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Polynomial set-valued functions

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Abstract. The aim of this paper is to give a necessary and sufficient condition for a set-valued function to be a polynomial s.v. function of order at most 2.

Let X, Z be vector spaces over $\mathbb{Q}$ and C be a $\mathbb{Q}$-convex subset of X. Let $f : C \to Z$ be an arbitrary function and $h \in X$. The difference operator Δ_h is given by the formula

$$\Delta_h f(x) := f(x+h) - f(x)$$

for $x \in C$ such that $x + h \in C$. The iterates Δ_h^n of Δ_h are given by the recurrence

$$\Delta_h^0 f := f, \quad \Delta_h^{n+1} f := \Delta_h(\Delta_h^n f), \quad n = 0, 1, 2, \ldots$$

The expression $\Delta_h^n f$ is a function defined for all $x \in C$ such that $x + nh \in C$. It is easy to see that $x + kh \in C$ for $k = 1, \ldots, n-1$ whenever $x, x+nh \in C$.

A function $f : C \to Z$ is said to be *a Jensen function* if it satisfies the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}[f(x) + f(y)]$$

for all $x, y \in C$.

A function $f : C \to Z$ is called *a polynomial function* of order at most n if

$$\Delta_h^{n+1} f(x) = 0$$

for every $x \in C$ and $h \in X$ such that $x + (n+1)h \in C$.

We have

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh)$$

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for $n \in \mathbb{N}$, $h \in X$ and $x \in C$ such that $x + nh \in C$ (see e.g. [3], Corollary 2, p. 368).

R. Ger has proved that every polynomial function $f : C \to Z$ of order at most n admits an extension to a polynomial function of order at most n on the whole X (see Theorem 2 of [2]). Therefore, due to Theorem 3 of [3] (p. 379), we can formulate the following theorem:

THEOREM 1. *Let X, Z be $\mathbb{Q}$-linear spaces and C be a nonempty $\mathbb{Q}$-convex subset of X. If $f : C \to Z$ is a polynomial function of order at most n then*

$$\Delta_{h_1 \ldots h_{n+1}} f(x) := \Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_{n+1}} f(x) = 0$$

for $x \in C$ and $h_1, \ldots, h_{n+1} \in X$ such that $x + \varepsilon_1 h_1 + \ldots + \varepsilon_{n+1} h_{n+1} \in C$, $\varepsilon_1, \ldots, \varepsilon_{n+1} \in \{0, 1\}$.

We are going to deal with polynomial set-valued functions (abbreviated to s.v. functions in the sequel). Let Y be a real Hausdorff topological vector space. The symbol $n(Y)$ will stand for the set of all non-empty subsets of Y. The set of all convex and compact members of $n(Y)$ will be denoted by $cc(Y)$.

Rådström's equivalence relation $\sim$ (see [5]) is defined on $(cc(Y))^2$ by stating $(A, B) \sim (C, D)$ if $A + D = B + C$. The equivalence class containing (A, B) is denoted by $[A, B]$. The quotient space $\mathcal{Z} = (cc(Y))^2/\sim$, with addition defined by

$$[A, B] + [D, E] := [A + D, B + E],$$

and scalar multiplication

$$\lambda[A, B] := \begin{cases} [\lambda A, \lambda B], & \lambda \geq 0, \\ [-\lambda B, -\lambda A], & \lambda < 0, \end{cases}$$

is a real vector space.

The following result of Rådström (see [5], Lemma 3) is useful.

LEMMA 1. *Let A, B be convex and closed sets in Y and let C be nonempty and bounded. Then $A + C = B + C$ implies $A = B$.*

Let Y be a topological vector space and let $\mathcal{W}$ be a base of neighbourhoods of zero in Y. The space $n(Y)$ may be considered as a topological space with the Hausdorff topology. In this topology the families of sets

$$N_W(A) := \{B \in n(Y) : A \subseteq B + W \text{ and } B \subseteq A + W\},$$

where W runs over the base $\mathcal{W}$, form a base of neighbourhoods of the set $A \in n(Y)$ (see [6]).

