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On a differential inequality for a viscous compressible heat conducting capillary fluid bounded by a free surface

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Abstract. We derive a global differential inequality for solutions of a free boundary problem for a viscous compressible heat concluding capillary fluid. The inequality is essential in proving the global existence of solutions.

1. Introduction. The motion of a viscous compressible heat conducting capillary fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ (which depends on time $t \in \mathbb{R}^1_+$) is described by the following system with the boundary and initial conditions (see [2], Chs. 2 and 5):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] + \nabla p - \mu\Delta v - \nu\nabla\operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \kappa\Delta\theta \\
&\qquad - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{i,x_j} + v_{j,x_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\phi_t/|\nabla\phi| && \text{on } \widetilde{S}^T,\\
&\partial\theta/\partial n = \theta_1 && \text{on } \widetilde{S}^T,\\
&v|_{t=0} = v_0,\quad \varrho|_{t=0} = \varrho_0,\quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

where $\phi(x,t) = 0$ describes S_t, $\overline{n}$ is the outward vector normal to the boundary (i.e. $\overline{n} = \nabla\phi/|\nabla\phi|$), $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)} \Omega_t\times\{t\}$, $\Omega_0 = \Omega$ is an initial domain, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t\times\{t\}$. Moreover, $v = v(x,t)$ $(v = (v_1, v_2, v_3))$ is the velocity

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of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $f = f(x,t)$ the external force field per unit mass, $r = r(x,t)$ the heat sources per unit mass, $\theta_1 = \theta_1(x,t)$ the heat flow per unit surface, $p = p(\varrho,\theta)$ the pressure, μ and ν the viscosity coefficients, κ the coefficient of heat conductivity, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, and p_0 the external (constant) pressure. We assume that the coefficients μ, ν, κ are constants such that $\kappa > 0$, $\nu \ge \frac{1}{3}\mu > 0$ and moreover $c_v > 0$, which results from thermodynamic considerations. Further, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\mathbb{T} = \{T_{ij}\} = \{-p\delta_{ij} + \mu(v_{i,x_j} + v_{j,x_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\} \equiv \{-p\delta_{ij} + D_{ij}(v)\},$$

where $i,j = 1,2,3$, and $\mathbb{D} = \mathbb{D}(v) = \{D_{ij}\}$ is the deformation tensor.

Finally, we denote by H the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\overline{n} = \Delta_{S_t}(t)x, \quad x = (x_1,x_2,x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be determined by $x = x(s^1,s^2,t)$, $(s^1,s^2) \in \mathbb{R}^2$. Then we have

$$\begin{aligned}\Delta_{S_t}(t) &= g^{-1/2}\frac{\partial}{\partial s^\alpha} g^{-1/2}\widehat{g}_{\alpha\beta}\frac{\partial}{\partial s^\beta}\\ &= g^{-1/2}\frac{\partial}{\partial s^\alpha} g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial s^\beta} \quad (\alpha,\beta = 1,2),\end{aligned}$$

where the convention summation over repeated indices is assumed, $g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = x_\alpha \cdot x_\beta$, $(x^\alpha = \partial x/\partial s^\alpha)$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements of $\{g_{\alpha\beta}\}$.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t), \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1,\xi_2,\xi_3).$$

Hence

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv X_u(\xi,t), \tag{1.2}$$

where $u(\xi,t) = v(X_u(\xi,t),t)$.

Formula (1.2) yields the relation between the Eulerian x and Lagrangian ξ coordinates. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface. Thus, if $\xi \in S = S_0$ then $X_u(\xi,t) \in S_t$ and $S_t = \{x : x = X_u(\xi,t),\ \xi \in S\}$.

By the equation of continuity $(1.1)_2$ and $(1.1)_5$ the total mass of the drop is conserved and the following relation between ϱ and Ω_t holds:

$$\int_{\Omega_t} \varrho(x,t)\,dx = M.$$

In this paper we prove a global differential inequality for problem (1.1) (see Theorem 3.13) which we shall use in the next paper to prove the global-in-time existence of a solution to problem (1.1) close to a constant state. The paper is divided into three sections. In Section 2 we introduce some notation. In Section 3 we formulate a series of lemmas (see Lemmas 3.1–3.12) which are used to derive the differential inequalities (3.46) and (3.47).

Problem (1.1) is also considered in [12]–[17]. In [12] we prove the local existence of a solution to problem (1.1) in Sobolev–Slobodetskiĭ spaces in two cases: $\sigma=0$ and $\sigma>0$. [14] and [15] are devoted to conservation laws for problem (1.1) in two cases: without surface tension and with it, respectively. In [14] and [15] we prove that we can choose $\varrho_0, v_0, \theta_0, \theta_1, p_0, \kappa, \sigma$ (in the case $\sigma > 0$) and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\varrho, \theta)$ in such a way that $\mathrm{var}_t\,|\Omega_t|$ is as small as we need. In [16] the global differential inequality in the case $\sigma = 0$, analogous to inequality (3.46) is obtained. [17] is concerned with the global-in-time existence of solutions to problem (1.1) when $\sigma = 0$. Finally, [13] contains a review of results from [12], [1]–[17] and this paper.

In order to prove the main result of the paper, i.e. Theorem 3.13, we apply the same method as in [18], [19] and [16], which is very close to the methods used in [10] and [11] (see also [4]–[7] and [8]).

Papers [18] and [19] of W. M. Zajączkowski and [9] of V. A. Solonnikov and A. Tani refer to the problem corresponding to (1.1) for a compressible barotropic fluid.

In [8] K. Pileckas and W. M. Zajączkowski proved the existence of stationary motion of a viscous compressible barotropic fluid bounded by a free surface governed by surface tension.

Finally, the motion of a viscous compressible heat conducting fluid in a fixed domain was considered by A. Matsumura and T. Nishida in [3]–[7] and by A. Valli and W. M. Zajączkowski in [11].

2. Notation. Let $Q=\Omega_t$ or $Q = S_t$ ($t\ge 0$). We denote by $\|\cdot\|_{l,Q}$ ($l\ge 0$) and $|\cdot|_{p,Q}$ ($1 \le p \le \infty$) the norms in the usual Sobolev spaces $W_2^l(Q)$ and $L_p(Q)$ spaces, respectively.

Next, we introduce the space $\Gamma_k^l(Q)$ of functions u with the norm

$$\|u\|_{\Gamma_k^l(Q)} = \sum_{i\le l-k} \|\partial_t^i u\|_{l-i,Q} \equiv |u|_{l,k,Q},$$

where $l > 0$ and $k \ge 0$.

We shall use the following notation for derivatives of u. If u is a scalar-valued function we denote by $D^k_{x,t}u$ or $u_{\underbrace{x\ldots xt\ldots t}_{k\text{ times}}}$ the vector $(D_x^\alpha \partial_t^i u)_{|\alpha|+i=k}$.

Similarly, if $u=(u_1,u_2,u_3)$ we denote by $D^k_{x,t}u$ or $u_{\underbrace{x\ldots xt\ldots t}_{k\text{ times}}}$ the vector $(D^\alpha_x\partial^i_t u_j)_{|\alpha|+i=k,\, j=1,2,3}$. Hence

$$|D^k_{x,t}u|=\sum_{|\alpha|+i=k}|D^\alpha_x\partial^i_t u|.$$

We use the following lemma:

LEMMA 2.1. *The following imbedding holds*:

$$W^l_r(Q)\subset L^\alpha_p(Q)\qquad (Q\subset\mathbb{R}^3),$$

where $|\alpha|+3/r-3/p\le l$, $l\in\mathbb{Z}$, $1\le p,r\le\infty$; $L^\alpha_p(Q)$ *is the space of functions* u *such that* $|D^\alpha_x u|_{p,Q}<\infty$, *and* $W^l_r(Q)$ *is the Sobolev space.*

Moreover, *the following interpolation inequalities are true*:

$$|D^\alpha_x u|_{p,Q}\le c\varepsilon^{1-\kappa}|D^l_x u|_{r,Q}+c\varepsilon^{-\kappa}|u|_{r,Q},\tag{2.1}$$

where $\kappa=|\alpha|/l+3/(lr)-3/(lp)<1$, ε *is a parameter*, $c>0$ *is a constant independent of* u *and* ε;

$$|D^\alpha_x u|_{q,\partial Q}\le c\varepsilon^{1-\kappa}|D^l_x u|_{r,Q}+c\varepsilon^{1-\kappa}|u|_{r,Q},\tag{2.2}$$

where $\kappa=|\alpha|/l+3/(lr)-2/(lq)<1$, ε *is a parameter*, $c>0$ *is a constant independent of* u *and* ε. ■

Lemma 2.1 follows from Theorem 10.2 of [1].

3. Global differential inequality. In this section we assume that $\nu>\frac{1}{3}\mu$. Further, assume that the existence of a sufficiently smooth local solution of problem (1.1) has been proved. To prove the desired differential inequality we assume that Ω_t ($t\le T$, T is the time of local existence) is diffeomorphic to a ball, so S_t can be described by

$$|x|\equiv r=R(\omega,t),\qquad \omega\in S^1,$$

where S^1 is the unit sphere.

Moreover, we consider the motion near the constant state $v_{\rm e}=0$, $p_{\rm e}=p_0+2\sigma/R_{\rm e}$, $\theta_{\rm e}=(1/|\Omega|)\int_\Omega\theta_0\,d\xi$, $\varrho_{\rm e}=M/((4\pi/3)R^3_{\rm e})$, where $R_{\rm e}$ is a solution of the equation

$$p\bigg(\frac{M}{(4\pi/3)R^3_{\rm e}},\theta_{\rm e}\bigg)=p_{\rm e}.$$

(Obviously, we assume that the above equation is solvable with respect to $R_{\rm e}>0$.)

Let

$$p_\sigma=p-p_0-q_0,\quad \varrho_\sigma=\varrho-\varrho_{\rm e},\quad \vartheta_0=\theta-\theta_{\rm e},\quad \vartheta=\theta-\theta_{\Omega_t},$$

where $q_0 = 2\sigma/R_{\rm e}$ and $\theta_{\Omega_t} = (1/|\Omega_t|)\int_{\Omega_t}\theta\,dx$. Then problem (1.1) takes the form

$$\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p_\sigma) = \varrho f && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho c_v(\varrho,\theta)(\vartheta_{0t} + v\cdot\nabla\vartheta_0) + \theta p_\theta(\varrho,\theta)\operatorname{div} v\\
&\qquad - \kappa\Delta\vartheta_0 - \frac{\mu}{2}\sum_{i,j}(\partial_{x_i}v_j + \partial_{x_j}v_i)^2\\
&\qquad - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \Omega_t,\ t\in[0,T],\\
&\mathbb{T}(v,p_\sigma)\overline{n} = \sigma\Delta_{S_t}x\cdot\overline{n}\,\overline{n} + q_0\overline{n} && \text{on } S_t,\ t\in[0,T],\\
&\partial\vartheta_0/\partial n = \theta_1 && \text{on } S_t,\ t\in[0,T],
\end{aligned}\tag{3.1}$$

where $\mathbb{T}(v,p_\sigma) = \{\mu(\partial_{x_i}v_j + \partial_{x_j}v_i) + (\nu-\mu)\delta_{ij}\operatorname{div} v - p_\sigma\delta_{ij}\}$ and T is the time of local existence.

