|\varphi_i\|_{\Omega,E})e^{r\sigma} \\ &+ \Big(\big(\max_{0\le j\le\bar s-1}\sigma^j\big) r e^{r\sigma} + \sum_{j=0}^{\bar s-1}\sigma^j\Big)\lambda^{1-k}\|f\|_{[-r,r]\times\Omega,E}, \\ & e^r\Big(\big(\max_{0\le j\le\bar s-1}\sigma^j\big)\big(\max_{0\le i\le k-1}\lambda^{-i}\|\varphi_i\|_{\Omega,E}\big) \\ &+ \Big(\sum_{j=0}^{\bar s-1}\sigma^j\Big)\lambda^{1-k}\|f\|_{\{0\}\times\Omega,E}\Big)\Big\}. \end{aligned}$$

Before giving the proof of Theorem 1, we need some preliminary results.

PROPOSITION 2. *Let $A\in\mathcal{L}(E)$, $B\in V(\mathbb{R},E)$ and $v\in C^1(\mathbb{R},E)$ be such that $\sum_{n=1}^\infty \|A^n\|_{\mathcal{L}(E)} < \infty$ and*

$$v'(t) = A(v(t)) + B(t) \quad \text{in } \mathbb{R}.$$

Then $v\in V(\mathbb{R},E)$ and for each $r\ge 0$ the following inequality holds:

$$\|v\|_{[-r,r],E} \le \min\Big\{(\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})\|v(0)\|_E e^{r\|A\|_{\mathcal{L}(E)}} + \Big((\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)}) r e^{r\|A\|_{\mathcal{L}(E)}} + \sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{[-r,r],E}, \; e^r\Big((\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})\|v(0)\|_E + \Big(\sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{0,E}\Big)\Big\}, \tag{3}$$

where we put $\|A^0\|_{\mathcal{L}(E)} = 1$.

Proof. First, observe that $\sup_{n\in\mathbb{N}}\|A^n\|_{\mathcal{L}(E)} \le \sum_{n=1}^{\infty}\|A^n\|_{\mathcal{L}(E)} < \infty$. Since $B \in C^\infty(\mathbb{R}, E)$ we get $v \in C^\infty(\mathbb{R}, E)$ and, arguing by induction, we have

$$v^{(m)}(t) = A^m(v(t)) + \sum_{j=1}^{m-1} A^j(B^{(m-j-1)}(t)) + B^{(m-1)}(t)$$

for all $t \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \ge 2$. Now, fix any $r \ge 0$. We get

$$\begin{aligned}\|v^{(m)}(t)\|_E &\le \|A^m\|_{\mathcal{L}(E)}\|v(t)\|_E + \sum_{j=1}^{m-1}\|A^j\|_{\mathcal{L}(E)}\|B^{(m-j-1)}(t)\|_E + \|B^{(m-1)}(t)\|_E \\ &\le \sup_{n\in\mathbb{N}}\|A^n\|_{\mathcal{L}(E)} \sup_{t\in[-r,r]}\|v(t)\|_E + \Big(\sum_{n=1}^{\infty}\|A^n\|_{\mathcal{L}(E)} + 1\Big)\|B\|_{[-r,r],E}\end{aligned}$$

for each $t \in [-r, r]$ and $m \ge 2$. It is easy to see that the last inequality also holds for $t \in [-r, r]$ and $m = 1$. Hence

$$\|v^{(m)}(t)\|_E \le \max\{1, \sup_{n\in\mathbb{N}}\|A^n\|_{\mathcal{L}(E)}\} \sup_{t\in[-r,r]}\|v(t)\|_E + \Big(\sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{[-r,r],E}$$

for each $t \in [-r, r]$ and $m \in \mathbb{N}_0$. Consequently, $v \in V(\mathbb{R}, E)$ and

$$\|v\|_{[-r,r],E} \le \sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)} \sup_{t\in[-r,r]}\|v(t)\|_E + \Big(\sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{[-r,r],E} \tag{4}$$

for each $r \ge 0$. In particular, by (4) we get

$$\|v\|_{0,E} \le (\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})\|v(0)\|_E + \Big(\sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{0,E}.$$

By Proposition 2 of [4] we have $\|v\|_{[-r,r],E} \le e^r\|v\|_{0,E}$, hence

(5) $\|v\|_{[-r,r],E}$

$$\le e^r\Big((\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})\|v(0)\|_E + \Big(\sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{0,E}\Big).$$