The three lemmas below can be found e.g. in [4] (Lemmas 5.6 and 3.2).

LEMMA 2. *Let Y be a topological vector space and $A_n, B_n, A, B \in n(Y)$ for $n \in \mathbb{N}$. If $A_n \to A$ and $B_n \to B$ (in the Hausdorff topology on $n(Y)$),* *then $A_n + B_n \to A + B$. If A is bounded, then the function $t \to tA$ is continuous.*

LEMMA 3. *If $\lambda_n \to 0$ and $A \in n(Y)$ is bounded, then $\lambda_n A \to \{0\}$.*

LEMMA 4. *If $\lambda_n \to \lambda_0$ and $A \in n(Y)$ is bounded, then $\lambda_n A \to \lambda_0 A$.*

The next lemma is proved in [1] for a metric space Y.

LEMMA 5. *Let Y be a topological vector space. If $A_n \to A$ (in the Hausdorff topology on $n(Y)$) and A is closed then $A = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m}$.*

Proof. Fix $n \in \mathbb{N}$ and $W \in \mathcal{W}$, where $\mathcal{W}$ denotes a base of neighbourhoods of zero in Y. Since $A_n \to A$, there is $n_0 \in \mathbb{N}$ such that $A \subseteq A_m + W$ for every $m \ge n_0$. Hence, $A \subseteq \bigcup_{m\ge n} A_m + W$ for $W \in \mathcal{W}$. Therefore, we have

$$A \subseteq \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m}.$$

Now, fix $W \in \mathcal{W}$. Let $V \in \mathcal{W}$ with $V + V \subseteq W$. There is $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, then

$$A_n \subseteq A + V. \tag{1}$$

Choose an $x \in \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m}$. Hence $x \in A_m + V$ for some $m \ge n_0$. Then by (1),

$$x \in A + V + V \subseteq A + W,$$

that is, $x \in \overline{A} = A$. Consequently, $\bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} \subseteq A$. ■

LEMMA 6. *Let Y be a topological vector space and $A_n, B, C \in cc(Y)$ for $n \in \mathbb{N}$. If $A_n + B =: C_n \to C$, then there exists $A \in cc(Y)$ such that $C = A + B$.*

Proof. By the last lemma, Lemma 5.3 of [4] and the fact that the algebraic sum of a compact set and a closed set is closed, we have

$$C = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} (A_m + B)} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m + B} = \bigcap_{n=1}^{\infty} \overline{\overline{\bigcup_{m\ge n} A_m} + B}$$
$$= \bigcap_{n=1}^{\infty} \Big(\overline{\bigcup_{m\ge n} A_m} + B\Big) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} + B.$$

Hence

$$C = \operatorname{clconv}\Big(\bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} + B\Big) = \operatorname{cl}\Big(\operatorname{conv} \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} + B\Big)$$
$$= \operatorname{cl}\Big(\operatorname{clconv} \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} + B\Big) = \operatorname{clconv} \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m} + B.$$

Put

$$A := \operatorname{clconv} \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\ge n} A_m}.$$

The set A is of course closed and convex and $A+B=C$. Since $A\subset C-B$, A is compact. ■

DEFINITION 1. A set $S\subseteq X$ is said to be a $\mathbb{Q}$-*convex cone* if $S+S\subseteq S$ and $\lambda S\subseteq S$ for all $\lambda\in\mathbb{Q}\cap\langle 0,\infty)$.

Now consider an s.v. function $F:S\to cc(Y)$, where $S\subseteq X$ denotes a $\mathbb{Q}$-convex cone. Define $f:S\to\mathcal{Z}$ as follows:

$$f(x) := [F(x),\{0\}]. \tag{2}$$

DEFINITION 2. Let $h\in X$. The difference operator of the function $f: S\to\mathcal{Z}$ given by (2) is called the *difference operator of the s.v. function* F, i.e. $\Delta_h F(x) := \Delta_h f(x) = [F(x+h),F(x)]$ for $x\in S$ and $h\in X$ such that $x+h\in S$, and $\Delta_h^n F(x) := \Delta_h^n f(x)$ for $x\in S$ and $h\in X$ such that $x+nh\in S$.

