Polynomial set-valued functions

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Abstract. The aim of this paper is to give a necessary and sufficient condition for a set-valued function to be a polynomial s.v. function of order at most 2.

Let $X, Z$ be vector spaces over $\mathbb{Q}$ and $C$ be a $\mathbb{Q}$-convex subset of $X$. Let $f : C \to Z$ be an arbitrary function and $h \in X$. The difference operator $\Delta_h$ is given by the formula

$$\Delta_h f(x) := f(x + h) - f(x)$$

for $x \in C$ such that $x + h \in C$. The iterates $\Delta^n_h$ of $\Delta_h$ are given by the recurrence

$$\Delta^0_h f := f, \quad \Delta^{n+1}_h f := \Delta_h(\Delta^n_h f), \quad n = 0, 1, 2, \ldots$$

The expression $\Delta^n_h f$ is a function defined for all $x \in C$ such that $x + nh \in C$. It is easy to see that $x + kh \in C$ for $k = 1, \ldots, n-1$ whenever $x, x + nh \in C$.

A function $f : C \to Z$ is said to be a Jensen function if it satisfies the Jensen functional equation

$$f\left(\frac{x + y}{2}\right) = \frac{1}{2}[f(x) + f(y)]$$

for all $x, y \in C$.

A function $f : C \to Z$ is called a polynomial function of order at most $n$ if

$$\Delta^{n+1}_h f(x) = 0$$

for every $x \in C$ and $h \in X$ such that $x + (n + 1)h \in C$.

We have

$$\Delta^n_h f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh)$$

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[55]
for $n \in \mathbb{N}$, $h \in X$ and $x \in C$ such that $x + nh \in C$ (see e.g. [3], Corollary 2, p. 368).

R. Ger has proved that every polynomial function $f : C \to Z$ of order at most $n$ admits an extension to a polynomial function of order at most $n$ on the whole $X$ (see Theorem 2 of [2]). Therefore, due to Theorem 3 of [3] (p. 379), we can formulate the following theorem:

**Theorem 1.** Let $X, Z$ be $\mathbb{Q}$-linear spaces and $C$ be a nonempty $\mathbb{Q}$-convex subset of $X$. If $f : C \to Z$ is a polynomial function of order at most $n$ on the whole $X$ (see Theorem 2 of [2]). Therefore, due to Theorem 3 of [3] (p. 379), we can formulate the following theorem:

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**Theorem 1.** Let $X, Z$ be $\mathbb{Q}$-linear spaces and $C$ be a nonempty $\mathbb{Q}$-convex subset of $X$. If $f : C \to Z$ is a polynomial function of order at most $n$ then

$$\Delta h_1 \ldots h_{n+1} f(x) := \Delta h_1 \Delta h_2 \ldots \Delta h_{n+1} f(x) = 0$$

for $x \in C$ and $h_1, \ldots, h_{n+1} \in X$ such that $x + \varepsilon_1 h_1 + \ldots + \varepsilon_{n+1} h_{n+1} \in C$, $\varepsilon_1, \ldots, \varepsilon_{n+1} \in \{0, 1\}$.

We are going to deal with polynomial set-valued functions (abbreviated to s.v. functions in the sequel). Let $Y$ be a real Hausdorff topological vector space. The symbol $n(Y)$ will stand for the set of all non-empty subsets of $Y$. The set of all convex and compact members of $n(Y)$ will be denoted by $cc(Y)$.

Rådström’s equivalence relation $\sim$ (see [5]) is defined on $(cc(Y))^2$ by stating $(A, B) \sim (C, D)$ if $A + D = B + C$. The equivalence class containing $(A, B)$ is denoted by $[A, B]$. The quotient space $Z = (cc(Y))^2 / \sim$, with addition defined by

$$[A, B] + [D, E] := [A + D, B + E],$$

and scalar multiplication

$$\lambda[A, B] := \begin{cases} [\lambda A, \lambda B], & \lambda \geq 0, \\ [-\lambda B, -\lambda A], & \lambda < 0, \end{cases}$$

is a real vector space.