On the other hand, since $v(t) = v(0) + \int_0^t A(v(\tau))\,d\tau + \int_0^t B(\tau)\,d\tau$ we get

$$\|v(t)\|_E \le \|v(0)\|_E + r\|B\|_{[-r,r],E} + \|A\|_{\mathcal{L}(E)}\Big|\int_0^t \|v(\tau)\|_E\,d\tau\Big|$$

for every $t \in [-r,r]$. By Gronwall's lemma, we get

$$\|v(t)\|_E \le (\|v(0)\|_E + r\|B\|_{[-r,r],E})\,e^{r\|A\|_{\mathcal{L}(E)}} \tag{6}$$

for all $t \in [-r,r]$. By (4) and (6) we get

(7) $\|v\|_{[-r,r],E}$

$$\le (\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})\|v(0)\|_E\,e^{r\|A\|_{\mathcal{L}(E)}}$$

$$+\Big((\sup_{n\in\mathbb{N}_0}\|A^n\|_{\mathcal{L}(E)})re^{r\|A\|_{\mathcal{L}(E)}} + \sum_{n=0}^{\infty}\|A^n\|_{\mathcal{L}(E)}\Big)\|B\|_{[-r,r],E}.$$

Our claim follows easily from (5) and (7). ∎

We point out that the operator A satisfies $\sum_{n=1}^{\infty}\|A^n\|_{\mathcal{L}(E)} < \infty$ if, for instance, $\|A\|_{\mathcal{L}(E)} < 1$ or if there is some $\overline{m} \in \mathbb{N}$ such that $A^m = 0$ for all $m \ge \overline{m}$. When the former situation occurs, Proposition 2 reduces to Proposition 4 of [4], while in the latter case from Proposition 2 we get

(8) $\|v\|_{[-r,r],E}$

$$\le \min\Big\{(\max_{0\le j\le \bar{m}-1}\|A^j\|_{\mathcal{L}(E)})\|v(0)\|_E\,e^{r\|A\|_{\mathcal{L}(E)}}$$

$$+\Big((\max_{0\le j\le \bar{m}-1}\|A^j\|_{\mathcal{L}(E)})re^{r\|A\|_{\mathcal{L}(E)}} + \sum_{j=0}^{\bar{m}-1}\|A^j\|_{\mathcal{L}(E)}\Big)\|B\|_{[-r,r],E},$$

$$e^r\Big((\max_{0\le j\le \bar{m}-1}\|A^j\|_{\mathcal{L}(E)})\|v(0)\|_E + \Big(\sum_{j=0}^{\bar{m}-1}\|A^j\|_{\mathcal{L}(E)}\Big)\|B\|_{0,E}\Big)\Big\}.$$

PROPOSITION 3. *Let $k \in \mathbb{N}$ with $k \ge 2$, and let $A_0, A_1, \ldots, A_{k-1} \in \mathcal{L}(E)$ be pairwise commuting operators. Assume that there exists $m^* \in \mathbb{N}$ such that $A_j^{m^*} = 0$ for each $j = 0, 1, \ldots, k-1$. Let $\lambda > 0$, and consider the operator $A : E^k \to E^k$ defined by*

$$A(y) = \Big(\lambda y_1, \lambda y_2, \ldots, \lambda y_{k-1}, \sum_{j=0}^{k-1}\lambda^{j-k+1}A_j(y_j)\Big)$$

for each $y = (y_0, y_1, \ldots, y_{k-1}) \in E^k$. Then $A^{\bar{m}} = 0$ for $\overline{m} = k^2 m^*$.

P r o o f. We divide the proof into several steps.

F i r s t s t e p. Let $y = (y_0, y_1, \ldots, y_{k-1}) \in E^k$ and $s \in \{1, \ldots, k\}$ be fixed. Let us show that if one puts $A^s(y) = (x_0, x_1, \ldots, x_{k-1})$, then the vector x_j can be represented in the following way:

$$x_j = \sum_{m=1}^{m_j} \mu_{j,m}(\lambda) A_0^{n_{j,m,0}} A_1^{n_{j,m,1}} \ldots A_{k-1}^{n_{j,m,k-1}}(y_{r(j,m)}) \tag{9}$$

if $j = k-s, \ldots, k-1$ (with $m_j \geq 1$, and $\sum_{l=0}^{k-1} n_{j,m,l} \geq 1$, $r(j,m) \in \{0, 1, \ldots, k-1\}$ for each $m = 1, \ldots, m_j$), and

$$x_j = \lambda^s y_{j+s}$$

if $j = 0, \ldots, k-s-1$ (if $s < k$). To prove our claim, we argue by induction on s. Of course, our claim is true for $s = 1$. Now, assume that it is true for $s = i$ (with $i < k$) and let us show that it remains true for $s = i+1$. By assumption, if we put $A^i(y) = (\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_{k-1})$, then we have

$$\widetilde{x}_j = \sum_{m=1}^{\tilde{m}_j} \widetilde{\mu}_{j,m}(\lambda) A_0^{\tilde{n}_{j,m,0}} A_1^{\tilde{n}_{j,m,1}} \ldots A_{k-1}^{\tilde{n}_{j,m,k-1}}(y_{\tilde{r}(j,m)})$$

if $j = k-i, \ldots, k-1$ (with $\widetilde{m}_j \geq 1$, and $\sum_{l=0}^{k-1} \widetilde{n}_{j,m,l} \geq 1$, $\widetilde{r}(j,m) \in \{0, 1, \ldots, k-1\}$ for each $m = 1, \ldots, \widetilde{m}_j$), and