DEFINITION 3. An s.v. function $F:S\to cc(Y)$ is called a *polynomial s.v. function* of order at most n if the function $f:S\to\mathcal{Z}$ given by (2) is a polynomial function of order at most n, i.e. $\Delta_h^{n+1}F(x)=0$ for $x\in S$ and $h\in X$ such that $x+(n+1)h\in S$.

Observe that if $F:S\to cc(Y)$ is polynomial of order 0, i.e. $\Delta_h F(x)=0$ for $x\in S$ and $h\in X$ such that $x+h\in S$, then F is constant.

Now, let F be a polynomial s.v. function of order at most one. Then

$$\Delta_h^2 F(x) = [F(x+2h)+F(x), 2F(x+h)] = 0$$

for $x\in S$ and $h\in X$ such that $x+2h\in S$. This means that

$$F(x+2h)+F(x) = 2F(x+h) \tag{3}$$

for $x\in S$ and $h\in X$ such that $x+2h\in S$.

Putting $h:=(y-x)/2\in X$ in (3), where x,y are arbitrary from S, we get $x+h=(x+y)/2\in S, x+2h=y\in S$ and

$$F(y)+F(x) = 2F\left(\frac{x+y}{2}\right), \qquad x,y\in S. \tag{4}$$

So, if $\Delta_h^2 F(x)=0$, then F satisfies the Jensen equation (4). Conversely, if F satisfies the above equation, then F is a polynomial function of order at most one.

If F is a polynomial s.v. function of order at most one then the function $g:S\to\mathcal{Z}$ given by

$$g(x) := \Delta_x f(0) = [F(x),F(0)]$$

is additive. Indeed, by Lemma 3 (p. 367) of [3] and Theorem 1,

$$g(x+y) = \Delta_{x+y} f(0) = \Delta_{x,y} f(0) + \Delta_x f(0) + \Delta_y f(0) = g(x) + g(y)$$

for all $x, y \in S$. Then $g(nx) = ng(x)$ for all $n \in \mathbb{N}$ and $x \in S$, which gives

$$[F(nx), F(0)] = n[F(x), F(0)].$$

Hence

$$\frac{1}{n} F(nx) + F(0) = \frac{1}{n} F(0) + F(x)$$

for $x \in S$ and $n \in \mathbb{N}$. Since the limit of the right-hand side exists, so does the limit of the left-hand side. By Lemma 6, there is a set $A(x) \in cc(Y)$ such that

$$A(x) + F(0) = \lim_{n\to\infty} \left(\frac{1}{n} F(0) + F(x) \right) = F(x)$$

for $x \in S$. It follows that

$$[A(x), \{0\}] = [F(x), F(0)],$$

so the s.v. function A is additive. Conversely, if $A : S \to cc(Y)$ is additive and $F(0) \in cc(Y)$, then the s.v. function F given by $F(x) = F(0) + A(x)$ is a polynomial s.v. function of order at most one. By the above considerations we can formulate a theorem proved by K. Nikodem [4] in a different way.

THEOREM 2. *Let X be a real vector space, S be a $\mathbb{Q}$-convex cone in X and let Y be a real topological vector space. Then $F : S \to cc(Y)$ is a polynomial s.v. function of order at most one if and only if there exists an additive s.v. function $A : S \to cc(Y)$ such that $F(x) = F(0) + A(x)$ for $x \in S$.*

An s.v. function F is a polynomial function of order 2 if and only if

$$F(x+3h) + 3F(x+h) = 3F(x+2h) + F(x)$$

for $x \in S$ and $h \in X$ such that $x + 3h \in S$. It is easily seen that if

$$F(x) = A_0 + A_1(x) + A_2(x) \tag{5}$$

for $x \in S$, where $A_0 \in cc(Y)$, $A_1, A_2 : S \to cc(Y)$, A_1 is additive and A_2 is the diagonalization of a biadditive s.v. function $\overline{A}_2 : S \times S \to cc(Y)$ (i.e. $A_2(x) = \overline{A}_2(x,x)$, $x \in S$) then F is a polynomial s.v. function of order at most 2.