In the sequel we shall use the following Taylor formula for p_σ:

$$\begin{aligned}
p_\sigma &= p(\varrho,\theta) - p(\varrho_{\rm e},\theta_{\rm e})\\
&= p(\varrho,\theta) - p(\varrho_{\rm e},\theta) + p(\varrho_{\rm e},\theta) - p(\varrho_{\rm e},\theta_{\rm e})\\
&= (\varrho-\varrho_{\rm e})\int_0^1 p_\varrho(\varrho_{\rm e} + s(\varrho-\varrho_{\rm e}),\theta)\,ds\\
&\quad + (\theta-\theta_{\rm e})\int_0^1 p_\theta(\varrho_{\rm e},\theta_{\rm e} + s(\theta-\theta_{\rm e}))\,ds\\
&\equiv p_1\varrho_\sigma + p_2\vartheta_0.
\end{aligned}\tag{3.2}$$

We shall also use the formula:

$$\begin{aligned}
p_\sigma &= p(\varrho,\theta) - p(\varrho_{\Omega_t},\theta_{\Omega_t})\\
&= (\varrho-\varrho_{\Omega_t})\int_0^1 p_\varrho(\varrho_{\Omega_t} + s(\varrho-\varrho_{\Omega_t}),\theta)\,ds\\
&\quad + (\theta-\theta_{\Omega_t})\int_0^1 p_\theta(\varrho_{\Omega_t},\theta_{\Omega_t} + s(\theta-\theta_{\Omega_t}))\,ds\\
&\equiv p_3\overline{\varrho}_{\Omega_t} + p_4\vartheta,
\end{aligned}\tag{3.3}$$

where the function $\varrho_{\Omega_t} = \varrho_{\Omega_t}(t)$ is a solution of the problem

$$p(\varrho_{\Omega_t},\theta_{\Omega_t}) = p_{\rm e},\qquad \varrho_{\Omega_t}|_{t=0} = \varrho_{\rm e}\tag{3.4}$$

and

$$\overline{\varrho}_{\Omega_t} = \varrho - \varrho_{\Omega_t}.$$

The functions p_i $(i=1,2,3,4)$ in (3.4) and (3.5) are positive.

Set

$$\varrho_* = \min_{\widetilde{\Omega}^T} \varrho(x,t), \quad \varrho^* = \max_{\widetilde{\Omega}^T} \varrho(x,t),$$
$$\theta_* = \min_{\widetilde{\Omega}^T} \theta(x,t), \quad \theta^* = \max_{\widetilde{\Omega}^T} \theta(x,t).$$

Now we point out the following facts concerning the estimates in Lemmas 3.1–3.12 and Theorem 3.13:

1. We denote by ε small constants and for simplicity we do not distinguish them.

2. We denote by C_1 and C_2 constants which depend on ϱ_*, ϱ^*, θ_*, θ^*, T, $\int_0^T \|v\|_{3,\Omega_{t'}}^2\,dt'$, $\|S\|_{4+1/2}$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi,t)$ and also on the constants of imbedding theorems and Korn inequalities. C_1 is always the coefficient of a linear term, while C_2 is the coefficient of a nonlinear term. For simplicity we do not distinguish different C_1's and C_2's.

3. We denote by c absolute constants which may depend on such parameters as μ, ν, κ, and by $c_0 < 1$ positive constants which may depend on $\mu,\nu,\kappa,\varrho_*,\varrho^*,\theta_*,\theta^*$. For simplicity we do not distinguish different C's and C_0's.

4. We underline that all the estimates are obtained under the assumption that there exists a local-in-time solution of (1.1), so all the quantities ϱ_*, ϱ^*, θ_*, θ^*, T, $\int_0^T \|v\|_{3,\Omega_{t'}}^2\,dt'$, $\|S\|_{4+1/2}$ are estimated by the data functions. Moreover, the existence of the inverse transformation to $x = x(\xi,t)$ is guaranteed by the estimates for the local solutions (see [12]).

Lemma 3.1. *Let v,ϱ,ϑ_0 be a sufficiently smooth solution of* (3.1). *Then*

$$\begin{aligned}(3.6)\quad & \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2 + \overline{\varrho}_{\Omega_t}^2 + \frac{p_2\varrho c_v}{p_\theta\theta}\vartheta_0^2\Big)dx \\ & + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\Big(g^{\alpha\beta}\overline{n}\cdot\int_0^t v_{s^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{s^\beta}\,dt'\Big)\,ds + c_0\|v\|_{1,\Omega_t}^2 \\ & + (\nu-\mu)\|\operatorname{div} v\|_{0,\Omega_t}^2 + c_0\|\vartheta_{0x}\|_{0,\Omega_t}^2 \\ \le{}& \varepsilon\Big(\|p_\sigma\|_{0,\Omega_t}^2 + \|\vartheta_{0tx}\|_{0,\Omega_t}^2 + \Big\|\int_0^t v_s\,dt'\Big\|_{0,S_t}^2 + \|H(\cdot,0)+2/R_{\mathrm{e}}\|_{0,S^1}^2\Big) \\ & + C_1(\|v\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t} + \|\theta_1\|_{1,\Omega_t}^2 \\ & + \|\theta_1\|_{1,\Omega_t} + \|f\|_{0,\Omega_t}^2) + C_2X_1Y_1,\end{aligned}$$

where $\varepsilon > 0$ *is sufficiently small,* $s = (s^1, s^2)$ *and*

$$X_1 = \|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\vartheta_0\|^2_{2,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},$$
$$Y_1 = X_1 + \Big\|\int_0^1 v\,dt'\Big\|^2_{2,S_t}.$$

Proof. Multiplying $(3.1)_1$ by v, integrating over Ω_t and using the continuity equation $(3.1)_2$, boundary condition $(3.1)_4$ and (3.2) we obtain

$$\begin{aligned}(3.7)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\varrho v^2\,dx + \frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t}\\ &-\int_{\Omega_t} p_1\varrho_\sigma \operatorname{div} v dx - \int_{\Omega_t} p_2\vartheta_0 \operatorname{div} v\,dx\\ &-\sigma\int_{S_t}(\Delta_{S_t}x\cdot\overline{n} + 2/R_{\rm e})\overline{n}\cdot v\,ds = \int_{\Omega_t}\varrho f v\,dx,\end{aligned}$$

where $E_{\Omega_t}(v) = \int_{\Omega_t}\sum_{i,j=1}^3(\partial_{x_i}v_j + \partial_{x_j}v_i)^2\,dx$.

First, we consider the sum of the second and third terms on the left-hand side of (3.7). We have

$$\begin{aligned}(3.8)\qquad &\frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t}\\ &= \frac{\mu}{2}\int_{\Omega_t}(v_{i,x_j} + v_{j,x_i})^2\,dx + (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx\\ &= \frac{\mu}{2}\sum_{i\ne j}\int_{\Omega_t}(v_{i,x_j} + v_{j,x_i})^2\,dx + \frac{\mu}{2}\sum_{i=j}\int_{\Omega_t}(v_{i,x_j} + v_{j,x_i})^2\,dx\\ &\quad + (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx\\ &= \frac{\mu}{2}\sum_{i\ne j}\int_{\Omega_t}(v_{i,x_j} + v_{j,x_i})^2\,dx + \frac{\mu}{2}\varepsilon_1\sum_{i=j}\int_{\Omega_t}(v_{i,x_j} + v_{j,x_i})^2\,dx\\ &\quad + \frac{\mu}{2}(1-\varepsilon_1)\cdot 4\sum_i\int_{\Omega_t}(v_{i,x_j})^2\,dx + (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx \equiv I,\end{aligned}$$

where $\varepsilon_1 \in (0,1)$. Since $(\xi_1+\xi_2+\xi_3)^2 \le 3(\xi_1^2+\xi_2^2+\xi_3^2)$ the last two terms in I are estimated from below by

$$[\nu - (1+2\varepsilon_1)\mu/2]\int_{\Omega_t}(\operatorname{div} v)^2\,dx.$$

Assuming that $\nu=(1+2\varepsilon_1)\mu/3$ we obtain $\varepsilon_1=\frac{3}{2\mu}(\nu-\mu/3)$, so

$$I\ge\frac{\mu}{2}\varepsilon_1\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx=\frac{3}{4}\left(\nu-\frac{\mu}{3}\right)\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx. \tag{3.9}$$

By the continuity equation $(3.1)_2$, energy equation $(3.1)_3$ and boundary condition $(3.1)_5$ we have

$$\begin{aligned}-\int_{\Omega_t}p_1\varrho_\sigma\operatorname{div}v\,dx&=\int_{\Omega_t}\frac{p_1}{\varrho}\varrho_\sigma(\varrho_{\sigma_t}+v\cdot\nabla\varrho_\sigma)\,dx\\&=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{p_1\varrho_\sigma^2}{\varrho}\,dx+I_1,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}|I_1|\le{}&\varepsilon(\|v_x\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2)+C_1(\|r\|_{0,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}^2)\\&+C_2(\|\varrho_\sigma\|_{1,\Omega_t}^4+\|v\|_{1,\Omega_t}^2\|\varrho_\sigma\|_{2,\Omega_t}^2+\|v\|_{1,\Omega_t}^2\|\vartheta_0\|_{2,\Omega_t}^2\\&+\|v\|_{2,\Omega_t}^2\|\varrho_\sigma\|_{1,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2\|\varrho_\sigma\|_{1,\Omega_t}^2).\end{aligned} \tag{3.11}$$

Now, we consider the boundary term in (3.7). In the same way as in Lemma 4.1 of [19] we obtain

$$\begin{aligned}&-\int_{S_t}(\Delta_{S_t}x\cdot\overline{n}+2/R_{\rm e})v\cdot\overline{n}\,ds\\&\qquad=\frac{1}{2}\frac{d}{dt}\int_{S_t}\Big(g^{\alpha\beta}\overline{n}\cdot\int_0^t v_{s^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{s^\beta}\,dt'\Big)ds+I_1,\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}|I_1|\le{}&\varepsilon\Big(\Big\|\int_0^t v_s\,dt'\Big\|_{0,S_t}^2+\|H(\cdot,0)+2/R_{\rm e}\|_{0,S^1}^2+\|v\|_{1,\Omega_t}^2\Big)\\&+C_1\|v\|_{0,\Omega_t}^2+C_2\Big\|\int_0^t v\,dt'\Big\|_{2,S_t}^2\|v\|_{2,\Omega_t}^2.\end{aligned} \tag{3.13}$$

Next, dividing $(3.1)_3$ by θp_θ, multiplying the result by $p_2\vartheta_0$ and integrating over Ω_t we get

$$\begin{aligned}&\int_{\Omega_t}\frac{p_2\varrho c_v}{\theta p_\theta}\left(\partial_t\frac{\vartheta_0^2}{2}+v\cdot\nabla\frac{\vartheta_0^2}{2}\right)dx+\int_{\Omega_t}p_2\vartheta_0\operatorname{div}v\,dx\\&\quad-\int_{\Omega_t}\frac{p_2\kappa\Delta\vartheta_0}{\theta p_\theta}\vartheta_0dx-\int_{\Omega_t}\frac{p_2\mu}{2\theta p_\theta}\sum_{i,j}(\partial_{x_j}v_i+\partial_{x_i}v_j)^2\vartheta_0\,dx\\&\quad-\int_{\Omega_t}\frac{p_2(\nu-\mu)}{\theta p_\theta}(\operatorname{div}v)^2\vartheta_0\,dx=\int_{\Omega_t}\frac{p_2\varrho r}{\theta p_\theta}\vartheta_0\,dx.\end{aligned} \tag{3.14}$$

Therefore, using the same argument as in Lemma 3.1 of [16], by (3.7)–(3.14) we have

$$
\begin{aligned}
(3.15)\quad & \frac{1}{2}\frac{d}{dt}\bigg(\varrho v^2 + \frac{p_1\varrho_\sigma^2}{\varrho} + \frac{p_2\varrho c_v}{\theta p_\theta}\vartheta_0^2\bigg)\,dx \\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t} g^{\alpha\beta}\overline{n}\cdot\int_0^t v_{s^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{s^\beta}\,dt'\,ds + c_0\|v\|_{1,\Omega_t}^2 \\
&\quad + (\nu-\mu)\|\operatorname{div} v\|_{0,\Omega_t}^2 + c_0\|\vartheta_{0x}\|_{0,\Omega_t}^2 \\
&\le \varepsilon\bigg(\bigg\|\int_0^t v_s\,dt'\bigg\|_{0,S_t}^2 + \|H(\cdot,0)+2/R_0\|_{0,S^1}^2\bigg) \\
&\quad + C_1(\|v\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t} \\
&\quad + \|\theta_1\|_{1,\Omega_t}^2 + \|\theta_1\|_{1,\Omega_t} + \|f\|_{0,\Omega_t}^2) + C_2X_1Y_1.
\end{aligned}
$$