$$\widetilde{x}_j = \lambda^i y_{j+i}$$

if $j = 0, \ldots, k-i-1$. Put $A^{i+1}(y) = (w_0, w_1, \ldots, w_{k-1})$. We get

$$\begin{aligned}
w_{k-1} &= \sum_{d=0}^{k-1} \lambda^{d-k+1} A_d(\widetilde{x}_d) \\
&= \sum_{d=0}^{k-i-1} \lambda^{d-k+1} A_d(\lambda^i y_{d+i}) \\
&\quad + \sum_{d=k-i}^{k-1} \lambda^{d-k+1} A_d\Big(\sum_{m=1}^{\tilde{m}_d} \widetilde{\mu}_{d,m}(\lambda) A_0^{\tilde{n}_{d,m,0}} A_1^{\tilde{n}_{d,m,1}} \ldots A_{k-1}^{\tilde{n}_{d,m,k-1}}(y_{\tilde{r}(d,m)})\Big) \\
&= \sum_{d=0}^{k-i-1} \lambda^{d-k+i+1} A_d(y_{d+i}) \\
&\quad + \sum_{d=k-i}^{k-1} \sum_{m=1}^{\tilde{m}_d} \lambda^{d-k+1} \widetilde{\mu}_{d,m}(\lambda) A_d(A_0^{\tilde{n}_{d,m,0}} A_1^{\tilde{n}_{d,m,1}} \ldots A_{k-1}^{\tilde{n}_{d,m,k-1}}(y_{\tilde{r}(d,m)})),
\end{aligned}$$

hence it is easily seen that w_{k-1} is of the form (9). Now, let $j \in \{k-i-1,$

$\dots, k-2\}$. We get

$$w_j = \lambda \widetilde{x}_{j+1} = \lambda \sum_{m=1}^{\tilde{m}_{j+1}} \widetilde{\mu}_{j+1,m}(\lambda) A_0^{\tilde{n}_{j+1,m,0}} A_1^{\tilde{n}_{j+1,m,1}} \dots A_{k-1}^{\tilde{n}_{j+1,m,k-1}} (y_{\tilde{r}(j+1,m)}),$$

hence w_j is of the form (9) even for $j = k-i-1, \dots, k-2$. Thus, if $i = k-1$, our claim follows. If $i < k-1$, for each $j = 0, \dots, k-i-2$ we have

$$w_j = \lambda \widetilde{x}_{j+1} = \lambda^{i+1} y_{j+i+1},$$

as desired.

Second step. We prove that for each fixed $y = (y_0, y_1, \dots, y_{k-1}) \in E^k$ and $s \in \mathbb{N}$, if we put $A^{sk}(y) = (z_0, z_1, \dots, z_{k-1})$, then for each $j \in \{0, 1, \dots, k-1\}$ the vector z_j can be represented in the following way:

$$z_j = \sum_{m=1}^{b_j} \sigma_{j,m}(\lambda) A_0^{p_{j,m,0}} A_1^{p_{j,m,1}} \dots A_{k-1}^{p_{j,m,k-1}} (y_{v(j,m)})$$

with $b_j \geq 1$ and $\sum_{l=0}^{k-1} p_{j,m,l} \geq s$, $v(j,m) \in \{0, 1, \dots, k-1\}$ for each $m = 1, \dots, b_j$. Again, we argue by induction. First, we observe that by the first part of the proof, if we put $A^k(y) = (u_0, u_1, \dots, u_{k-1})$, then for each $j = 0, 1, \dots, k-1$ the vector u_j can be represented in the form

$$u_j = \sum_{m=1}^{\bar{m}_j} \overline{\mu}_{j,m}(\lambda) A_0^{\bar{n}_{j,m,0}} A_1^{\bar{n}_{j,m,1}} \dots A_{k-1}^{\bar{n}_{j,m,k-1}} (y_{\bar{r}(j,m)})$$

with $\overline{m}_j \geq 1$ and $\sum_{l=0}^{k-1} \overline{n}_{j,m,l} \geq 1$, $\overline{r}(j,m) \in \{0, 1, \dots, k-1\}$ for each $m = 1, \dots, \overline{m}_j$, hence our claim is true for $s = 1$. Assume that it is true for $s = i$, and let us show that it is true for $s = i+1$. Thus, if we put $A^{ik}(y) = (\widehat{z}_0, \widehat{z}_1, \dots, \widehat{z}_{k-1})$, then for each $j = 0, 1, \dots, k-1$ the vector $\widehat{z}_j$ can be represented in the following way:

$$\widehat{z}_j = \sum_{m=1}^{\hat{b}_j} \widehat{\sigma}_{j,m}(\lambda) A_0^{\hat{p}_{j,m,0}} A_1^{\hat{p}_{j,m,1}} \dots A_{k-1}^{\hat{p}_{j,m,k-1}} (y_{\hat{v}(j,m)}) \tag{10}$$