The following result of Rådström (see [5], Lemma 3) is useful.

**Lemma 1.** Let $A, B$ be convex and closed sets in $Y$ and let $C$ be nonempty and bounded. Then $A + C = B + C$ implies $A = B$.

Let $Y$ be a topological vector space and let $\mathcal{W}$ be a base of neighbourhoods of zero in $Y$. The space $n(Y)$ may be considered as a topological space with the Hausdorff topology. In this topology the families of sets $N_W(A) := \{ B \in n(Y) : A \subseteq B + W \text{ and } B \subseteq A + W \}$, where $W$ runs over the base $\mathcal{W}$, form a base of neighbourhoods of the set $A \in n(Y)$ (see [6]).

The three lemmas below can be found e.g. in [4] (Lemmas 5.6 and 3.2).

**Lemma 2.** Let $Y$ be a topological vector space and $A_n, B_n, A, B \in n(Y)$ for $n \in \mathbb{N}$. If $A_n \to A$ and $B_n \to B$ (in the Hausdorff topology on $n(Y)$),
then $A_n + B_n \to A + B$. If $A$ is bounded, then the function $t \to tA$ is continuous.

**Lemma 3.** If $\lambda_n \to 0$ and $A \in n(Y)$ is bounded, then $\lambda_n A \to \{0\}$.

**Lemma 4.** If $\lambda_n \to \lambda_0$ and $A \in n(Y)$ is bounded, then $\lambda_n A \to \lambda_0 A$.

The next lemma is proved in [1] for a metric space $Y$.

**Lemma 5.** Let $Y$ be a topological vector space. If $A_n \to A$ (in the Hausdorff topology on $n(Y)$) and $A$ is closed then $A = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$.

**Proof.** Fix $n \in \mathbb{N}$ and $W \in \mathcal{W}$, where $\mathcal{W}$ denotes a base of neighbourhoods of zero in $Y$. Since $A_n \to A$, there is $n_0 \in \mathbb{N}$ such that $A \subseteq A_m + W$ for every $m \geq n_0$. Hence, $A \subseteq \bigcup_{m \geq n_0} A_m + W$ for $W \in \mathcal{W}$. Therefore, we have

$$A \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m.$$

Now, fix $W \in \mathcal{W}$. Let $V \in \mathcal{W}$ with $V + V \subseteq W$. There is $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$A_n \subseteq A + V.$$

Choose an $x \in \bigcap_{m=1}^{\infty} \bigcup_{m \geq n} A_m$. Hence $x \in A_m + V$ for some $m \geq n_0$. Then by (1),

$$x \in A + V + V \subseteq A + W,$$

that is, $x \in A$. Consequently, $\bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m \subseteq A$. \hfill \blacksquare

**Lemma 6.** Let $Y$ be a topological vector space and $A_n, B, C \in cc(Y)$ for $n \in \mathbb{N}$. If $A_n + B =: C_n \to C$, then there exists $A \in cc(Y)$ such that $C = A + B$.

**Proof.** By the last lemma, Lemma 5.3 of [4] and the fact that the algebraic sum of a compact set and a closed set is closed, we have

$$C = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} (A_m + B) = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B$$

$$= \bigcap_{n=1}^{\infty} \left( \bigcup_{m \geq n} A_m + B \right) = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B.$$

Hence

$$C = \text{clconv} \left( \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B \right) = \text{cl} \left( \text{conv} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B \right)$$

$$= \text{cl} \left( \text{clconv} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B \right) = \text{clconv} \left( \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m + B \right).$$
Put
\[ A := \text{cleovn} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m. \]
The set \( A \) is of course closed and convex and \( A + B = C \). Since \( A \subset C - B \), \( A \) is compact.

**Definition 1.** A set \( S \subseteq X \) is said to be a \( \mathbb{Q} \)-convex cone if \( S + S \subseteq S \) and \( \lambda S \subseteq S \) for all \( \lambda \in \mathbb{Q} \cap (0, \infty) \).