Finally, by $(3.1)_2$ and (3.5) we have

$$
(3.16)\qquad \partial_t\overline{\varrho}_{\Omega_t} + v\cdot\nabla\overline{\varrho}_{\Omega_t} + \varrho\operatorname{div} v + \partial_t\varrho_{\Omega_t} = 0,
$$

where in view of (3.4) we get

$$
(3.17)\qquad \partial_t\varrho_{\Omega_t} = -\frac{p_{\theta_{\Omega_t}}}{p_{\varrho_{\Omega_t}}}\partial_t\theta_{\Omega_t}.
$$

Using the definition of θ_{Ω_t} we calculate

$$
\begin{aligned}
(3.18)\qquad \partial_t\theta_{\Omega_t} &= \frac{1}{|\Omega_t|}\int_{\Omega_t}\vartheta_{0t}\,dx + \frac{1}{|\Omega_t|}\int_{\Omega_t}\theta\operatorname{div} v\,dx \\
&\quad - \frac{1}{|\Omega_t|^2}\Big(\int_{\Omega_t}\theta\,dx\Big)\Big(\int_{\Omega_t}\operatorname{div} v\,dx\Big).
\end{aligned}
$$

Equation (3.16) and formulas (3.17), (3.18) yield

$$
\begin{aligned}
(3.19)\qquad \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\overline{\varrho}_{\Omega_t}^2\,dx &\le \varepsilon(\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta_{0tx}\|_{0,\Omega_t}^2) \\
&\quad + C_1(\|v_x\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2 + \|r\|_{0,\Omega_t}^2 + \|\theta_1\|_{1,\Omega_t}^2) \\
&\quad + C_2(\|v\|_{1,\Omega_t}^2\|\vartheta_0\|_{2,\Omega_t}^2 + \|v\|_{2,\Omega_t}^4 + \|\overline{\varrho}_{\Omega_t}\|_{1,\Omega_t}^4 \\
&\quad + \|\varrho_\sigma\|_{2,\Omega_t}^2\|\vartheta_0\|_{2,\Omega_t}^2 + \|\vartheta_0\|_{2,\Omega_t}^4).
\end{aligned}
$$

By (3.3) and the Poincaré inequality

$$
(3.20)\qquad \|\vartheta\|_{0,\Omega_t} \le \|\vartheta_{0x}\|_{0,\Omega_t}
$$

we get

$$
(3.21)\qquad \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t} \le C_1(\|\vartheta_{0x}\|_{0,\Omega_t} + \|p_\sigma\|_{0,\Omega_t}).
$$

The estimates (3.15), (3.19) and (3.21) yield (3.6). ■

LEMMA 3.2. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.1). *Then*

$$\begin{aligned} (3.22)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2+\frac{\varrho c_v}{\theta}\vartheta_{0t}^2\right)dx \\ &+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t} g^{\alpha\beta}v_{s^\alpha}\cdot\overline{n}v_{s^\beta}\cdot\overline{n}\,ds+c_0\|v_t\|_{1,\Omega_t}^2 \\ &+(\nu-\mu)\|\mathrm{div}v_t\|_{0,\Omega_t}^2+c_0\|\vartheta_{0t}\|_{1,\Omega_t}^2 \\ &\le\varepsilon(\|v\|_{1,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2) \\ &+C_1(|f|_{1,0,\Omega_t}^2+|r|_{1,0,\Omega_t}^2+\|r\|_{0,\Omega_t}+|\theta_1|_{2,1,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}) \\ &+C_2X_2^2(1+X_2), \end{aligned}$$

where $X_2=|v|_{2,1,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+|\vartheta_0|_{2,1,\Omega_t}^2$.

Proof. By the same argument as in Lemma 3.2 of [16] we have

$$\begin{aligned} (3.23)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2+\frac{\varrho c_v}{\theta}\vartheta_{0t}^2\right)dx+\|v_t\|_{1,\Omega_t}^2 \\ &+(\nu-\mu)\|\mathrm{div}v_t\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2-\int\limits_{S_t}[\mathbb{T}(v,p_\sigma)]_{,t}\overline{n}\cdot v_t\,ds \\ &\le\varepsilon(\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|v_t\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|\vartheta_{0tx}\|_{0,\Omega_t}^2) \\ &+C_1(\|r\|_{0,\Omega_t}^2+\|r_t\|_{0,\Omega_t}^2+|\theta_1|_{2,1,\Omega_t}^2)+C_2X_2^2(1+X_2). \end{aligned}$$

By the boundary condition $(3.1)_4$ and the same argument as in Lemma 4.2 of [19] we get

$$(3.24)\qquad -\int\limits_{S_t}[\mathbb{T}(v,p_\sigma)]_{,t}\overline{n}\cdot v_t\,ds=\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}g^{\alpha\beta}v_{s^\alpha}\cdot\overline{n}v_{s^\beta}\cdot\overline{n}\,ds+I_2,$$

where

$$(3.25)\qquad |I_2|\le\varepsilon(\|v_t\|_{1,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2)+C_1\|v\|_{0,\Omega_t}^2+C_2X_2^2.$$

From the continuity equation $(3.1)_2$ it follows that

$$(3.26)\qquad \|\varrho_{\sigma t}\|_{0,\Omega_t}^2\le C_1\|v\|_{1,\Omega_t}^2+C_2\|v\|_{1,\Omega_t}^2\|\varrho_\sigma\|_{2,\Omega_t}^2$$

and equation $(3.1)_3$ yields

$$\begin{aligned} (3.27)\quad \|\vartheta_{0t}\|_{0,\Omega_t}^2&\le\varepsilon\|\vartheta_{0xt}\|_{0,\Omega_t}^2 \\ &+C_1(\|v_x\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2+\|r\|_{0,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}^2) \\ &+C_2(\|v\|_{1,\Omega_t}^2\|\vartheta_0\|_{2,\Omega_t}^2+\|v\|_{1,\Omega_t}^4 \\ &+\|\varrho_\sigma\|_{2,\Omega_t}^2\|\vartheta_0\|_{2,\Omega_t}^2+\|\vartheta_0\|_{2,\Omega_t}^4). \end{aligned}$$

Therefore, taking into account (3.23)–(3.27) we obtain (3.22). ∎

Lemmas 3.1 and 3.2 imply

LEMMA 3.3. *Let v,ϱ,ϑ_0 be a sufficiently smooth solution of* (3.1). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg[\varrho(v^2+v_t^2)+\frac{1}{\varrho}(p_1\varrho_\sigma^2+p_{\sigma\varrho}\varrho_{\sigma t}^2)+\overline{\varrho}_{\Omega_t}^2+\frac{\varrho c_v}{\theta}\bigg(\frac{p_2}{p_\theta}\vartheta_0^2+\vartheta_{0t}^2\bigg)\bigg]\,dx\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}g^{\alpha\beta}\bigg[\overline{n}\cdot\int\limits_0^t v_{s^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t v_{s^\beta}\,dt'+\overline{n}\cdot v_{s^\alpha}\overline{n}\cdot v_{s^\beta}\bigg]\,ds\\
&\quad+c_0(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2)+(\nu-\mu)(\|\operatorname{div}v\|_{0,\Omega_t}^2+\|\operatorname{div}v_t\|_{0,\Omega_t}^2)\\
&\quad+c_0(\|\vartheta_{0x}\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2)\\
&\le\varepsilon\bigg(\|\varrho_\sigma\|_{0,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2+\bigg\|\int\limits_0^t v_s\,dt'\bigg\|_{0,S_t}^2+\|H(\cdot,0)+2/R_{\mathrm e}\|_{0,S^1}^2\bigg)\\
&\quad+C_1(\|v\|_{0,\Omega_t}^2+|r|_{1,0,\Omega_t}^2+\|r\|_{0,\Omega_t}+|\theta_1|_{2,1,\Omega_t}^2+\|\theta_1\|_{1,\Omega_t}+|f|_{1,0,\Omega_t}^2)\\
&\quad+C_2[X_1Y_1+X_2^2(1+X_2)].\ \blacksquare
\end{aligned}$$

In order to obtain an inequality for derivatives with respect to x we rewrite problem (3.1) in the Lagrangian coordinates and next we introduce a partition of unity in the fixed domain Ω. Thus we have

$$\begin{aligned}
&\eta u_{it}-\nabla_{u_j}\mathbb{T}_u^{ij}(u,p_\sigma)=\eta g_i,\qquad i=1,2,3,\\
&\eta_{\sigma_t}+\eta\nabla_u\cdot u=0,\\
&\eta c_v(\eta,\Gamma)\gamma_{0t}-\kappa\nabla_u^2\gamma_0=\eta k-\Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot u\\
&\qquad+\frac{\mu}{2}\sum_{i,j=1}^3(\xi_{kx_i}\partial_{\xi_k}u_j+\xi_{kx_j}\partial_{\xi_k}u_i)^2\\
&\qquad+(\nu-\mu)(\nabla_u\cdot u)^2,\\
&\mathbb{T}_u(u,p_\sigma)\overline{n}=\sigma\Delta_{S_t}x(\xi,t)\cdot\overline{n}\,\overline{n}+q_0\overline{n},\\
&\overline{n}\cdot\nabla_u\gamma_0=\Gamma_1,
\end{aligned}\tag{3.28}$$

where $\eta(\xi,t)=\varrho(x(\xi,t),t)$, $u(\xi,t)=v(x(\xi,t),t)$, $g(\xi,t)=f(x(\xi,t),t)$, $\Gamma(\xi,t)=\theta(x(\xi,t),t)$, $\gamma_0(\xi,t)=\vartheta_0(x(\xi,t),t)$, $\Gamma_1(\xi,t)=\theta_1(x(\xi,t),t)$, $\overline{n}=\overline{n}(\xi,t)$ and

$$\mathbb{T}_u(u,p_\sigma)=\{T_u^{ij}(u,p_\sigma)\}=\{-p_\sigma\delta_{ij}+\mu(\nabla_{u_i}u_j+\nabla_{u_j}u_i)+(\nu-\mu)\delta_{ij}\nabla_u\cdot u\},$$

$\nabla_{u_i}=\xi_{kx_i}\partial_{\xi_k}$ and $\operatorname{div}\mathbb{T}_u(u,p_\sigma)=\nabla_u\cdot\mathbb{T}_u(u,p_\sigma)$.

By (3.4) and (3.5) we have respectively

$$p_\sigma=p_1\eta_\sigma+p_2\gamma_0$$

and

$$p_\sigma = p_3\overline{\eta}_{\Omega_t} + p_4\gamma,$$

where $\eta_\sigma = \eta - \varrho_{\rm e}$, $\gamma_0 = \Gamma - \theta_{\rm e}$, $\overline{\eta}_{\Omega_t} = \eta - \varrho_{\Omega_t}$, $p_1 = \int_0^1 p_\eta(\varrho_{\rm e} + s\eta_\sigma, \Gamma)\,ds$, $p_2 = \int_0^1 p_\Gamma(\varrho_{\rm e}, \theta_{\rm e} + s\gamma_0)\,ds$, $p_3 = \int_0^1 p_\eta(\varrho_{\Omega_t} + s\overline{\eta}_{\Omega_t}, \Gamma)\,ds$, $p_4 = \int_0^1 p_\Gamma(\varrho_{\Omega_t}, \theta_{\Omega_t} + s\gamma)\,ds$, $p_i > 0$ $(i = 1,2,3,4)$.