with $\widehat{b}_j \geq 1$ and $\sum_{l=0}^{k-1} \widehat{p}_{j,m,l} \geq i$, $\widehat{v}(j,m) \in \{0, 1, \dots, k-1\}$ for each $m = 1, \dots, \widehat{b}_j$. From the first part of the proof, if we put $A^{(i+1)k}(y) = A^k(A^{ik}(y)) = (\widehat{w}_0, \widehat{w}_1, \dots, \widehat{w}_{k-1})$, then for each $j = 0, 1, \dots, k-1$ we have

$$\widehat{w}_j = \sum_{m=1}^{\hat{m}_j} \widehat{\mu}_{j,m}(\lambda)\, A_0^{\hat{n}_{j,m,0}} A_1^{\hat{n}_{j,m,1}} \dots A_{k-1}^{\hat{n}_{j,m,k-1}} (\widehat{z}_{\hat{r}(j,m)})$$

with $\widehat{m}_j \geq 1$ and $\sum_{l=0}^{k-1} \widehat{n}_{j,m,l} \geq 1$, $\widehat{r}(j,m) \in \{0, 1, \dots, k-1\}$ for each

$m = 1, \dots, \widehat{m}_j$. By (10), for each $j = 0, 1, \dots, k-1$ we get

$$\widehat{w}_j = \sum_{m=1}^{\hat{m}_j} \sum_{d=1}^{\hat{b}_{\hat{r}(j,m)}} \widehat{\mu}_{j,m}(\lambda)\, \widehat{\sigma}_{\hat{r}(j,m),d}(\lambda) \cdot A_0^{\hat{n}_{j,m,0}+\hat{p}_{\hat{r}(j,m),d,0}} A_1^{\hat{n}_{j,m,1}+\hat{p}_{\hat{r}(j,m),d,1}} \dots A_{k-1}^{\hat{n}_{j,m,k-1}+\hat{p}_{\hat{r}(j,m),d,k-1}} (y_{\hat{v}(\hat{r}(j,m),d)}).$$

Since for each $m \in \{1, \dots, \widehat{m}_j\}$ and $d \in \{1, \dots, b_{\hat{r}(j,m)}\}$ we have $\sum_{l=0}^{k-1} \widehat{n}_{j,m,l} + \widehat{p}_{\hat{r}(j,m),d,l} \ge i+1$, our claim follows.

Third step. We claim that $A^{k^2 m^*} = 0$. To see this, choose any $y = (y_0, y_1, \dots, y_{k-1}) \in E^k$. If we put $A^{k^2 m^*}(y) = (\widetilde{w}_0, \widetilde{w}_1, \dots, \widetilde{w}_{k-1})$, from the second part of the proof we see that for each $j = 0, 1, \dots, k-1$ the vector $\widetilde{w}_j$ can be represented in the following way:

$$\widetilde{w}_j = \sum_{m=1}^{\tilde{b}_j} \widetilde{\sigma}_{j,m}(\lambda) A_0^{\tilde{p}_{j,m,0}} A_1^{\tilde{p}_{j,m,1}} \dots A_{k-1}^{\tilde{p}_{j,m,k-1}} (y_{\tilde{v}(j,m)})$$

with $\widetilde{b}_j \ge 1$ and $\sum_{l=0}^{k-1} \widetilde{p}_{j,m,l} \ge km^*$, $\widetilde{v}(j,m) \in \{0, 1, \dots, k-1\}$ for each $m = 1, \dots, \widetilde{b}_j$. Now, it is easy to see that for each fixed $j \in \{0, 1, \dots, k-1\}$ and $m \in \{1, \dots, \widetilde{b}_j\}$ there exists $\widetilde{l} \in \{0, 1, \dots, k-1\}$ such that $\widetilde{p}_{j,m,\tilde{l}} \ge m^*$. Hence, we conclude that $\widetilde{w}_j = 0$ for all $j = 0, 1, \dots, k-1$. This completes the proof. ∎

PROPOSITION 4. *Let $k \in \mathbb{N}$ and let $A_0, A_1, \dots, A_{k-1} \in \mathcal{L}(E)$ be pairwise commuting operators. Assume that there exists $m^* \in \mathbb{N}$ such that $A_j^{m^*} = 0$ for each $j = 0, 1, \dots, k-1$. Then for each $B \in V(\mathbb{R}, E)$ and for each $w_0, w_1, \dots, w_{k-1} \in E$, there exists a unique $v \in V(\mathbb{R}, E)$ such that*

$$(11) \qquad \begin{cases} v^{(k)}(t) = \sum_{j=0}^{k-1} A_j(v^{(j)}(t)) + B(t) & \text{for all } t \in \mathbb{R}, \\ v^{(j)}(0) = w_j & \text{for } j = 0, 1, \dots, k-1. \end{cases}$$