Now consider an s.v. function \( F : S \to \text{cc}(Y) \), where \( S \subseteq X \) denotes a \( \mathbb{Q} \)-convex cone. Define \( f : S \to Z \) as follows:
\[ f(x) := [F(x), \{0\}] \]

**Definition 2.** Let \( h \in X \). The difference operator of the function \( f : S \to Z \) given by (2) is called the difference operator of the s.v. function \( F \), i.e. \( \Delta_h F(x) := \Delta_h f(x) = [F(x + h), F(x)] \) for \( x \in S \) and \( h \in X \) such that \( x + h \in S \), and \( \Delta_h^n F(x) := \Delta_h^n f(x) \) for \( x \in S \) and \( h \in X \) such that \( x + nh \in S \).

**Definition 3.** An s.v. function \( F : S \to \text{cc}(Y) \) is called a polynomial s.v. function of order at most \( n \) if the function \( f : S \to Z \) given by (2) is a polynomial function of order at most \( n \), i.e. \( \Delta_h^{n+1} F(x) = 0 \) for \( x \in S \) and \( h \in X \) such that \( x + (n + 1)h \in S \).

Observe that if \( F : S \to \text{cc}(Y) \) is polynomial of order 0, i.e. \( \Delta_h F(x) = 0 \) for \( x \in S \) and \( h \in X \) such that \( x + h \in S \), then \( F \) is constant.

Now, let \( F \) be a polynomial s.v. function of order at most one. Then
\[ \Delta_2^2 F(x) = [F(x + 2h) + F(x), 2F(x + h)] = 0 \]
for \( x \in S \) and \( h \in X \) such that \( x + 2h \in S \). This means that
\[ F(x + 2h) + F(x) = 2F(x + h) \]
for \( x \in S \) and \( h \in X \) such that \( x + 2h \in S \).

Putting \( h := (y - x)/2 \in X \) in (3), where \( x, y \) are arbitrary from \( S \), we get \( x + h = (x + y)/2 \in S, x + 2h = y \in S \) and
\[ F(y) + F(x) = 2F \left( \frac{x + y}{2} \right), \quad x, y \in S. \]
So, if \( \Delta_2^2 F(x) = 0 \), then \( F \) satisfies the Jensen equation (4). Conversely, if \( F \) satisfies the above equation, then \( F \) is a polynomial function of order at most one.

If \( F \) is a polynomial s.v. function of order at most one then the function \( g : S \to Z \) given by
\[ g(x) := \Delta_x f(0) = [F(x), F(0)] \]
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is additive. Indeed, by Lemma 3 (p. 367) of [3] and Theorem 1, 
\[ g(x + y) = \Delta_{x+y} f(0) = \Delta_x f(0) + \Delta_y f(0) = g(x) + g(y) \]
for all \( x, y \in S \). Then \( g(nx) = ng(x) \) for all \( n \in \mathbb{N} \) and \( x \in S \), which gives 
\[ [F(nx), F(0)] = n[F(x), F(0)]. \]
Hence 
\[ \frac{1}{n} F(nx) + F(0) = \frac{1}{n} F(0) + F(x) \]
for \( x \in S \) and \( n \in \mathbb{N} \). Since the limit of the right-hand side exists, so does the limit of the left-hand side. By Lemma 6, there is a set \( A(x) \in cc(Y) \) such that 
\[ A(x) + F(0) = \lim_{n \to \infty} \left( \frac{1}{n} F(0) + F(x) \right) = F(x) \]
for \( x \in S \). It follows that 
\[ [A(x), \{0\}] = [F(x), F(0)], \]
so the s.v. function \( A \) is additive. Conversely, if \( A : S \to cc(Y) \) is additive and \( F(0) \in cc(Y) \), then the s.v. function \( F \) given by 
\[ F(x) = F(0) + A(x) \]
is a polynomial s.v. function of order at most one. By the above considerations we can formulate a theorem proved by K. Nikodem [4] in a different way.