Now, let us consider an example. Let $S = \langle 0, \infty)$ and $F : S \to cc(\mathbb{R})$ be given by the formula $F(x) := \langle 2x, x^2 + 1 \rangle$, $x \in S$. Obviously, F is a polynomial function of order at most 2 but we cannot present it in the form (5). In fact, putting $x = 0$ in (5), we get $A_0 = \langle 0, 1 \rangle$. Next, putting $x = 1$ in (5), we obtain

$$\langle 0, 1 \rangle + A_1(1) + A_2(1) = \langle 2, 2 \rangle,$$

which is not possible.

Remark 1. Let $F : S \to cc(Y)$ be a polynomial s.v. function of order at most 2 and let $\bar{f} : X \to \mathcal{Z}$ denote an extension of the function f defined by (2). The function $g : X \times X \to \mathcal{Z}$ given by

$$g(x,y) := \tfrac{1}{2}\Delta_{x,y}\bar{f}(0)$$

is biadditive and

$$(6) \quad g(x,y) = \tfrac{1}{2}\Delta_{x,y} f(0) = \tfrac{1}{2}[F(x+y)+F(0), F(x)+F(y)] \quad \text{for } x,y \in S.$$

Proof. A polynomial extension $\bar{f} : X \to \mathcal{Z}$ of order at most 2 of the function f exists in view of Theorem 2 of [2]. Note that g is symmetric. Fix $x,y,z \in X$. By Lemma 3 of [3] (p. 367) and Theorem 1 we have

$$\begin{aligned} g(x+z,y) &= \tfrac{1}{2}\Delta_{x+z,y} f(0) = \tfrac{1}{2}\Delta_y \Delta_{x+z} f(0) \\ &= \tfrac{1}{2}\Delta_y(\Delta_{x,z} f(0) + \Delta_x f(0) + \Delta_z f(0)) \\ &= \tfrac{1}{2}\Delta_{x,y,z} f(0) + \tfrac{1}{2}\Delta_{x,y} f(0) + \tfrac{1}{2}\Delta_{z,y} f(x)(0) \\ &= g(x,y) + g(z,y). \end{aligned}$$

By (2), the equation (6) is obvious. ■

THEOREM 3. *Let $F : S \to cc(Y)$ be a polynomial function of order at most 2. Then there exists a polynomial s.v. function $A : S \to cc(Y)$ of order at most 2 such that*

$$\begin{aligned} &\tfrac{1}{2}F(0) + \tfrac{1}{2}F(2x) = A(x) + F(x), \qquad x \in S, \\ &A(\lambda x) = \lambda^2 A(x), \qquad x \in S,\ \lambda \in \mathbb{Q} \cap \langle 0, \infty), \end{aligned}$$

and the function

$$x \to [F(x), F(0) + A(x)], \qquad x \in S,$$

is additive.

Proof. By Remark 1 the function $g : X \times X \to \mathcal{Z}$ given by $g(x,y) := \frac{1}{2}\Delta_{x,y}\bar{f}(0)$ is biadditive, where $\bar{f}$ denotes an extension of f.