Moreover, if $\overline{m} := k^2 m^*$, then for each fixed $r \ge 0$ and $\lambda > 0$, if one puts

$$c_A = \max\Big\{\lambda, \sum_{j=0}^{k-1} \lambda^{j-k+1} \|A_j\|_{\mathcal{L}(E)}\Big\},$$

one has

$$(12) \qquad \max_{0 \le i \le k-1} \lambda^{-i} \|v^{(i)}\|_{[-r,r],E} \le \min\Big\{\big(\max_{0 \le j \le \bar{m}-1} c_A^j\big) \max_{0 \le i \le k-1} \lambda^{-i} \|w_i\|_E \, e^{rc_A}$$

$$+\Big(\big(\max_{0\le j\le \bar m-1} c_A^j\big) r e^{rc_A} + \sum_{j=0}^{\bar m-1} c_A^j\Big)\lambda^{1-k}\|B\|_{[-r,r],E},$$

$$e^r\Big(\big(\max_{0\le j\le \bar m-1} c_A^j\big)\max_{0\le i\le k-1}\lambda^{-i}\|w_i\|_E + \Big(\sum_{j=0}^{\bar m-1} c_A^j\Big)\lambda^{1-k}\|B\|_{0,E}\Big)\Big\}.$$

P r o o f. If $k=1$, our claim follows by Picard–Lindelöf's theorem and Proposition 2. Now, let $k\ge 2$, and consider the space E^k endowed with the norm

$$\|y\|_{E^k} = \max_{0\le j\le k-1}\|y_i\|_E,$$

where $y=(y_0,y_1,\ldots,y_{k-1})$. Fix $\lambda>0$, and let $A: E^k\to E^k$ be defined by setting

$$A(y_0,y_1,\ldots,y_{k-1}) = \Big(\lambda y_1,\lambda y_2,\ldots,\lambda y_{k-1},\sum_{j=0}^{k-1}\lambda^{j-k+1}A_j(y_j)\Big)$$

for each $y\in E^k$. Now, observe that for each $y\in E^k$ one has

$$\|A(y)\|_{E^k} \le \max\Big\{\lambda,\sum_{j=0}^{k-1}\lambda^{j-k+1}\|A_j\|_{\mathcal{L}(E)}\Big\}\|y\|_{E^k} = c_A\|y\|_{E^k}.$$

Thus, $A\in\mathcal{L}(E^k)$ and $\|A\|_{\mathcal{L}(E^k)}\le c_A$. By Proposition 3 one has $A^{\bar m}=0$, where $\overline{m}=k^2m^*$. By Picard–Lindelöf's theorem, there exists a unique $v\in C^k(\mathbb{R},E)$ such that

$$\begin{cases} v^{(k)}(t)=\sum_{j=0}^{k-1}A_j(v^{(j)}(t))+B(t) & \text{for } t\in\mathbb{R},\\ v^{(j)}(0)=w_j & \text{for all } j=0,1,\ldots,k-1.\end{cases}$$

Let $\Gamma:\mathbb{R}\to E^k$ and $\omega:\mathbb{R}\to E^k$ be defined by setting for each $t\in\mathbb{R}$,

$$\Gamma(t) = (0,0,\ldots,\lambda^{1-k}B(t)),$$
$$\omega(t) = (v(t),\lambda^{-1}v'(t),\lambda^{-2}v''(t),\ldots,\lambda^{1-k}v^{(k-1)}(t)).$$

It is easy to see that $\Gamma\in V(\mathbb{R},E^k)$, $\omega\in C^1(\mathbb{R},E^k)$ and

$$\omega'(t) = A(\omega(t))+\Gamma(t)\quad \text{for all } t\in\mathbb{R}.$$

By Proposition 2 we get $\omega\in V(\mathbb{R},E^k)$, hence $v\in V(\mathbb{R},E)$. Moreover, (8) gives

$$\max_{0\le i\le k-1}\lambda^{-i}\|v^{(i)}\|_{[-r,r],E} = \|\omega\|_{[-r,r],E^k}$$
$$\le \min\Big\{\big(\max_{0\le j\le \bar m-1}\|A^j\|_{\mathcal{L}(E^k)}\big)\|\omega(0)\|_{E^k}\, e^{r\|A\|_{\mathcal{L}(E^k)}}$$

$$+\Big((\max_{0\le j\le \bar m-1}\|A^j\|_{\mathcal{L}(E^k)})re^{r\|A\|_{\mathcal{L}(E^k)}}+\sum_{j=0}^{\bar m-1}\|A^j\|_{\mathcal{L}(E^k)}\Big)\|\Gamma\|_{[-r,r],E^k},$$

$$e^r\Big((\max_{0\le j\le \bar m-1}\|A^j\|_{\mathcal{L}(E^k)})\|\omega(0)\|_{E^k}+\Big(\sum_{j=0}^{\bar m-1}\|A^j\|_{\mathcal{L}(E^k)}\Big)\|\Gamma\|_{0,E^k}\Big)\Big\}$$

for every $r\ge 0$. Since $\|A^j\|_{\mathcal{L}(E^k)}\le \|A\|^j_{\mathcal{L}(E^k)}\le c_A^j$ for each $j=1,\dots,\overline{m}-1$, our claim follows at once. ∎