**Theorem 2.** Let \( X \) be a real vector space, \( S \) be a \( \mathbb{Q} \)-convex cone in \( X \) and let \( Y \) be a real topological vector space. Then \( F : S \to cc(Y) \) is a polynomial s.v. function of order at most one if and only if there exists an additive s.v. function \( A : S \to cc(Y) \) such that 
\[ F(x) = F(0) + A(x) \] for \( x \in S \).

An s.v. function \( F \) is a polynomial function of order 2 if and only if 
\[ F(x + 3h) + 3F(x + h) = 3F(x + 2h) + F(x) \]
for \( x \in S \) and \( h \in X \) such that \( x + 3h \in S \). It is easily seen that if 
\[ F(x) = A_0 + A_1(x) + A_2(x) \]
for \( x \in S \), where \( A_0 \in cc(Y), A_1, A_2 : S \to cc(Y) \), \( A_1 \) is additive and \( A_2 \) is the diagonalization of a biadditive s.v. function \( A_2 : S \times S \to cc(Y) \) (i.e. \( A_2(x) = A_2(x, x), x \in S \)) then \( F \) is a polynomial s.v. function of order at most 2.

Now, let us consider an example. Let \( S = \langle 0, \infty \rangle \) and \( F : S \to cc(\mathbb{R}) \) be given by the formula 
\[ F(x) := \langle 2x, x^2 + 1 \rangle, x \in S. \]
Obviously, \( F \) is a polynomial function of order at most 2 but we cannot present it in the form (5). In fact, putting \( x = 0 \) in (5), we get \( A_0 = \{0, 1\} \). Next, putting \( x = 1 \) in (5), we obtain 
\[ \langle 0, 1 \rangle + A_1(1) + A_2(1) = (2, 2), \]
which is not possible.
Remark 1. Let $F : S \to \mathbb{C}(Y)$ be a polynomial s.v. function of order at most 2 and let $\tilde{f} : X \to Z$ denote an extension of the function $f$ defined by (2). The function $g : X \times X \to Z$ given by
\[
g(x, y) := \frac{1}{2} \Delta_{x, y} f(0)
\]
is biadditive and
\[
(6) \quad g(x, y) = \frac{1}{2} \Delta_{x, y} f(0) = \frac{1}{2} [F(x + y) + F(0), F(x) + F(y)] \quad \text{for } x, y \in S.
\]

Proof. A polynomial extension $\tilde{f} : X \to Z$ of order at most 2 of the function $f$ exists in view of Theorem 2 of [2]. Note that $g$ is symmetric. Fix $x, y, z \in X$. By Lemma 3 of [3] (p. 367) and Theorem 1 we have
\[
g(x + z, y) = \frac{1}{2} \Delta_{x + z, y} f(0) = \frac{1}{2} \Delta_y \Delta_{x + z} f(0)
\]
\[
= \frac{1}{2} \Delta_y (\Delta_{x, z} f(0) + \Delta_{x, f(0)} + \Delta_{z, f(0)})
\]
\[
= \frac{1}{2} \Delta_{x, y, z} f(0) + \frac{1}{2} \Delta_{x, y} f(0) + \frac{1}{2} \Delta_{z, y} f(x)(0)
\]
\[
= g(x, y) + g(z, y).
\]
By (2), the equation (6) is obvious. ■

Theorem 3. Let $F : S \to \mathbb{C}(Y)$ be a polynomial function of order at most 2. Then there exists a polynomial s.v. function $A : S \to \mathbb{C}(Y)$ of order at most 2 such that
\[
\frac{1}{2} F(0) + \frac{1}{2} F(2x) = A(x) + F(x), \quad x \in S,
\]
\[
A(\lambda x) = \lambda^2 A(x), \quad x \in S, \quad \lambda \in \mathbb{Q} \cap (0, \infty),
\]
and the function
\[
x \to [F(x), F(0) + A(x)], \quad x \in S,
\]
is additive.

Proof. By Remark 1 the function $g : X \times X \to Z$ given by $g(x, y) := \frac{1}{2} \Delta_{x, y} \tilde{f}(0)$ is biadditive, where $\tilde{f}$ denotes an extension of $f$.