First, we prove that

$$(7) \quad F\Big(\sum_{k=1}^{n} x_k\Big) + (n-2)\sum_{k=1}^{n} F(x_k) = \frac{(n-2)(n-1)}{2}F(0) + \sum_{1 \le k < l \le n} F(x_k + x_l),$$

where $n \ge 2$ and $x_1, \ldots, x_n \in S$. If $n = 2$, then (7) is trivial. Now, assume that (7) holds for $n \ge 2$. Let $x_1, \ldots, x_{n+1} \in S$. Since

$$g\Big(\sum_{k=1}^{n} x_k, x_{n+1}\Big) = \sum_{k=1}^{n} g(x_k, x_{n+1}),$$

we have

$$\Big[F\Big(\sum_{k=1}^{n} x_k + x_{n+1}\Big) + F(0), F\Big(\sum_{k=1}^{n} x_k\Big) + F(x_{n+1})\Big]$$
$$= \sum_{k=1}^{n}[F(x_k + x_{n+1}) + F(0), F(x_k) + F(x_{n+1})],$$

whence

$$F\Big(\sum_{k=1}^{n+1} x_k\Big) + F(0) + \sum_{k=1}^{n} F(x_k) + nF(x_{n+1})$$
$$= F\Big(\sum_{k=1}^{n} x_k\Big) + F(x_{n+1}) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + nF(0).$$

By the Rådström lemma

$$F\Big(\sum_{k=1}^{n+1} x_k\Big) + \sum_{k=1}^{n} F(x_k) + (n-1)F(x_{n+1})$$
$$= F\Big(\sum_{k=1}^{n} x_k\Big) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0).$$

Hence and by the induction hypothesis we have

$$F\Big(\sum_{k=1}^{n+1} x_k\Big) + (n-1)\sum_{k=1}^{n+1} F(x_k)$$
$$= F\Big(\sum_{k=1}^{n} x_k\Big) + (n-2)\sum_{k=1}^{n} F(x_k) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0)$$
$$= \frac{(n-2)(n-1)}{2}F(0) + \sum_{1\le k<l\le n} F(x_k + x_l)$$
$$+ \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0)$$
$$= \sum_{1\le k<l\le n+1} F(x_k + x_l) + \frac{(n-1)n}{2}F(0),$$

which ends the induction.

Putting $x = x_1 = \ldots = x_n$ in (7), we have

$$F(nx) + n(n-2)F(x) = \binom{n-1}{2}F(0) + \binom{n}{2}F(2x), \qquad n \ge 3,\ x \in S,$$

and

$$\text{(8)}\qquad \frac{F(nx)}{n(n-2)} + F(x) = \frac{\binom{n-1}{2}}{n(n-2)}F(0) + \frac{\binom{n}{2}}{n(n-2)}F(2x), \qquad n \geq 3.$$

By Lemmas 4 and 2 the limit of the right-hand side of (8) exists; consequently, so does the limit of the left-hand side, and by Lemma 6, for all $x \in S$, there is a set $A(x) \in cc(Y)$ such that

$$\tfrac{1}{2}F(0) + \tfrac{1}{2}F(2x) = A(x) + F(x), \qquad x \in S.$$

This means that

$$[A(x), \{0\}] = \tfrac{1}{2}[F(2x) + F(0), 2F(x)] = g(x, x), \qquad x \in S.$$

Therefore, the function $a : X \to \mathcal{Z}$ defined by $a(x) := g(x, x)$ is the diagonalization of the biadditive function g and

$$a(x) = [A(x), \{0\}] \qquad \text{for } x \in S.$$

By Definition 3, A is a polynomial function of order at most 2. Since g is biadditive, for $x \in S$ and $\lambda \in \mathbb{Q} \cap \langle 0, \infty)$,

$$[A(\lambda x), \{0\}] = g(\lambda x, \lambda x) = \lambda^2 g(x, x) = \lambda^2 [A(x), \{0\}],$$

which means that $A(\lambda x) = \lambda^2 A(x)$.