Let us introduce a partition of unity $(\{\widetilde{\Omega}_i\}, \{\zeta_i\})$, $\Omega = \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_{i,s}$ and $\zeta(\xi) = \zeta_i(\xi)$ be the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain then let $\widetilde{\omega}$ be a set such that $\overline{\widetilde{\omega}} \subset \widetilde{\Omega}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$. Otherwise we assume that $\overline{\widetilde{\Omega}} \cap S = \emptyset$, $\overline{\widetilde{\omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$. Take any $\beta \in \overline{\widetilde{\omega}} \cap S \subset \overline{\widetilde{\Omega}} \cap S = \overline{\widetilde{S}}\partial$ and introduce local coordinates $\{y\}$ associated with $\{\xi\}$ by the relation

$$y_k = \alpha_{kl}(\xi_l - \beta_l), \quad \alpha_{3k} = n_k(\beta), \quad k = 1,2,3, \tag{3.29}$$

where $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\widetilde{S}$ is described by the equation $y_3 = F(y_1, y_2)$, $F \in W_2^{4-1/2}$ and

$$\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1,2,\ F(y') < y_3 < F(y') + d,\ y' = (y_1, y_2)\}.$$

Next introduce functions u', η', Γ', γ_0', γ', Γ_1' by means of the formulas

$$\begin{aligned} u_i'(y) &= \alpha_{ij}u_j(\xi)|_{\xi=\xi(y)}, & \eta'(y) &= \eta(\xi)|_{\xi=\xi(y)},\\ \Gamma'(y) &= \Gamma(\xi)|_{\xi=\xi(y)}, & \gamma_0'(y) &= \gamma_0(\xi)|_{\xi=\xi(y)},\\ \gamma'(y) &= \gamma(\xi)|_{\xi=\xi(y)}, & \Gamma_1'(y) &= \Gamma_1(\xi)|_{\xi=\xi(y)}, \end{aligned}$$

where $\xi = \xi(y)$ is the inverse transformation to (3.29). Further, we introduce new variables by

$$z_i = y_i \quad (i = 1,2), \quad z_3 = y_3 - \widetilde{F}(y), \quad y \in \widetilde{\Omega},$$

which will be denoted by $z = \Phi(y)$, where $\widetilde{F}$ is an extension of F, so $\widetilde{F} \in W_2^4$.

Let $\widehat{\Omega} = \Phi(\widetilde{\Omega}) = \{z : |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\}$ and $\widehat{S} = \Phi(\widetilde{S})$. Define

$$\begin{aligned} \widehat{u}(z) &= u'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\eta}(z) &= \eta'(y)|_{y=\Phi^{-1}(z)},\\ \widehat{\Gamma}(z) &= \Gamma'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\gamma}_0(z) &= \gamma_0'(y)|_{y=\Phi^{-1}(z)},\\ \widehat{\gamma}(z) &= \gamma'(y)|_{y=\Phi^{-1}(z)}, & \widehat{\Gamma}_1(z) &= \Gamma_1(y)|_{y=\Phi^{-1}(z)}. \end{aligned}$$

Set $\widehat{\nabla}_k = \xi_{lx_k}(\xi) z_{i\xi_l}\nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is described by (3.29). We also introduce the following notation:

$$\begin{aligned} \widetilde{u}(\xi) &= u(\xi)\zeta(\xi), & \widetilde{\eta}(\xi) &= \eta(\xi)\zeta(\xi),\\ \widetilde{\Gamma}(\xi) &= \Gamma(\xi)\zeta(\xi), & \widetilde{\gamma}_0(\xi) &= \gamma_0(\xi)\zeta(\xi),\\ \widetilde{\gamma}(\xi) &= \gamma(\xi)\zeta(\xi), & \widetilde{\Gamma}_1(\xi) &= \Gamma_1(\xi)\zeta(\xi) \end{aligned}$$

for $\xi \in \widetilde{\Omega}$, $\widetilde{\Omega} \cap S = \emptyset$ and

$$\widetilde{u}(z) = \widehat{u}(z)\widehat{\zeta}(z), \qquad \widetilde{\eta}(z) = \widehat{\eta}(z)\widehat{\zeta}(z),$$
$$\widetilde{\Gamma}(z) = \widehat{\Gamma}(z)\widehat{\zeta}(z), \qquad \widetilde{\gamma}_0(z) = \widehat{\gamma}_0(z)\widehat{\zeta}(z),$$
$$\widetilde{\gamma}(z) = \widehat{\gamma}(z)\widehat{\zeta}(z), \qquad \widetilde{\Gamma}_1(z) = \widehat{\Gamma}_1(z)\widehat{\zeta}(z)$$

for $z \in \widehat{\Omega} = \Phi(\widetilde{\Omega})$, $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$.

Using the above notation we can rewrite problem (3.28) in the following form in an interior subdomain :

$$\begin{aligned}
&\eta\widetilde{u}_{it} - \nabla_{u_j} T_u^{ij}(\widetilde{u}, \widetilde{p}_\sigma) = \eta\widetilde{g}_i - \nabla_{u_j} B_u^{ij}(u,\zeta) - T_u^{ij}(u, p_\sigma)\nabla_{u_j}\zeta \\
&\qquad\qquad \equiv \eta\widetilde{g}_i + k_1, \qquad i = 1,2,3, \\
&\widetilde{\eta}_{\sigma t} + \eta\nabla_u \cdot \widetilde{u} = \eta u \cdot \nabla_u \zeta \equiv k_2, \\
&\eta c_v(\eta,\Gamma)\widetilde{\gamma}_t - \kappa\nabla_u^2\widetilde{\gamma} + \Gamma p_\Gamma(\eta,\Gamma)\nabla_u \cdot \widetilde{u} \\
&\qquad = \eta\widetilde{k} + \bigg[\frac{\mu}{2}\sum_{i,j=1}^{3}(\xi_{kx_i}\partial_{\xi_k}u_j + \xi_{kx_j}\partial_{\xi_k}u_i)^2 \\
&\qquad\quad + (\nu-\mu)(\nabla_u \cdot u)^2\bigg]\zeta + \Gamma p_\Gamma(\eta,\Gamma)u \cdot \nabla_u\zeta \\
&\qquad\quad - \kappa(\nabla_u^2\zeta\gamma + 2\nabla_u\zeta \cdot \nabla_u\gamma) - \eta c_v(\eta,\Gamma)\zeta\partial_t\theta_{\Omega_t} \\
&\qquad \equiv \eta\widetilde{k} + k_3,
\end{aligned}$$

where $\widetilde{p}_\sigma = p_\sigma\zeta$ and

$$\mathbb{B}_u(u,\zeta) = \{B_u^{ij}(u,\zeta)\} = \{\mu(u_i\nabla_{u_j}\zeta + u_j\nabla_{u_i}\zeta) + (\nu-\mu)\delta_{ij}u \cdot \nabla_u\zeta\}.$$

In boundary subdomains we have

$$\begin{aligned}
(3.30)\quad &\widehat{\eta}\widetilde{u}_{it} - \widehat{\nabla}_j\widehat{T}^{ij}(\widetilde{u}, \widetilde{p}_\sigma) = \widehat{\eta}\widetilde{g}_i - \widehat{\nabla}_j\widehat{B}^{ij}(\widehat{u}, \widehat{\zeta}) - \widehat{T}^{ij}(\widehat{u}, p_\sigma)\widehat{\nabla}_j\widehat{\zeta} \\
&\qquad \equiv \widehat{\eta}\widetilde{g}_i + k_4^i, \\
&\widetilde{\eta}_{\sigma t} + \widehat{\eta}\widehat{\nabla} \cdot \widetilde{u} = \widehat{\eta}\widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} \equiv k_5, \\
&\widehat{\eta}c_v(\widehat{\eta},\widehat{\Gamma})\widetilde{\gamma}_t - \kappa\widehat{\nabla}^2\widetilde{\gamma} + \widehat{\Gamma}p_{\widehat{\Gamma}}(\widehat{\eta},\widehat{\Gamma})\widehat{\nabla} \cdot \widetilde{u} \\
&= \widehat{\eta}\widetilde{k} + \bigg[\frac{\mu}{2}\sum_{i,j=1}^{3}(\widehat{\nabla}_i\widehat{u}_j + \widehat{\nabla}_j\widehat{u}_i)^2 + (\nu-\mu)(\widehat{\nabla} \cdot \widehat{u})^2\bigg]\widehat{\zeta} \\
&\quad + \widehat{\Gamma}p_{\widehat{\Gamma}}(\widehat{\eta},\widehat{\Gamma})\widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} - \kappa(\widehat{\nabla}^2\widehat{\zeta}\widehat{\gamma} + \widehat{\nabla}\widehat{\zeta} \cdot \widehat{\nabla}\widehat{\gamma}) \\
&\quad - \widehat{\eta}c_v(\widehat{\eta},\widehat{\Gamma})\partial_t\theta_{\Omega_t}\widehat{\zeta} \equiv \widehat{\eta}\widetilde{k} + k_6, \\
&\widehat{\mathbb{T}}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n} - \sigma\widehat{\Delta}_{\hat{S}}\widehat{\xi} \cdot \widehat{n}\widehat{n}\widehat{\zeta} - \sigma\widehat{\Delta}_{\hat{S}}\int_0^t \widetilde{u}\,dt' \cdot \widehat{n}\widehat{n} = \frac{2\sigma}{R_0}\widehat{\zeta}\widehat{n} + k_7 + k_8, \\
&\widehat{n} \cdot \widehat{\nabla}\widetilde{\gamma} = \widetilde{\Gamma}_1 + k_9,
\end{aligned}$$

where $k_7^i = \widehat{B}^{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j$, $k_8 = -\sigma(2\widehat{\nabla}\int_0^t \widehat{u}\,dt'\,\widehat{\nabla}\widehat{\zeta} + \int_0^t \widehat{u}\,dt'\,\widehat{\nabla}^2\widehat{\zeta})\cdot\widehat{n}\widehat{n}$, $k_9 = \widehat{n}\cdot\widehat{\nabla}\widehat{\zeta}\widehat{\gamma}$ and $\widehat{\mathbb{T}}$, $\widehat{\mathbb{B}}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$.

In the considerations below we denote z_1, z_2 by τ and z_3 by n.