First, we prove that
\[
(7) \quad F \left( \sum_{k=1}^{n} x_k \right) + (n - 2) \sum_{k=1}^{n} F(x_k)
\]
\[
= \frac{(n - 2)(n - 1)}{2} F(0) + \sum_{1 \leq k < l \leq n} F(x_k + x_l),
\]
where $n \geq 2$ and $x_1, \ldots, x_n \in S$. If $n = 2$, then (7) is trivial. Now, assume that (7) holds for $n \geq 2$. Let $x_1, \ldots, x_{n+1} \in S$. Since
\[
g \left( \sum_{k=1}^{n} x_k, x_{n+1} \right) = \sum_{k=1}^{n} g(x_k, x_{n+1}),
\]
we have
\[ F \left( \sum_{k=1}^{n} x_k + x_{n+1} \right) + F(0), F \left( \sum_{k=1}^{n} x_k \right) + F(x_{n+1}) \]
\[ = \sum_{k=1}^{n} \left[ F(x_k + x_{n+1}) + F(0), F(x_k) + F(x_{n+1}) \right], \]
whence
\[ F \left( \sum_{k=1}^{n+1} x_k \right) + F(0) + \sum_{k=1}^{n} F(x_k) + nF(x_{n+1}) \]
\[ = F \left( \sum_{k=1}^{n} x_k \right) + F(x_{n+1}) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + nF(0). \]
By the Rådström lemma
\[ F \left( \sum_{k=1}^{n+1} x_k \right) + \sum_{k=1}^{n} F(x_k) + (n-1)F(x_{n+1}) \]
\[ = F \left( \sum_{k=1}^{n} x_k \right) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0). \]
Hence and by the induction hypothesis we have
\[ F \left( \sum_{k=1}^{n+1} x_k \right) + (n-1) \sum_{k=1}^{n+1} F(x_k) \]
\[ = F \left( \sum_{k=1}^{n} x_k \right) + (n-2) \sum_{k=1}^{n} F(x_k) + \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0) \]
\[ = \frac{(n-2)(n-1)}{2} F(0) + \sum_{1 \leq k < l \leq n} F(x_k + x_l) \]
\[ + \sum_{k=1}^{n} F(x_k + x_{n+1}) + (n-1)F(0) \]
\[ = \sum_{1 \leq k < l \leq n+1} F(x_k + x_l) + \frac{(n-1)n}{2} F(0), \]
which ends the induction.
Putting \( x = x_1 = \ldots = x_n \) in (7), we have
\[ F(nx) + n(n-2)F(x) = \binom{n-1}{2} F(0) + \binom{n}{2} F(2x), \quad n \geq 3, \ x \in S, \]
and
\[
\frac{F(nx)}{n(n-2)} + F(x) = \left(\frac{n-1}{2}\right) F(0) + \left(\frac{n}{n(n-2)}\right) F(2x), \quad n \geq 3.
\]

By Lemmas 4 and 2 the limit of the right-hand side of (8) exists; consequently, so does the limit of the left-hand side, and by Lemma 6, for all \( x \in S \), there is a set \( A(x) \in cc(Y) \) such that
\[
\frac{1}{2} F(0) + \frac{1}{2} F(2x) = A(x) + F(x), \quad x \in S.
\]
This means that
\[
[A(x), \{0\}] = \frac{1}{2} [F(2x) + F(0), 2F(x)] = g(x,x), \quad x \in S.
\]

Therefore, the function \( a : X \to \mathbb{Z} \) defined by \( a(x) := g(x,x) \) is the diagonalization of the biadditive function \( g \) and
\[
a(x) = [A(x), \{0\}] \quad \text{for} \quad x \in S.
\]

By Definition 3, \( A \) is a polynomial function of order at most 2. Since \( g \) is biadditive, for \( x \in S \) and \( \lambda \in \mathbb{Q} \cap (0, \infty) \),
\[
[A(\lambda x), \{0\}] = g(\lambda x, \lambda x) = \lambda^2 g(x,x) = \lambda^2 [A(x), \{0\}],
\]
which means that \( A(\lambda x) = \lambda^2 A(x) \).