Finally, observe that the function $x \to f(x) - a(x)$, $x \in S$, is a Jensen function. Indeed, let $x \in S$ and $h \in X$ with $x + 2h \in S$. Then

$$\Delta_h^2(f(x) - a(x)) = \Delta_h^2 f(x) - 2g(h, h) = \Delta_h^2 f(x) - \Delta_h^2 f(0) = \Delta_h^2 \Delta_x f(0) = 0,$$

by Theorem 1 and biadditivity of g. Define $\overline{g} : S \to \mathcal{Z}$ by

$$\overline{g}(x) = f(x) - a(x) - [F(0), \{0\}] = [F(x), A(x) + F(0)].$$

Then the considerations above and the fact that $\overline{g}(0) = 0$ imply the additivity of $\overline{g}$. ∎

Definition 4 (cf. [3]). Let S be a convex cone in a vector space X over $\mathbb{Q}$. A set $\mathcal{E}$ is called a *base* of S if $\mathcal{E}$ is linearly independent and the cone is spanned by $\mathcal{E}$, i.e., the set

$$\Big\{x \in X : x = \sum_{k=1}^{n} \lambda_k e_k, \quad e_1, \ldots, e_n \in \mathcal{E},\ \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \cap \langle 0, \infty),\ n \in \mathbb{N}\Big\}$$

coincides with S.

Theorem 4. *Let X be a vector space over $\mathbb{Q}$ and Y be a topological vector space, and let $S \subseteq X$ be a cone with a base. Then $F : S \to cc(Y)$ is a polynomial s.v. function of order at most 2 if and only if there exist additive s.v. functions $\overline{B}, \overline{C} : S \to cc(Y)$ and biadditive s.v. functions $\overline{D}, \overline{H} : S \times S \to cc(Y)$ such that*

(9) $$F(x)+\overline{C}(x)+\overline{H}(x,x)=F(0)+\overline{D}(x,x)+\overline{B}(x)$$

for $x\in S$.

P r o o f. Since a cone with a base is $\mathbb{Q}$-convex, by Theorem 3 there is an s.v. function $A:S\to cc(Y)$ such that

$$x\to[F(x),F(0)+A(x)],\qquad x\in S,$$

is additive. There exist (see Theorem 1 of [7]) additive s.v. functions $\overline{B},\overline{C}:S\to cc(Y)$ such that

$$[F(x),F(0)+A(x)]=[\overline{B}(x),\overline{C}(x)],\qquad x\in S,$$

which gives

(10) $$F(x)+\overline{C}(x)=F(0)+\overline{B}(x)+A(x),\qquad x\in S.$$

In view of Remark 1,

$$g(x,y)=\tfrac{1}{2}[F(x+y)+F(0),F(x)+F(y)]$$

is biadditive. Set

$$D(x,y):=\tfrac{1}{2}(F(x+y)+F(0)),\qquad H(x,y):=\tfrac{1}{2}(F(x)+F(y)),$$

and let $\mathcal{E}$ be a base of S. Fix $x,y\in S$. There exist $n\in\mathbb{N}$, $\lambda_1,\ldots,\lambda_n\in\mathbb{Q}\cap\langle 0,\infty)$ and $e_1,\ldots,e_n\in\mathcal{E}$ such that $x=\sum_{i=1}^n\lambda_i e_i$, and

$$D(x,y)+\sum_{i=1}^n\lambda_i H(e_i,y)=H(x,y)+\sum_{i=1}^n\lambda_i D(e_i,y).$$

Similarly

$$H(e_i,y)+\sum_{j=1}^m\mu_j D(e_i,\overline{e}_j)=D(e_i,y)+\sum_{j=1}^m\mu_j H(e_i,\overline{e}_j),$$

where $y=\sum_{j=1}^m\mu_j\overline{e}_j$, $\overline{e}_1,\ldots,\overline{e}_m\in\mathcal{E}$ and $\mu_1,\ldots,\mu_m\in\mathbb{Q}\cap\langle 0,\infty)$. Hence

$$\begin{aligned}D(x,y)&+\sum_{i=1}^n\lambda_i D(e_i,y)+\sum_{i=1}^n\sum_{j=1}^m\lambda_i\mu_j H(e_i,\overline{e}_j)\\&=D(x,y)+\sum_{i=1}^n\lambda_i\Big[D(e_i,y)+\sum_{j=1}^m\mu_j H(e_i,\overline{e}_j)\Big]\\&=D(x,y)+\sum_{i=1}^n\lambda_i\Big[H(e_i,y)+\sum_{j=1}^m\mu_j D(e_i,\overline{e}_j)\Big]\\&=D(x,y)+\sum_{i=1}^n\lambda_i H(e_i,y)+\sum_{i=1}^n\sum_{j=1}^m\lambda_i\mu_j D(e_i,\overline{e}_j).\end{aligned}$$