Lemma 3.4. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of* (3.1). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_x^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_x^2 + \frac{\varrho c_v}{\theta}\vartheta_{0x}^2\right)dx \\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int_0^t v_{pp^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{pp^\beta}\,dt'\,ds \\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\left|\overline{n}\cdot\int_0^t v_{p^1p^2}\,dt'\right|^2 ds \\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\sum_{i=1}^2\left(\frac{1}{2}\overline{n}\cdot\int_0^t v_{p^ip^i}\,dt' + 2(H(\cdot,0)+2/R_{\rm e})\right)^2 ds \\
&\quad + c_0(\|v_x\|_{1,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{0,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{0,\Omega_t}^2 + \|\vartheta_{0xx}\|_{0,\Omega_t}^2) \\
&\le \varepsilon\bigg(\|v_{xt}\|_{0,\Omega_t}^2 + \|\vartheta_{0xt}\|_{0,\Omega_t}^2 + \bigg\|\int_0^t v\,dt'\bigg\|_{0,\Omega_t}^2 \\
&\quad + \|H(\cdot,0)+2/R_{\rm e}\|_{0,S^1}^2 + \|R(\cdot,t)-R(\cdot,0)\|_{2,S^1}^2\bigg) \\
&\quad + C_1(|v|_{1,0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2 \\
&\quad + \|\vartheta_{0t}\|_{0,\Omega_t}^2 + \|f\|_{1,\Omega_t}^2 + \|r\|_{1,\Omega_t}^2 + \|\theta_1\|_{2,\Omega_t}^2) \\
&\quad + C_2(X_3Y_3 + \|H(\cdot,0)+2/R_{\rm e}\|_{0,S^1}^4),
\end{aligned}$$

where the summation over the repeated indices $(\alpha,\beta = 1,2)$ *and coordinates* $(x, p = (p^1,p^2))$ *is assumed*, $\widetilde{\delta}^{\alpha\beta}$ *on each boundary part* $\Sigma_t = S_t\cap\{\zeta(x)\ne 0\}$ (ζ *belongs to a partition of unity of* Ω_t) *is of the form* $\widetilde{\delta}^{\alpha\beta} = \delta^{\alpha\beta} + 2\varepsilon^{\alpha\beta}$, $\varepsilon^{\alpha\beta} = -\overline{F}_{p^\alpha}\overline{F}_{p^\beta}(1+\overline{F}_{p^1}^2+\overline{F}_{p^2}^2)^{-1}$, $\overline{F}$ *is the function such that in the local coordinates* $\{y\}$, $\sum_t$ *is described by the formula*

$$y_i = p^i \quad (i=1,2), \qquad y_3 = \overline{F}(p^1,p^2,t) \tag{3.31}$$

and $\operatorname{supp}\zeta$ *is so small that* $|\overline{F}_p| \le 1/2$. *Moreover*,

$$X_3 = |v|_{2,1,\Omega_t}^2 + |\varrho_\sigma|_{2,1,\Omega_t}^2 + |\vartheta_0|_{2,1,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2,$$

$$Y_3 = X_3 + \|v\|_{3,\Omega_t}^2 + \|\vartheta_{0x}\|_{2,\Omega_t}^2 + \|\vartheta\|_{0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \int_0^t \|v\|_{3,\Omega_{t'}}^2\,dt'.$$

P r o o f. Similarly to [16] (see the proof of Lemma 3.4) we obtain the following estimate for interior subdomains:

$$
\begin{aligned}
(3.32)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\bigg(\eta\widetilde{u}_\xi^2+\frac{p_{\sigma\eta}}{\eta}\widetilde{\overline{\eta}}_{\Omega_t\xi}^2+\frac{\eta c_v}{\varGamma}\widetilde{\gamma}_\xi^2\bigg)A\,d\xi\\
&+\frac{\mu}{2}\|\widetilde{u}_\xi\|_{1,\widetilde{\Omega}}^2+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\xi\xi}\|_{0,\widetilde{\Omega}}^2+\|\widetilde{\overline{\eta}}_{\Omega_t}\|_{1,\widetilde{\Omega}}^2\\
&\le\varepsilon(\|\widetilde{u}_{\xi\xi}\|_{0,\widetilde{\Omega}}^2+\|\eta_{\sigma\xi}\|_{0,\widetilde{\Omega}}^2+\|\widetilde{\gamma}_{\xi\xi}\|_{0,\widetilde{\Omega}}^2)\\
&\quad+C_1(|u|_{1,0,\widetilde{\Omega}}^2+\|v\|_{1,\Omega_t}^2+\|\gamma_{0\xi}\|_{0,\widetilde{\Omega}}^2+\|\gamma\|_{0,\widetilde{\Omega}}^2\\
&\quad+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|\overline{\eta}_{\Omega_t}\|_{0,\widetilde{\Omega}}^2+\|\widetilde{g}\|_{0,\widetilde{\Omega}}^2+\|\widetilde{k}\|_{0,\widetilde{\Omega}}^2)\\
&\quad+C_2\Big[\Big(X_3(\widetilde{\Omega})+\int\limits_0^t\|u\|_{3,\widetilde{\Omega}}^2\,dt'\Big)Y_3(\widetilde{\Omega})+\|\gamma\|_{2,\widetilde{\Omega}}^2(\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
X_3(\widetilde{\Omega})&=|u|_{2,1,\widetilde{\Omega}}^2+|\varrho_\sigma|_{2,1,\widetilde{\Omega}}^2+|\gamma_0|_{2,1,\widetilde{\Omega}}^2+\|\overline{\eta}_{\Omega_t}\|_{0,\widetilde{\Omega}}^2,\\
Y_3(\widetilde{\Omega})&=X_3(\widetilde{\Omega})+\|u\|_{3,\widetilde{\Omega}}^2+\|\gamma\|_{3,\widetilde{\Omega}}^2+\|\overline{\eta}_{\Omega_t}\|_{0,\widetilde{\Omega}}^2+\int\limits_0^t\|u\|_{3,\widetilde{\Omega}}^2\,dt'.
\end{aligned}
$$

Now, we consider subdomains near the boundary. Differentiate $(3.30)_1$ with respect to τ, multiply the result by $\widetilde{u}_\tau J$ and integrate over $\widehat{\Omega}$ (J is the Jacobian of the transformation $x=x(z)$). Next, divide $(3.30)_3$ by $\widehat{\varGamma}$, differentiate the result with respect to τ, multiply by $\widetilde{\gamma}_\tau J$ and integrate over $\widehat{\Omega}$. Hence using Lemma 5.1 of [18] we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\bigg(\widehat{\eta}\widetilde{u}_\tau^2+\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t\tau}^2+\frac{\widehat{\eta}c_v}{\widehat{\varGamma}}\widetilde{\gamma}_\tau^2\bigg)J\,dz+\frac{\mu}{2}\|\widetilde{u}_\tau\|_{1,\widehat{\Omega}}^2\\
&\quad+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\tau z}\|_{0,\widehat{\Omega}}^2-\int\limits_{\hat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma))_{,\tau}\widetilde{u}_\tau J\,dz'-\kappa\int\limits_{\hat{S}}\bigg(\widehat{n}\frac{1}{\widehat{\varGamma}}\widehat{\nabla}\widetilde{\gamma}\bigg)_{,\tau}\widetilde{\gamma}_\tau J\,dz'\\
&\le\varepsilon(\|\widetilde{u}_{zz}\|_{0,\widehat{\Omega}}^2+\|\widehat{\eta}_{\sigma z}\|_{0,\widehat{\Omega}}^2+\|\widehat{\gamma}_{0zz}\|_{0,\widehat{\Omega}}^2)\\
&\quad+C_1(|\widehat{u}|_{1,0,\widehat{\Omega}}^2+\|v\|_{1,\Omega_t}^2+\|\widehat{\gamma}_{0\tau}\|_{0,\widehat{\Omega}}^2\\
&\quad+\|\widehat{\gamma}\|_{0,\widehat{\Omega}}^2+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|\widehat{\overline{\eta}}_{\Omega_t}\|_{0,\widehat{\Omega}}^2+\|\widetilde{g}\|_{1,\widehat{\Omega}}^2+\|\widetilde{k}\|_{1,\widehat{\Omega}}^2)\\
&\quad+C_2\Big[\Big(X_2(\widehat{\Omega})+\int\limits_0^t\|\widehat{u}\|_{3,\widehat{\Omega}}^2\,dt'\Big)Y_2(\widehat{\Omega})+\|\widehat{\gamma}\|_{2,\widehat{\Omega}}^2(\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2)\Big],
\end{aligned}
$$

where $X_2(\widehat{\Omega})$ and $Y_2(\widehat{\Omega})$ are defined analogously to $X_2(\widetilde{\Omega})$ and $Y_2(\widetilde{\Omega})$.

Using the boundary condition $(3.30)_4$ we have

$$
\begin{aligned}
(3.33)\qquad & -\int_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma))_{,\tau}\widetilde{u}_\tau J\,dz' \\
&\le -\frac{\sigma}{2}\frac{d}{dt}\int_{\widehat{S}} g^{\alpha\beta}\widehat{n}\cdot\int_0^t \widetilde{u}_{pp^\alpha}\,dt'\,\widehat{n}\cdot\int_0^t\widetilde{u}_{pp^\beta}\,dt'\,J\,dz' \\
&\quad -\sigma\int_{\widehat{S}}(\widehat{H}(\cdot,0)+2/R_{\rm e})\widehat{\zeta}\cdot\widetilde{u}_{pp}\cdot\widehat{n}J\,dz' \\
&\quad +\varepsilon\Big(\Big\|\int_0^t\widetilde{u}\,dt'\Big\|^2_{2,\widehat{S}}+\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}}+\|(\widehat{H}(\cdot,0)+2/R_{\rm e})\widehat{\zeta}\|^2_{0,\widehat{S}} \\
&\quad +\|R(\cdot,t)-R(\cdot,0)\|^2_{2,S^1}\Big) \\
&\quad +C_2\Big(\|\widehat{u}\|^2_{0,\widehat{\Omega}}+\|\widehat{u}\|^2_{2,\widehat{\Omega}}\Big\|\int_0^t\widetilde{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}\Big).
\end{aligned}
$$

By the boundary condition $(3.30)_5$ we get

$$
\begin{aligned}
(3.34)\qquad & -\kappa\int_{\widehat{S}}\bigg(\widehat{n}\cdot\frac{1}{\widehat{\Gamma}}\widehat{\nabla}\widehat{\gamma}\bigg)_{,\tau}\widetilde{\gamma}_\tau J\,dz' \\
&\le \varepsilon\|\widehat{\gamma}_{0zz}\|^2_{0,\widehat{\Omega}}+C_1(\|\widehat{\gamma}\|^2_{0,\widehat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{0,\widehat{\Omega}}+\|\widetilde{\Gamma}_1\|^2_{2,\widehat{\Omega}}) \\
&\quad +C_2\|\widehat{\gamma}\|^2_{2,\widehat{\Omega}}\Big(\|\widehat{\gamma}_0\|^2_{2,\widehat{\Omega}}+\|\widehat{\gamma}\|^2_{2,\widehat{\Omega}}+\|\widehat{\eta}_\sigma\|^2_{2,\widehat{\Omega}}+\Big\|\int_0^t\widehat{u}\,dt'\Big\|^2_{3,\widehat{\Omega}}\Big).
\end{aligned}
$$

To obtain (3.33) and (3.34) we have applied the interpolation inequality (2.2) (see Lemma 2.1).

For the quantities

$$
\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}^2_{\Omega_t n}J\,dz+c_0\|\widetilde{\overline{\eta}}_{\Omega_t n}\|^2_{0,\widehat{\Omega}},
$$

$$
\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\widehat{\eta}\widetilde{u}^2_{3n}Jdz+c_0\|\widetilde{u}_{3nn}\|^2_{0,\widehat{\Omega}},
$$

$$
\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{0,\Omega_t},\quad \|\widetilde{u}'_{z\tau}\|^2_{0,\widehat{\Omega}},\quad \|\widetilde{\overline{\eta}}_{\Omega_t\tau}\|^2_{0,\widehat{\Omega}},\quad \|\widetilde{u}'_{nn}\|^2_{0,\widehat{\Omega}},\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\widehat{\eta}\widetilde{u}^2_n J\,dz,
$$

$$
\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}^2_n Jdz+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{nn}\|^2_{0,\widehat{\Omega}}
$$

we obtain the same estimates as in the proof of Lemma 3.4 of [16]. Therefore, we have

$$
\begin{aligned}
(3.35)\qquad & \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\Big(\widehat{\eta}\widetilde{u}_z^2+\frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t z}^2+\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_z^2\Big)J\,dz\\
&+\frac{\sigma}{2}\frac{d}{dt}\int_{\hat{S}}\Big[g^{\alpha\beta}\widehat{n}\cdot\int_0^t\widetilde{u}_{pp^\alpha}\,dt'\,\widehat{n}\cdot\int_0^t\widetilde{u}_{pp^\beta}\,dt'\\
&+2(\widehat{H}(\cdot,0)+2/R_{\rm e})\widehat{\zeta}\widehat{n}\cdot\int_0^t\widetilde{u}_{pp}\,dt'\Big]J\,dz'+\frac{\mu}{2}\|\widetilde{u}_z\|^2_{1,\widehat{\Omega}}\\
&+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{zz}\|^2_{0,\widehat{\Omega}}+c_0\|\widetilde{\overline{\eta}}_{\Omega_t}\|^2_{1,\widehat{\Omega}}\\
&\le\varepsilon\Big(\|\widetilde{u}_{zz}\|^2_{0,\widehat{\Omega}}+\|\widehat{\eta}_{\sigma z}\|^2_{0,\widehat{\Omega}}+\|\widehat{\gamma}_{0zz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{u}_{zt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{\gamma}_{0zt}\|^2_{0,\widehat{\Omega}}\\
&+\Big\|\int_0^t\widetilde{u}\,dt'\Big\|^2_{2,\widehat{S}}+\|(\widehat{H}(\cdot,0)+2/R_{\rm e})\widehat{\zeta}\|^2_{0,\widehat{S}}+\|R(\cdot,t)-R(\cdot,0)\|^2_{2,S^1}\Big)\\
&+C_1(|\widehat{u}|^2_{1,0,\widehat{\Omega}}+\|v\|^2_{1,\Omega_t}+\|\widehat{\gamma}_{0\tau}\|^2_{0,\widehat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\widehat{\Omega}}\\
&+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\widehat{\Omega}}+\|\widetilde{g}\|^2_{1,\widehat{\Omega}}+\|\widetilde{k}\|^2_{1,\widehat{\Omega}})\\
&+C_2\Big[\Big(X_2(\widehat{\Omega})+\int_0^t\|\widehat{u}\|^2_{3,\widehat{\Omega}}\,dt'\Big)Y_2(\widehat{\Omega})+\|\widehat{\gamma}\|^2_{2,\widehat{\Omega}}(\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})\Big].
\end{aligned}
$$