Finally, observe that the function \( x \to f(x) - a(x) \), \( x \in S \), is a Jensen function. Indeed, let \( x \in S \) and \( h \in X \) with \( x + 2h \in S \). Then
\[
\Delta^2_h (f(x) - a(x)) = \Delta^2_h f(x) - 2g(h,h) = \Delta^2_h f(x) - \Delta^2_h f(0) = \Delta^2_h \Delta_x f(0) = 0,
\]
by Theorem 1 and biadditivity of \( g \). Define \( \overline{g} : S \to \mathbb{Z} \) by
\[
\overline{g}(x) = f(x) - a(x) - [F(0), \{0\}] = [F(x), A(x) + F(0)].
\]
Then the considerations above and the fact that \( \overline{g}(0) = 0 \) imply the additivity of \( \overline{g} \).

**Definition 4** (cf. [3]). Let \( S \) be a convex cone in a vector space \( X \) over \( \mathbb{Q} \). A set \( E \) is called a base of \( S \) if \( E \) is linearly independent and the cone is spanned by \( E \), i.e., the set
\[
\left\{ x \in X : x = \sum_{k=1}^{n} \lambda_k e_k, \quad e_1, \ldots, e_n \in E, \quad \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \cap (0, \infty), \quad n \in \mathbb{N} \right\}
\]
coinsides with \( S \).

**Theorem 4.** Let \( X \) be a vector space over \( \mathbb{Q} \) and \( Y \) be a topological vector space, and let \( S \subseteq X \) be a cone with a base. Then \( F : S \to cc(Y) \) is a polynomial s.v. function of order at most 2 if and only if there exist additive s.v. functions \( B, C : S \to cc(Y) \) and biadditive s.v. functions \( D, H : S \times S \to cc(Y) \) such that
(9) \[ F(x) + \overline{C}(x) + \overline{H}(x, x) = F(0) + \overline{D}(x, x) + \overline{B}(x) \]
for \( x \in S \).

**Proof.** Since a cone with a base is \( \mathbb{Q} \)-convex, by Theorem 3 there is an s.v. function \( A : S \to \text{cc}(Y) \) such that
\[
x \mapsto [F(x), F(0) + A(x)], \quad x \in S,
\]
is additive. There exist (see Theorem 1 of [7]) additive s.v. functions \( \overline{B}, \overline{C} : S \to \text{cc}(Y) \) such that
\[
[F(x), F(0) + A(x)] = [\overline{B}(x), \overline{C}(x)], \quad x \in S,
\]
which gives
(10) \[ F(x) + \overline{C}(x) = F(0) + \overline{B}(x) + A(x), \quad x \in S. \]