Define

$$\overline{D}(x,y) := \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i\mu_j D(e_i,\overline{e}_j), \qquad \overline{H}(x,y) := \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i\mu_j H(e_i,\overline{e}_j),$$

where $x = \sum_{i=1}^{n} \lambda_i e_i$, $y = \sum_{j=1}^{m} \mu_j \overline{e}_j$, $x, y \in S$. It is clear that $\overline{D}$ and $\overline{H}$ are biadditive and

$$\tfrac{1}{2}F(x+y) + \tfrac{1}{2}F(0) + \overline{H}(x,y) = \tfrac{1}{2}F(x) + \tfrac{1}{2}F(y) + \overline{D}(x,y).$$

Setting $y = x$, we have

$$\tfrac{1}{2}F(2x) + \tfrac{1}{2}F(0) + \overline{H}(x,x) = F(x) + \overline{D}(x,x).$$

Hence and by Theorem 3,

$$A(x) + \overline{H}(x,x) = \overline{D}(x,x)$$

and by (10),

$$\begin{aligned} F(x) + \overline{C}(x) + \overline{H}(x,x) &= F(0) + \overline{B}(x) + A(x) + \overline{H}(x,x) \\ &= F(0) + \overline{D}(x,x) + \overline{B}(x), \qquad x \in S. \end{aligned}$$

Thus (9) holds true. To end the proof it suffices to prove that F is a polynomial s.v. function of order at most 2 if (9) is satisfied. By (9),

$$\begin{aligned} &\Delta_h^3 F(x) \\ &\quad= [F(x+3h) + 3F(x+h), 3F(x+2h) + F(x)] \\ &\quad= [\overline{D}(x+3h,x+3h) + \overline{B}(x+3h) + 3\overline{D}(x+h,x+h) + 3\overline{B}(x+h), \\ &\qquad\qquad \overline{H}(x+3h,x+3h) + \overline{C}(x+3h) + 3\overline{H}(x+h,x+h) + 3\overline{C}(x+h)] \\ &\qquad - [3\overline{D}(x+2h,x+2h) + 3\overline{B}(x+2h) + \overline{D}(x,x) + \overline{B}(x), \\ &\qquad\qquad 3\overline{H}(x+2h,x+2h) + 3\overline{C}(x+2h) + \overline{H}(x,x) + \overline{C}(x)] \\ &\quad= [\overline{D}(x+3h,x+3h) + 3\overline{D}(x+h,x+h), 3\overline{D}(x+2h,x+2h) + \overline{D}(x,x)] \\ &\quad= [\overline{H}(x+3h,x+3h) + 3\overline{H}(x+h,x+h), 3\overline{H}(x+2h,x+2h) + \overline{H}(x,x)] \\ &\qquad + [\overline{B}(x+3h) + 3\overline{B}(x+h), 3\overline{B}(x+2h) + \overline{B}(x)] \\ &\qquad + [\overline{C}(x+3h) + 3\overline{C}(x+h), 3\overline{C}(x+2h) + \overline{C}(x)] \\ &\quad= \Delta_h^3\overline{D}(x,x) - \Delta_h^3\overline{H}(x,x) + \Delta_h^3\overline{B}(x) - \Delta_h^3\overline{C}(x) = 0, \end{aligned}$$

for $x \in S$ and $h \in X$ such that $x + 3h \in S$, because $\overline{D}$ and $\overline{H}$ are biadditive and $\overline{B}y$ and $\overline{C}$ are additive. So, the proof is complete. ∎

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