We estimate the second term on the left-hand side of (3.35) in the same way as in the proof of Lemma 4.4 of [19]. Going back to the variables ξ in (3.35), next from the resulting estimate and (3.32), after summing over all neighbourhoods of the partition of unity and finally going back to the variables x and using (3.26) we get

$$
\begin{aligned}
(3.36)\qquad & \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v_x^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma x}^2+\frac{\varrho c_v}{\theta}\vartheta_{0x}^2\Big)dx\\
&\qquad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int_0^t v_{pp^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{pp^\beta}\,dt'\,ds\\
&\qquad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\Big|\overline{n}\cdot\int_0^t v_{p^1p^2}\,dt'\Big|^2\,ds\\
&\qquad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\sum_{i=1}^2\Big(\frac{1}{2}\overline{n}\cdot\int_0^t v_{p^ip^i}\,dt'+2(\widehat{H}(\cdot,0)+2/R_{\rm e})\Big)^2\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ c_0(\|v_x\|_{1,\Omega_t}^2 + \|\vartheta_{0xx}\|_{0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{0,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{0,\Omega_t}^2) \\
&\le \varepsilon\Big(\|v_{xt}\|_{0,\Omega_t}^2 + \|\vartheta_{0xt}\|_{0,\Omega_t}^2 + \Big\|\int_0^t v\,dt'\Big\|_{2,S_t}^2 \\
&+ \|H(\cdot,0) + 2/R_{\rm e}\|_{0,S_t}^2 + \|R(\cdot,t) - R(\cdot,0)\|_{2,S^1}^2\Big) \\
&+ C_1(|v|_{1,0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2 \\
&+ \|\vartheta_{0t}\|_{0,\Omega_t}^2 + \|f\|_{1,\Omega_t}^2 + \|r\|_{1,\Omega_t}^2 + \|\theta_1\|_{2,\Omega_t}^2) \\
&+ C_2X_3Y_3 + 4\sigma\frac{d}{dt}\int_{S_t}(H(\cdot,0) + 2/R_{\rm e})^2\,ds.
\end{aligned}
$$

In virtue of the interpolation inequality (2.2) we have

$$
\begin{aligned}
(3.37)\qquad &\Big|\frac{d}{dt}\int_{S_t}(H(\cdot,0) + 2/R_{\rm e})^2\,ds\Big| \\
&\le \varepsilon\|v_{xx}\|_{0,\Omega_t}^2 + C_1\|v\|_{0,\Omega_t}^2 + C_2\|H(\cdot,0) + 2/R_{\rm e}\|_{0,S^1}^4.
\end{aligned}
$$

Writing the boundary condition $(3.1)_4$ locally we obtain

$$
(3.38)\qquad \sigma\widehat{\Delta}_{\widehat{S}}\int_0^t\widetilde{u}\,dt' = -\sigma\Big(\widehat{\Delta}_{\widehat{S}}\widehat{\xi} + \frac{2}{R_{\rm e}}\widehat{n}\Big)\widehat{\zeta} - \widehat{\mathbb{T}}_u(\widetilde{u},\widetilde{p}_\sigma)\widehat{n} + I_1 + I_2,
$$

where

$$
I_1^i = -\widehat{B}^{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j, \qquad I_2 = \sigma\Big(2\widehat{\nabla}\int_0^t\widehat{u}\,dt'\,\widehat{\nabla}\widehat{\zeta} + \int_0^t\widehat{u}\,dt'\,\widehat{\nabla}^2\widehat{\zeta}\Big).
$$

Multiply (3.38) by $\int_0^t\widetilde{u}\,dt'$, next differentiate with respect to τ and multiply by $\int_0^t\widetilde{u}_\tau\,dt'$. Integrating the result over $\widehat{S}$ and summing over all neighbourhoods of the partition of unity we get

$$
\begin{aligned}
(3.39)\qquad &\Big\|\int_0^t v\,dt'\Big\|_{2,S_t}^2 \\
&\le \varepsilon(\|v\|_{2,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2) \\
&\quad + C_1\Big(\|v\|_{0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta\|_{0,\Omega_t}^2 \\
&\quad + \Big\|\int_0^t v\,dt'\Big\|_{0,\Omega_t}^2 + \|H(\cdot,0) + 2/R_{\rm e}\|_{0,S^1}^2 + \|R(\cdot,t) - R(\cdot,0)\|_{2,S^1}^2\Big) \\
&\quad + C_2(\|v\|_{2,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{1,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \|\vartheta_0\|_{2,\Omega_t}^2)\int_0^t\|v\|_{3,\Omega_{t'}}^2\,dt'.
\end{aligned}
$$

From (3.36), (3.37) and (3.39) we obtain (3.31). ■

Now, we formulate Lemmas 3.5–3.7, the proofs of which are similar to the proofs of Lemmas 3.5–3.7 of [16]. The boundary terms associated with the boundary condition $(3.1)_4$ are estimated in the same way as in Lemmas 4.5–4.7 of [19] and similarly to Lemmas 4.1, 4.2 and 4.4.

Lemma 3.5. *Let v,ϱ,ϑ_0 be a sufficiently smooth solution of problem* (3.1). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v_{xx}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xx}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xx}^2\Big)\,dx\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int_0^t v_{p^\gamma p^\delta p^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{p^\gamma p^\delta p^\beta}\,dt'\,ds\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\Big|\overline{n}\cdot\int_0^t v_{p^1p^2p}\,dt'\Big|^2\,ds\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\sum_{i=1}^2\Big(\frac{1}{2}\overline{n}\cdot\int_0^t v_{pp^ip^i}\,dt'+2(H(\cdot,0)+2/R_{\rm e})_{,p}\Big)^2\,ds\\
&\quad+c_0(\|v_{xx}\|_{1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{1,\Omega_t}^2+\|\vartheta_{0xxx}\|_{0,\Omega_t}^2)\\
&\le\varepsilon(\|v_{xxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxt}\|_{0,\Omega_t}^2+\|H(\cdot,0)+2/R_{\rm e}\|_{1,S^1}^2\\
&\quad+\|R(\cdot,t)-R(\cdot,0)\|_{3,S^1}^2)\\
&\quad+C_1(|v|_{2,1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{1,\Omega_t}^2\\
&\quad+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\|f\|_{1,\Omega_t}^2+\|r\|_{1,\Omega_t}^2+\|\theta_1\|_{3,\Omega_t}^2)\\
&\quad+C_2[X_4(1+X_4)Y_4+\|H(\cdot,0)+2/R_{\rm e}\|_{1,S^1}^4],
\end{aligned}$$

where the summation over the repeated indices $(\alpha,\beta,\gamma,\delta)$ and coordinates x, $p=(p^1,p^2)$ is assumed and

$$\begin{aligned}
X_4&=\|v\|_{3,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+|\vartheta_0|_{2,1,\Omega_t}^2+\|\vartheta_0\|_{3,\Omega_t}^2\\
&\quad+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\int_0^t\|v\|_{3,\Omega_{t'}}^2\,dt',\\
Y_4&=\|v\|_{4,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma x}\|_{2,\Omega_t}^2+|\varrho_\sigma|_{2,1,\Omega_t}^2+\|\vartheta_{0x}\|_{3,\Omega_t}^2\\
&\quad+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'. \ \blacksquare
\end{aligned}$$

Lemma 3.6. *Let v,ϱ,ϑ_0 be a sufficiently smooth solution of problem* (3.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v_{xt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xt}^2\Big)\,dx+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}g^{\alpha\beta}\overline{n}\cdot v_{pp^\alpha}\overline{n}\cdot v_{pp^\beta}\,ds$$

$$
\begin{aligned}
&+ c_0(\|v_t\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t} + \|\vartheta_{0xxt}\|^2_{0,\Omega_t})\\
&\le \varepsilon(\|v_{xtt}\|^2_{0,\Omega_t} + \|\vartheta_{0xtt}\|^2_{0,\Omega_t} + \|H(\cdot,0) + 2/R_{\rm e}\|^2_{0,S^1})\\
&\quad + C_1(|v|^2_{2,0,\Omega_t} + \|\varrho_{\sigma x}\|^2_{0,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}\\
&\quad + \|\vartheta_{0x}\|^2_{1,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t} + |f|^2_{1,0,\Omega_t} + |r|^2_{1,0,\Omega_t}\\
&\quad + \|\theta_{1t}\|^2_{2,\Omega_t} + \|\theta_1\|^2_{1,\Omega_t}) + C_2X_5(1+X_5)Y_5,
\end{aligned}
$$

where to describe S_t *we have used formula* (3.31) *and*

$$
\begin{aligned}
X_5 &= |v|^2_{3,1,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t} + |\vartheta_0|^2_{3,1,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt',\\
Y_5 &= |v|^2_{4,2,\Omega_t} + |\varrho_\sigma|^2_{3,1,\Omega_t} + |\vartheta_{0t}|^2_{3,2,\Omega_t} + \|\vartheta_{0x}\|^2_{3,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t}\\
&\quad + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \int_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'.
\end{aligned}
$$

Lemma 3.7. *Let* v, ϱ, ϑ_0 *be a sufficiently smooth solution of problem* (3.1). *Then*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_{tt}^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma tt} + \frac{\varrho c_v}{\theta}\vartheta^2_{0tt}\right)dx + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t} g^{\alpha\beta}\overline{n}\cdot v_{s^\alpha t}\overline{n}\cdot v_{s^\beta t}\,ds\\
&\quad + c_0(\|v_{tt}\|^2_{1,\Omega_t} + +\|\varrho_{\sigma tt}\|^2_{0,\Omega_t} + \|\vartheta_{0tt}\|^2_{1,\Omega_t})\\
&\le \varepsilon\|H(\cdot,0) + 2/R_{\rm e}\|^2_{0,S^1} + C_1(\|v_t\|^2_{1,\Omega_t} + \|\vartheta_{0t}\|^2_{1,\Omega_t} + |f|^2_{1,0,\Omega_t} + \|f_{tt}\|^2_{0,\Omega_t}\\
&\quad + |r|^2_{1,0,\Omega_t} + \|r_{tt}\|^2_{0,\Omega_t} + \|r\|_{0,\Omega_t} + |\theta_1|^2_{3,1,\Omega_t}) + +C_2X_6(1+X_6)Y_6,
\end{aligned}
$$

where

$$
\begin{aligned}
X_6 &= |v|^2_{3,1,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t} + |\vartheta_0|^2_{3,1,\Omega_t},\\
Y_6 &= |v|^2_{4,2,\Omega_t} + |\varrho_\sigma|^2_{3,1,\Omega_t} + |\vartheta_0|^2_{4,2,\Omega_t}. \ \blacksquare
\end{aligned}
$$