Proof of Theorem 1. First, we denote by $F_1,\dots,F_p$ the elements of the set $\{A_{j,\alpha}: A_{j,\alpha}\ne 0,\ j=0,\dots,k-1,\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le m\}$. Fix $f\in V(\mathbb{R}^{n+1},E)$ and $\varphi_0,\varphi_1,\dots,\varphi_{k-1}\in V(\mathbb{R}^n,E)$, where the space $V(\mathbb{R}^n,E)$ will be considered with any norm $\|\cdot\|_{\Omega,E}$. For each $j=0,1,\dots,k-1$, $v\in V(\mathbb{R}^n,E)$ and $x\in\mathbb{R}^n$, let

$$T_j(v)(x)=-\sum_{|\alpha|\le m}A_{j,\alpha}(D^\alpha v(x)).$$

By Proposition 6 of [4] we have $T_j\in\mathcal{L}(V(\mathbb{R}^n,E))$ and $\|T_j\|_{\mathcal{L}(V(\mathbb{R}^n,E))}\le \sum_{|\alpha|\le m}\|A_{j,\alpha}\|_{\mathcal{L}(E)}$. Consider the problem

$$(13)\qquad \begin{cases}\omega^{(k)}(t)=\sum_{j=0}^{k-1}T_j(\omega^{(j)}(t))+\Psi^{-1}(f)(t) & \text{in } \mathbb{R},\\ \omega^{(j)}(0)=\varphi_j & \text{for } j=0,1,\dots,k-1,\end{cases}$$

where $\Psi: V(\mathbb{R},V(\mathbb{R}^n,E))\to V(\mathbb{R}^{n+1},E)$ is the function defined as in Proposition 3 of [4]. Namely, $\Psi(g)(t,x)=g(t)(x)$ for $g\in V(\mathbb{R},V(\mathbb{R}^n,E))$, $t\in\mathbb{R}$, and $x\in\mathbb{R}^n$. Now, it is easily seen that the operators T_j are pairwise commuting. We claim that

$$\|T_j^{p\bar q}\|_{\mathcal{L}(V(\mathbb{R}^n,E))}=0\qquad\text{for each } j=0,1,\dots,k-1.$$

To see this, let $j\in\{0,1,\dots,k-1\}$, $v\in V(\mathbb{R}^n,E)$ and $x\in\mathbb{R}^n$ be fixed, and let $\{A_{j,\alpha(j,i)}\}_{i=1}^{s_j}$ be the elements of the family $\{A_{j,\alpha}\}_{\alpha\in\mathbb{N}_0^n,|\alpha|\le m}$ that are different from the origin of $\mathcal{L}(E)$. Thus, we have

$$T_j(v)(x)=-\sum_{i=1}^{s_j}A_{j,\alpha(j,i)}(D^{\alpha(j,i)}v(x)).$$

Now we show that for each $h\in\mathbb{N}$ the vector $T_j^h(v)(x)$ can be represented as follows:

$$(14)\qquad T_j^h(v)(x)=(-1)^h\sum_{l=1}^{b(h)}A^{r(h,l,1)}_{j,\alpha(j,1)}A^{r(h,l,2)}_{j,\alpha(j,2)}\cdots$$
$$\cdots A^{r(h,l,s_j)}_{j,\alpha(j,s_j)}(D^{r(h,l,1)\alpha(j,1)+r(h,l,2)\alpha(j,2)+\dots+r(h,l,s_j)\alpha(j,s_j)}v(x))$$

for suitable $b(h) \in \mathbb{N}$ and $r(h,l,1), r(h,l,2), \dots, r(h,l,s_j) \in \mathbb{N}$ with

$$\sum_{i=1}^{s_j} r(h,l,i) = h \quad \text{for each } l \in \{1, \dots, b(h)\}.$$

We argue by induction. Of course, our claim is true for $h=1$, with $r(1,l,d)= 1$ if $d = l$, while $r(1,l,d) = 0$ if $d \neq l$. Now, assume that our claim is true for some $h \in \mathbb{N}$. We have

$$\begin{aligned}
&T_j^{h+1}(v)(x) \\
&\quad = T_j(T_j^h(v))(x) = -\sum_{i=1}^{s_j} A_{j,\alpha(j,i)}(D^{\alpha(j,i)} T_j^h(v)(x)) \\
&\quad = \sum_{i=1}^{s_j} (-1)^{h+1} \sum_{l=1}^{b(h)} A_{j,\alpha(j,1)}^{r(h,l,1)} A_{j,\alpha(j,2)}^{r(h,l,2)} \dots \\
&\qquad \dots A_{j,\alpha(j,s_j)}^{r(h,l,s_j)} A_{j,i}(D^{\alpha(j,i)} D^{r(h,l,1)\alpha(j,1)+r(h,l,2)\alpha(j,2)+\dots+r(h,l,s_j)\alpha(j,s_j)} v(x)).
\end{aligned}$$