In view of Remark 1,
\[
g(x, y) = \frac{1}{2} [F(x + y) + F(0), F(x) + F(y)]
\]
is biadditive. Set
\[
D(x, y) := \frac{1}{2} (F(x + y) + F(0)), \quad H(x, y) := \frac{1}{2} (F(x) + F(y)),
\]
and let \( \mathcal{E} \) be a base of \( S \). Fix \( x, y \in S \). There exist \( n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \cap (0, \infty) \) and \( e_1, \ldots, e_n \in \mathcal{E} \) such that \( x = \sum_{i=1}^{n} \lambda_i e_i \), and
\[
D(x, y) + \sum_{i=1}^{n} \lambda_i H(e_i, y) = H(x, y) + \sum_{i=1}^{n} \lambda_i D(e_i, y).
\]
Similarly
\[
H(e_i, y) + \sum_{j=1}^{m} \mu_j D(e_i, \tau_j) = D(e_i, y) + \sum_{j=1}^{m} \mu_j H(e_i, \tau_j),
\]
where \( y = \sum_{j=1}^{m} \mu_j \tau_j, \quad \tau_1, \ldots, \tau_m \in \mathcal{E} \) and \( \mu_1, \ldots, \mu_m \in \mathbb{Q} \cap (0, \infty) \). Hence
\[
D(x, y) + \sum_{i=1}^{n} \lambda_i D(e_i, y) + \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j H(e_i, \tau_j)
\]
\[
= D(x, y) + \sum_{i=1}^{n} \lambda_i \left[ D(e_i, y) + \sum_{j=1}^{m} \mu_j H(e_i, \tau_j) \right]
\]
\[
= D(x, y) + \sum_{i=1}^{n} \lambda_i \left[ H(e_i, y) + \sum_{j=1}^{m} \mu_j D(e_i, \tau_j) \right]
\]
\[
= D(x, y) + \sum_{i=1}^{n} \lambda_i H(e_i, y) + \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j D(e_i, \tau_j).
\]
Define
\[ D(x, y) := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j D(e_i, e_j), \quad H(x, y) := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j H(e_i, e_j), \]
where \( x = \sum_{i=1}^{n} \lambda_i e_i, y = \sum_{j=1}^{m} \mu_j e_j, x, y \in S \). It is clear that \( \overline{D} \) and \( \overline{H} \) are biadditive and
\[ \frac{1}{2} F(x + y) + \frac{1}{2} F(0) + \overline{H}(x, y) = \frac{1}{2} F(x) + \frac{1}{2} F(y) + \overline{D}(x, y). \]
Setting \( y = x \), we have
\[ \frac{1}{2} F(2x) + \frac{1}{2} F(0) + \overline{H}(x, x) = F(x) + \overline{D}(x, x). \]
Hence and by Theorem 3,
\[ A(x) + \overline{H}(x, x) = \overline{D}(x, x) \]
and by (10),
\[ F(x) + \overline{C}(x) + \overline{H}(x, x) = F(0) + \overline{B}(x) + A(x) + \overline{H}(x, x) \]
\[ = F(0) + \overline{D}(x, x) + \overline{B}(x), \quad x \in S. \]
Thus (9) holds true. To end the proof it suffices to prove that \( F \) is a polynomial s.v. function of order at most 2 if (9) is satisfied. By (9),
\[ \Delta_3^h F(x) \]
\[ = [F(x + 3h) + 3F(x + h), 3F(x + 2h) + F(x)] \]
\[ = [\overline{D}(x + 3h, x + 3h) + \overline{B}(x + 3h) + 3\overline{D}(x, x + h) + 3\overline{B}(x + h), \]
\[ \overline{H}(x + 3h, x + 3h) + \overline{C}(x + 3h) + 3\overline{H}(x, x + h) + 3\overline{C}(x + h)] \]
\[ - [3\overline{D}(x + 2h, x + 2h) + 3\overline{B}(x) + 2h) + \overline{D}(x, x) + \overline{B}(x), \]
\[ 3\overline{H}(x + 2h, x + 2h) + 3\overline{C}(x + 2h) + \overline{H}(x, x) + \overline{C}(x)] \]
\[ = [\overline{D}(x + 3h, x + 3h) + 3\overline{D}(x + h, x + h), 3\overline{D}(x, x + 2h) + \overline{D}(x, x)] \]
\[ = [\overline{H}(x + 3h, x + 3h) + 3\overline{H}(x + h, x + h), 3\overline{H}(x, x + 2h) + \overline{H}(x, x)] \]
\[ + [\overline{B}(x + 3h) + 3\overline{B}(x + h), 3\overline{B}(x + 2h) + \overline{B}(x)] \]
\[ + [\overline{C}(x + 3h) + 3\overline{C}(x + h), 3\overline{C}(x + 2h) + \overline{C}(x)] \]
\[ = \Delta_3^h D(x, x) - \Delta_3^h H(x, x) + \Delta_3^h B(x) - \Delta_3^h C(x) = 0, \]
for \( x \in S \) and \( h \in X \) such that \( x + 3h \in S \), because \( \overline{D} \) and \( \overline{H} \) are biadditive and \( \overline{B} \) and \( \overline{C} \) are additive. So, the proof is complete. \( \blacksquare \)
References


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