Summarizing, from Lemmas 3.5–3.7 we obtain

Lemma 3.8. *Let* v, ϱ, ϑ_0 *be a sufficiently smooth solution of problem* (3.1). *Then*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho|D^2_{x,t}v|^2 + \frac{p_{\sigma\varrho}}{\varrho}|D^2_{x,t}\varrho_\sigma|^2 + \frac{\varrho c_v}{\theta}|D^2_{x,t}\vartheta_0|^2\right)dx\\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int_0^t v_{p^\gamma p^\delta p^\alpha}\,dt'\,\overline{n}\cdot\int_0^t v_{p^\gamma p^\delta p^\beta}\,dt'\,ds\\
&\quad + \frac{\sigma}{2}\frac{d}{dt}\int_{S_t}\left|\overline{n}\cdot\int_0^t v_{pp^1p^2}\,dt'\right|^2 ds
\end{aligned}
$$

$$+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\sum_{i=1}^{2}\bigg(\frac{1}{2}\overline{n}\cdot\int\limits_0^t v_{p^1p^2p}\,dt'+2(H(\cdot,0)+2/R_{\rm e})_{,p}\bigg)^2 ds$$
$$+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t} g^{\alpha\beta}\overline{n}\cdot v_{pp^\alpha}\overline{n}\cdot v_{pp^\beta}\,ds+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t} g^{\alpha\beta}\overline{n}\cdot v_{s^\alpha t}\overline{n}\cdot v_{s^\beta t}\,ds$$
$$+c_0(|v|^2_{3,1,\Omega_t}+|\varrho_{\sigma t}|^2_{1,0,\Omega_t}+\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+|\vartheta_{0t}|^2_{2,1,\Omega_t}+\|\vartheta_{0xxx}\|^2_{0,\Omega_t})$$
$$\le\varepsilon(\|H(\cdot,0)+2/R_{\rm e}\|^2_{1,S^1}+\|R(\cdot,t)-R(\cdot,0)\|^2_{3,S^1})$$
$$+C_1(|v|^2_{2,0,\Omega_t}+\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}$$
$$+\|\vartheta_{0x}\|^2_{1,\Omega_t}+\|\vartheta_{0t}\|^2_{1,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+|f|^2_{1,0,\Omega_t}+\|f_{tt}\|^2_{0,\Omega_t}$$
$$+|r|^2_{1,0,\Omega_t}+\|r_{tt}\|^2_{0,\Omega_t}+|\theta_1|^2_{3,1,\Omega_t})$$
$$+C_2[X_7(1+X_7)Y_7+\|H(\cdot,0)+2/R_{\rm e}\|^4_{1,S_1}],$$

where

$$X_7=|v|^2_{3,1,\Omega_t}+|\varrho_\sigma|^2_{2,0,\Omega_t}+|\vartheta_0|^2_{3,1,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{3,\Omega_{t'}}\,dt',$$
$$Y_7=|v|^2_{4,2,\Omega_t}+|\varrho_\sigma|^2_{3,1,\Omega_t}+|\vartheta_{0t}|^2_{3,2,\Omega_t}+\|\vartheta_{0x}\|^2_{3,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}$$
$$+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'.\ \blacksquare$$

LEMMA 3.9. *Let v,ϱ,ϑ_0 be a sufficiently smooth solution of problem* (3.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\bigg(\varrho v^2_{xxx}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xxx}+\frac{\varrho c_v}{\theta}\vartheta^2_{0xxx}\bigg)dx \tag{3.40}$$
$$+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int\limits_0^t v_{p^\gamma p^\delta p^\eta p^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t v_{p^\gamma p^\delta p^\eta p^\beta}\,dt'\,ds$$
$$+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\bigg|\overline{n}\cdot\int\limits_0^t v_{ppp^1p^2}\,dt'\bigg|^2 ds$$
$$+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\sum_{i=1}^{2}\bigg(\frac{1}{2}\overline{n}\cdot\int\limits_0^t v_{ppp^ip^i}\,dt'+2(H(\cdot,0)+2/R_{\rm e})_{,pp}\bigg)^2 ds$$
$$+c_0(\|v_{xxx}\|^2_{1,\Omega_t}+\|\varrho_{\sigma xxx}\|^2_{0,\Omega_t}+\|\vartheta_{0xxxx}\|^2_{0,\Omega_t})$$
$$\le\varepsilon(\|v_{xxxt}\|^2_{0,\Omega_t}+\|\vartheta_{0xxxt}\|^2_{0,\Omega_t}+\|H(\cdot,0)+2/R_0\|^2_{2,S^1}$$
$$+\|R(\cdot,t)-R(\cdot,0)\|^2_{4,S^1})+C_1\Big(|v|^2_{3,2,\Omega_t}++\|\varrho_{\sigma x}\|^2_{1,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}$$
$$+\|\vartheta_{0x}\|^2_{2,\Omega_t}+\|\vartheta_{0t}\|^2_{2,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+\|f\|^2_{2,\Omega_t}+\|r\|^2_{2,\Omega_t}$$

$$+ \|\theta_1\|^2_{4,\Omega_t} + \Big\|\int\limits_0^t v\,dt'\Big\|^2_{0,\Omega_t} + \|R(\cdot,t) - R(\cdot,0)\|^2_{0,S^1}\Big)$$

$$+ C_2\Big[\|H(\cdot,0) + 2/R_0\|^4_{2,S_1} + \|R(\cdot,t) - R(\cdot,0)\|^2_{3,S^1}\Big\|\int\limits_0^t v\,dt'\Big\|^2_{4,S_t}$$

$$+ \|R(\cdot,t) - R(\cdot,0)\|^2_{4+1/2,S^1}\Big\|\int\limits_0^t v\,dt'\Big\|^2_{3,S_t} + X_8(1 + X_8^2)Y_8\Big],$$

where the summation over repeated indices $(\alpha,\beta,\gamma,\delta = 1,2)$ *and coordinates* $(x, p = (p^1,p^2),\ \ i = 1,2)$ *is assumed and*

$$X_8 = |v|^2_{3,2,\Omega_t} + |\varrho_\sigma|^2_{3,2,\Omega_t} + |\vartheta_0|^2_{3,2,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \int\limits_0^t \|v\|^2_{3,\Omega_{t'}}\,dt',$$

$$Y_8 = |v|^2_{4,3,\Omega_t} + |\varrho_\sigma|^2_{3,2,\Omega_t} + \|\vartheta_{0x}\|^2_{3,\Omega_t} + \|\vartheta_{0t}\|^2_{3,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t}$$

$$+ \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \int\limits_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'.$$

Proof. For interior subdomains we obtain the estimate (see [16], proof of Lemma 3.9)

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde\Omega}\Big(\eta\widetilde{u}^2_{\xi\xi\xi} + \frac{p_{\sigma\eta}}{\eta}\widetilde{\overline{\eta}}^2_{\Omega_t\xi\xi\xi} + \frac{\eta c_v}{\Gamma}\widetilde{\gamma}^2_{\xi\xi\xi}\Big)A\,d\xi + \frac{\mu}{2}\|\widetilde{u}_{\xi\xi\xi}\|^2_{1,\widetilde\Omega} \tag{3.41}$$

$$+ \|\widetilde{\overline{\eta}}_{\Omega_t\xi\xi\xi}\|^2_{0,\widetilde\Omega} + \frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\xi\xi\xi\xi}\|^2_{0,\widetilde\Omega}$$

$$\le \varepsilon(\|\widetilde{u}_{\xi\xi\xi\xi}\|^2_{0,\widetilde\Omega} + \|\widetilde{\gamma}_{\xi\xi\xi\xi}\|^2 + \|\widetilde{\eta}_{\sigma\xi\xi\xi}\|^2_{0,\widetilde\Omega})$$

$$+ C_1(|u|^2_{3,2,\widetilde\Omega} + \|\gamma_{0\xi}\|^2_{2,\widetilde\Omega} + \|\gamma\|^2_{0,\widetilde\Omega} + \|\eta_{\sigma\xi}\|^2_{1,\widetilde\Omega}$$

$$+ \|\overline{\eta}_{\Omega_t}\|^2_{0,\widetilde\Omega} + \|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t} + \|\widetilde{g}\|^2_{2,\widetilde\Omega} + \|\widetilde{k}\|^2_{2,\widetilde\Omega})$$

$$+ C_2\Big[\Big(X_8(\widetilde\Omega) + \int\limits_0^t \|u\|^2_{3,\widetilde\Omega}\,dt'\Big)(1 + X_8^2(\widetilde\Omega))Y_8(\widetilde\Omega)$$

$$+ \|\gamma\|^2_{4,\widetilde\Omega}(\|\vartheta_{0t}\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t})\Big],$$

where

$$X_8(\widetilde\Omega) = |u|^2_{3,2,\widetilde\Omega} + |\overline{\eta}_{\Omega_t}|^2_{3,2,\widetilde\Omega} + |\gamma|^2_{3,2,\widetilde\Omega} + |\eta_\sigma|^2_{3,2,\widetilde\Omega} + |\gamma_0|^2_{3,2,\widetilde\Omega},$$

$$Y_8(\widetilde\Omega) = |u|^2_{4,3,\widetilde\Omega} + |\overline{\eta}_{\Omega_t}|^2_{3,2,\widetilde\Omega} + |\gamma|^2_{4,3,\widetilde\Omega} + \|\eta_{\sigma t}\|^2_{2,\widetilde\Omega}$$

$$+ \|\gamma_{0t}\|^2_{3,\widetilde\Omega} + \int\limits_0^t \|u\|^2_{4,\widetilde\Omega}\,dt.$$