Now, it is easy to see that $T_j^{h+1}(v)(x)$ is also of the form (14). Hence, our claim is true for $h+1$, hence it is true for all $h \in \mathbb{N}$. In particular, if $h = p\overline{q}$, the representation (14) holds, with $\sum_{i=1}^{s_j} r(h,l,i) = p\overline{q}$ for each fixed $l \in \{1, \dots, b(p\overline{q})\}$. Observe that, for each fixed $l \in \{1, \dots, b(p\overline{q})\}$, we have

$$A_{j,\alpha(j,1)}^{r(p\bar{q},l,1)} A_{j,\alpha(j,2)}^{r(p\bar{q},l,2)} \dots A_{j,\alpha(j,s_j)}^{r(p\bar{q},l,s_j)} = F_1^{q_1} \dots F_p^{q_p}$$

with

$$\sum_{i=1}^{p} q_i = \sum_{d=1}^{s_j} r(p\overline{q}, l, d) = p\overline{q}.$$

Of course, this implies that there exists $i \in \{1, \dots, p\}$ such that $q_i \geq \overline{q}$, hence $F_i^{q_i} = 0$. Therefore, for each fixed $l \in \{1, \dots, b(\overline{q}h)\}$, the operator

$$A_{j,\alpha(j,1)}^{r(p\bar{q},l,1)} A_{j,\alpha(j,2)}^{r(p\bar{q},l,2)} \dots A_{j,\alpha(j,s_j)}^{r(p\bar{q},l,s_j)}$$

identically vanishes, hence $T_j^h(v)(x) = 0_E$. The arbitrariness of $v \in V(\mathbb{R}^n, E)$ and $x \in \mathbb{R}^n$ gives

$$\|T_j^{p\bar{q}}\|_{\mathcal{L}(V(\mathbb{R}^n,E))} = 0 \quad \text{for each } j = 0, 1, \dots, k-1.$$

By Proposition 4 there exists a unique $\omega \in V(\mathbb{R}, V(\mathbb{R}^n, E))$ satisfying (13). Now, if $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\omega^{(k)}(t)(x) = \Psi(\omega^{(k)})(t,x) &= \sum_{j=0}^{k-1} \Psi(T_j \circ \omega^{(j)})(t,x) + f(t,x) \\ &= -\sum_{j=0}^{k-1} \sum_{|\alpha|\le m} A_{j,\alpha}(D^\alpha \omega^{(j)}(t)(x)) + f(t,x), \\ \omega^{(j)}(0)(x) = \varphi_j(x) \quad & \text{for all } j = 0,1,\dots,k-1.\end{aligned}$$

If we put $u = \Psi(\omega)$, observing that by Proposition 3 of [4] we have $D_t^j \Psi(\omega) = \Psi(D^j \omega)$, for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we get

$$\begin{cases} D_t^k u(t,x) + \sum_{j=0}^{k-1} \sum_{|\alpha|\le m} A_{j,\alpha}(D_t^j D_x^\alpha u(t,x)) = f(t,x), \\ D_t^j u(0,x) = \varphi_j(x) \quad \text{for all } j = 0,1,\dots,k-1. \end{cases}$$

Hence, u is a solution of problem (2). Conversely, reasoning as in [4] one can show that if $\widetilde{u}$ solves (2), then $\widetilde{u} = u$. By Proposition 4, if $\bar{s} := k^2 p\overline{q}$, $r \ge 0$ and $\lambda > 0$, we get

$$\begin{aligned}&\max_{0\le i\le k-1} \lambda^{-i} \|\omega^{(i)}\|_{[-r,r],V(\mathbb{R}^n,E)} \\ &\quad\le \min\Big\{ \big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) \max_{0\le i\le k-1} (\lambda^{-i}\|\varphi_i\|_{V(\mathbb{R}^n,E)}) e^{r\sigma} \\ &\qquad + \Big(\big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) r e^{r\sigma} + \sum_{j=0}^{\bar{s}-1} \sigma^j\Big) \lambda^{1-k} \|\Psi^{-1}(f)\|_{[-r,r],V(\mathbb{R}^n,E)}, \\ &\qquad e^r \Big(\big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) \max_{0\le i\le k-1} (\lambda^{-i}\|\varphi_i\|_{V(\mathbb{R}^n,E)}) \\ &\qquad + \Big(\sum_{j=0}^{\bar{s}-1} \sigma^j\Big) \lambda^{1-k} \|\Psi^{-1}(f)\|_{0,V(\mathbb{R}^n,E)}\Big)\Big\} \\ &\quad\le \min\Big\{ \big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) \max_{0\le i\le k-1} (\lambda^{-i}\|\varphi_i\|_{\Omega,E}) e^{r\sigma} \\ &\qquad + \Big(\big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) r e^{r\sigma} + \sum_{j=0}^{\bar{s}-1} \sigma^j\Big) \lambda^{1-k} \|f\|_{[-r,r]\times\Omega,E}, \\ &\qquad e^r \Big(\big(\max_{0\le j\le \bar{s}-1} \sigma^j\big) \max_{0\le i\le k-1} (\lambda^{-i}\|\varphi_i\|_{\Omega,E}) \\ &\qquad + \Big(\sum_{j=0}^{\bar{s}-1} \sigma^j\Big) \lambda^{1-k} \|f\|_{\{0\}\times\Omega,E}\Big)\Big\},\end{aligned}$$

where Ω is any non-empty bounded subset of $\mathbb{R}^n$. Since

$$\|D_t^i u\|_{[-r,r]\times\Omega,E} = \|\omega^{(i)}\|_{[-r,r],V(\mathbb{R}^n,E)},$$

our claim follows. ∎

To conclude, we now present a simple example of application of Theorem 1 to integro-differential equations. Let $Y \subseteq \mathbb{R}^m$ ($m \in \mathbb{N}$) be a nonempty compact set. Following [4], denote by $V_0(\mathbb{R}^n \times Y)$ the space of all functions $u : \mathbb{R}^n \times Y \to \mathbb{R}$ such that $u(\cdot\,, y) \in C^\infty(\mathbb{R}^n)$ for each $y \in Y$, $D_x^\alpha u \in C^0(\mathbb{R}^n \times Y)$ for each $\alpha \in \mathbb{N}_0^n$ and

$$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{(x,y) \in \Omega \times Y} |D_x^\alpha u(x,y)| < \infty$$

for each bounded set $\Omega \subseteq \mathbb{R}^n$. Also recall ([4], Proposition 8) that if, for $u \in V(\mathbb{R}^n, C^0(Y))$ ($C^0(Y)$ is endowed with the usual sup-norm), $\Psi_n(u)$ denotes the function, from $\mathbb{R}^n \times Y$ into $\mathbb{R}$, defined by

$$\Psi_n(u)(x,y) = u(x)(y) \quad (x \in \mathbb{R}^n,\ y \in Y),$$

then $\Psi_n(u) \in V_0(\mathbb{R}^n \times Y)$, the mapping $u \to \Psi_n(u)$ is surjective, and, for each $\alpha \in \mathbb{N}_0^n$, one has $D_x^\alpha \Psi_n(u) = \Psi_n(D^\alpha u)$.

THEOREM 2. *Let $k, n, m \in \mathbb{N}$ and let $\{g_{j,\alpha}\}_{j=0,\ldots,k-1,\, \alpha \in \mathbb{N}_0^n, |\alpha| \le m}$ be a family of continuous real functions on Y each of which satisfies $\int_Y g_{j,\alpha}(y)\,dy = 0$. Then, for each $f \in V_0(\mathbb{R}^{n+1} \times Y)$ and $\varphi_0, \varphi_1, \ldots, \varphi_{k-1} \in V_0(\mathbb{R}^n \times Y)$, there exists a unique function $u \in V_0(\mathbb{R}^{n+1} \times Y)$ such that, for each $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $y \in Y$, one has*

$$\begin{cases} D_t^k u(t,x,y) + \sum_{j=0}^{k-1} \sum_{|\alpha| \le m} \Big(\int_Y D_t^j D_x^\alpha u(t,x,\xi)\,d\xi\Big) g_{j,\alpha}(y) = f(t,x,y), \\ D_t^j u(0,x,y) = \varphi_j(x,y), \quad j = 0, \ldots, k-1. \end{cases}$$

Proof. For each $j = 0, \ldots, k-1$, $\alpha \in \mathbb{N}_0^n$, let $A_{j,\alpha}$ be the element of $\mathcal{L}(C^0(Y))$ defined by putting

$$A_{j,\alpha}(v)(y) = \Big(\int_Y v(\xi)\,d\xi\Big) g_{j,\alpha}(y)$$

for all $v \in C^0(Y)$ and $y \in Y$. Clearly, $A_{j,\alpha} \circ A_{j',\alpha'} = A_{j',\alpha'} \circ A_{j,\alpha} = 0$. Consequently, by Theorem 1, there exists a unique function $w \in V(\mathbb{R}^{n+1}, C^0(Y))$ such that, for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, one has

$$\begin{cases} D_t^k w(t,x) + \sum_{j=0}^{k-1} \sum_{|\alpha| \le m} A_{j,\alpha}(D_t^j D_x^\alpha w(t,x)) = \Psi_{n+1}^{-1}(f)(t,x), \\ D_t^j w(0,x) = \Psi_n^{-1}(\varphi_j)(x), \quad j = 0, \ldots, k-1. \end{cases}$$

Then the function $u = \Psi_{n+1}(w)$ satisfies the conclusion. ∎

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