For boundary subdomains we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\bigg(\widehat{\eta}\widetilde{u}_{\tau\tau\tau}^2+\frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}}\widetilde{\overline{\eta}}_{\Omega_t\tau\tau\tau}^2+\frac{\widehat{\eta}c_v}{\widehat{\varGamma}}\widetilde{\gamma}_{\tau\tau\tau}^2\bigg)J\,dz\\
&\qquad+\frac{\mu}{2}\|\widetilde{u}_{\tau\tau\tau}\|_{1,\widehat{\Omega}}^2+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{\tau\tau\tau z}\|_{0,\widehat{\Omega}}^2\\
&\qquad-\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma))_{,\tau\tau\tau}\widetilde{u}_{\tau\tau\tau}J\,dz'-\kappa\int\limits_{\widehat{S}}\bigg(\widehat{n}\frac{1}{\widehat{\varGamma}}\widehat{\nabla}\widetilde{\gamma}\bigg)_{,\tau\tau\tau}\widetilde{\gamma}_{\tau\tau\tau}J\,dz'\\
&\quad\le\varepsilon(\|\widehat{u}_{zzzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{\gamma}_{0zzzz}\|_{0,\widehat{\Omega}}^2+\|\widehat{\eta}_{\sigma zzz}\|_{0,\widehat{\Omega}}^2)\\
&\qquad+C_1(|\widehat{u}|_{3,2,\widehat{\Omega}}^2+\|\widehat{\gamma}_{0z}\|_{2,\widehat{\Omega}}^2+\|\widehat{\gamma}\|_{0,\widehat{\Omega}}^2+\|\widehat{\eta}_{\sigma z}\|_{1,\widehat{\Omega}}^2+\|\overline{\eta}_{\Omega_t}\|_{0,\widehat{\Omega}}^2\\
&\qquad+\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2+\|\widetilde{g}\|_{2,\widehat{\Omega}}^2+\|\widetilde{k}\|_{2,\widehat{\Omega}}^2+\|\widetilde{\varGamma}_1\|_{4,\widehat{\Omega}}^2)\\
&\qquad+C_2\bigg[\bigg(X_8(\widehat{\Omega})+\int_0^t\|\widehat{u}\|_{3,\widehat{\Omega}}^2\,dt'\bigg)(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega})\\
&\qquad+\|\widehat{\gamma}\|_{4,\widehat{\Omega}}^2(\|\vartheta_{0t}\|_{0,\Omega_t}^2+\|v\|_{1,\Omega_t}^2)\bigg],
\end{aligned}
$$

where $X_8(\widehat{\Omega})$ and $Y_8(\widehat{\Omega})$ are defined analogously to $X_8(\widetilde{\Omega})$ and $Y_8(\widetilde{\Omega})$. The boundary conditions $(3.30)_4$ and $(3.30)_5$ yield

$$
\begin{aligned}
&-\int\limits_{\widehat{S}}(\widehat{n}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma))_{,\tau\tau\tau}\widetilde{u}_{\tau\tau\tau}J\,dz'\\
&\quad=-\sigma\int\limits_{\widehat{S}}(\widehat{\Delta}_{\widehat{S}}\widehat{\xi}\cdot\widehat{n}\widehat{n}\widehat{\zeta}+(2/R_{\mathrm{e}})\widehat{n}\widehat{\zeta})_{,\tau\tau\tau}\widetilde{u}_{\tau\tau\tau}\,dz'\\
&\qquad-\sigma\int\limits_{\widehat{S}}\bigg(\widehat{\Delta}_{\widehat{S}}\int_0^t\widetilde{u}\,d\tau\cdot\widehat{n}\widehat{n}\bigg)_{,\tau\tau\tau}\widetilde{u}_{\tau\tau\tau}J\,dz'+\int\limits_{\widehat{S}}(k_7+k_8)_{,\tau\tau\tau}\widetilde{u}_{\tau\tau\tau}J\,dz'\\
&\quad\le-\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}g^{\alpha\beta}\overline{n}\cdot\int_0^t\widetilde{u}_{\tau\tau\tau p^\alpha}\,dt'\,\overline{n}\cdot\int_0^t\widetilde{u}_{\tau\tau\tau p^\beta}\,dt'J\,dz'\\
&\qquad-\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}(\widehat{H}(\cdot,0)+2/R_{\mathrm{e}})_{,\tau\tau}\widehat{\zeta}\widehat{n}\cdot\int_0^t\widetilde{u}_{\tau\tau\tau_i\tau_i}\,dt'J\,dz'\\
&\qquad+\varepsilon\bigg(\|\widetilde{u}_{\tau\tau\tau}\|_{1,\widehat{\Omega}}^2+\bigg\|\int_0^t\widetilde{u}\,dt'\bigg\|_{4,\widehat{S}}^2+\|\widehat{H}(\cdot,0)+2/R_{\mathrm{e}}\|_{2,\widehat{S}}^2\\
&\qquad+\|R(\cdot,t)-R(\cdot,0)\|_{4,S^1}^2\bigg)\\
&\qquad+C_1\bigg(\|\widehat{u}\|_{3,\widehat{\Omega}}^2+\bigg\|\int_0^t\widehat{u}\,dt'\bigg\|_{0,\widehat{\Omega}}^2+\|R(\cdot,t)-R(\cdot,0)\|_{0,S^1}^2\bigg)
\end{aligned}
$$

$$+ C_2\Big(\|R(\cdot,t)-R(\cdot,0)\|_{3,S^1}^2\Big\|\int_0^t \widehat{u}\,dt'\Big\|_{4,\widehat{S}}^2$$

$$+ \|R(\cdot,t)-R(\cdot,0)\|_{4+1/2,S^1}^2\Big\|\int_0^t \widehat{u}\,dt'\Big\|_{3,\widehat{S}}^2 + \|\widehat{u}\|_{3,\widehat{\Omega}}^2\Big\|\int_0^t \widehat{u}\,dt'\Big\|_{4,\widehat{S}}^2\Big),$$

and

$$-\kappa\int_{\widehat{S}}\bigg(\widehat{n}\cdot\frac{1}{\widehat{\Gamma}}\widehat{\nabla}\widehat{\gamma}\bigg)_{,\tau\tau\tau}\widetilde{\gamma}_{\tau\tau\tau}J\,dz'$$

$$\le \varepsilon\|\widehat{\gamma}_{0zzzz}\|_{0,\widehat{\Omega}}^2 + C_1(\|\widehat{\gamma}\|_{0,\widehat{\Omega}}^2 + \|\widehat{\gamma}_{0z}\|_{2,\widehat{\Omega}}^2 + \|\widetilde{\Gamma}_1\|_{4,\widehat{\Omega}}^2)$$

$$+ C_2\Big[\|\widehat{\gamma}\|_{4,\widehat{\Omega}}^2(\|\widehat{\gamma}_0\|_{3,\widehat{\Omega}}^2 + \|\widehat{\gamma}\|_{3,\widehat{\Omega}}^2 + \|\widehat{\eta}_\sigma\|_{3,\widehat{\Omega}}^2) + \|\widehat{\gamma}\|_{3,\widehat{\Omega}}^2\Big\|\int_0^t \widehat{u}\,dt'\Big\|_{3,\widehat{\Omega}}^2\Big].$$

For the quantities

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t n\tau\tau}^2 J\,dz + c_0\|\widetilde{\widetilde{\eta}}_{\Omega_t n\tau\tau}\|_{0,\widehat{\Omega}}^2,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\widehat{\eta}\widetilde{u}_{3n\tau\tau}^2 J\,dz + c_0\|\widetilde{u}_{3nn\tau\tau}\|_{0,\widehat{\Omega}}^2,$$

$$\|\widetilde{u}'_{z\tau\tau\tau}\|_{0,\widehat{\Omega}}^2 + \|\widetilde{\widetilde{\eta}}_{\Omega_t\tau\tau\tau}\|_{0,\widehat{\Omega}}^2,\quad \|\widetilde{u}'_{nn\tau\tau}\|_{0,\widehat{\Omega}}^2,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{n\tau\tau}^2 J\,dz + \frac{\kappa}{\theta^*}\int_{\widehat{\Omega}}\widetilde{\gamma}_{nn\tau\tau}^2 J\,dz,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\widetilde{\eta}}_{\Omega_t nn\tau}J\,dz + \|\widetilde{\widetilde{\eta}}_{\Omega_t nn\tau}\|_{0,\widehat{\Omega}}^2,$$

$$\|\widetilde{u}_{nnn\tau}\|_{0,\widehat{\Omega}}^2,\quad \|\widetilde{\gamma}_{nnn\tau}\|_{0,\widehat{\Omega}}^2,\quad \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\widehat{\eta}\widetilde{u}_{zzz}^2 J\,dz,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{zzz}^2 J\,dz$$

we obtain the same estimates as in the proof of Lemma 3.9 of [16]. Therefore, from the above considerations we get

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\bigg(\widehat{\eta}\widetilde{u}_{zzz}^2 + \frac{p_{\sigma\hat{\eta}}}{\widehat{\eta}}\widetilde{\eta}_{\sigma zzz} + \frac{\widehat{\eta}c_v}{\widehat{\Gamma}}\widetilde{\gamma}_{zzz}^2\bigg)J\,dz \tag{3.42}$$

$$+ \frac{\sigma}{2}\frac{d}{dt}\int_{\widehat{S}} g^{\alpha\beta}\widehat{n}\cdot\int_0^t \widetilde{u}_{\tau\tau\tau s^\alpha}\,dt'\,\widehat{n}\cdot\int_0^t \widetilde{u}_{\tau\tau\tau s^\beta}\,dt' J\,dz'$$

$$
\begin{aligned}
&+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}(\widehat{H}(\cdot,0)+2/R_{\mathrm{e}})_{,\tau\tau}\widehat{\zeta n}\cdot\int\limits_0^t\widetilde{u}_{\tau\tau\tau_i\tau_i}\,dt'J\,dz'\\
&+\frac{\mu}{2}\|\widetilde{u}_{zzz}\|^2_{1,\widehat{\Omega}}+\frac{\kappa}{\theta^*}\|\widetilde{\gamma}_{zzzz}\|^2_{0,\widehat{\Omega}}+c_0\|\widetilde{\overline{\eta}}_{\Omega_t zzz}\|^2_{0,\widehat{\Omega}}\\
\le{}&(\varepsilon+cd)(\|\widetilde{u}_{zzzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{\overline{\eta}}_{\Omega_t zzz}\|^2_{0,\widehat{\Omega}}+\|\widetilde{\gamma}_{zzzz}\|^2_{0,\widehat{\Omega}})\\
&+\varepsilon\Big(\|\widetilde{u}_{zzzt}\|^2_{0,\widehat{\Omega}}+\|\widetilde{\gamma}_{zzzt}\|^2_{0,\widehat{\Omega}}+\Big\|\int\limits_0^t\widetilde{u}\,dt'\Big\|^2_{4,\widehat{S}}\\
&+\|H(\cdot,0)+2/R_{\mathrm{e}}\|^2_{2,S^1}+\|R(\cdot,t)-R(\cdot,0)\|^2_{4,S^1}\Big)\\
&+C_1\Big(|\widehat{u}|^2_{3,2,\widehat{\Omega}}+\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{0,\widehat{\Omega}}+\|\widehat{\gamma}_{0z}\|^2_{2,\widehat{\Omega}}+\|\widehat{\gamma}\|^2_{0,\widehat{\Omega}}\\
&+\|\widehat{\eta}_{\sigma z}\|^2_{1,\widehat{\Omega}}+\|\widehat{\overline{\eta}}_{\Omega_t}\|^2_{0,\widehat{\Omega}}+\|\vartheta_{0t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}\\
&+\|R(\cdot,t)-R(\cdot,0)\|^2_{0,S^1}+\|\widetilde{g}\|^2_{2,\widehat{\Omega}}+\|\widetilde{k}\|^2_{2,\widehat{\Omega}}+\|\widetilde{\Gamma}_1\|^2_{4,\widehat{\Omega}}\Big)\\
&+C_2\Big[\Big(X_8(\widehat{\Omega})+\int\limits_0^t\|\widehat{u}\|^2_{3,\widehat{\Omega}}\,dt'\Big)(1+X_8^2(\widehat{\Omega}))Y_8(\widehat{\Omega})\\
&+(\|\widehat{\eta}_\sigma\|^2_{3,\widehat{\Omega}}+\|\widehat{\gamma}_0\|^2_{3,\widehat{\Omega}}+\|\widehat{\eta}_\sigma\|^4_{3,\widehat{\Omega}}+\|\widehat{\eta}_\sigma\|^2_{3,\widehat{\Omega}}\|\widehat{\gamma}_0\|^2_{3,\widehat{\Omega}})\\
&\times(\|\vartheta_{0t}\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t})\\
&+\|\widehat{u}\|^2_{3,\widehat{\Omega}}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{4,\widehat{S}}+\|R(\cdot,t)-R(\cdot,0)\|^2_{3,S^1}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{4,\widehat{S}}\\
&+\|R(\cdot,t)-R(\cdot,0)\|^2_{4+1/2,S^1}\Big\|\int\limits_0^t\widehat{u}\,dt'\Big\|^2_{3,\widehat{S}}\Big].
\end{aligned}
$$

Since the sum of the second and third terms on the left-hand side of (3.42) is equal to

$$
\begin{aligned}
&\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int\limits_0^t\widetilde{u}_{p^\gamma p^\delta p^\eta p^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t\widetilde{u}_{p^\gamma p^\delta p^\eta p^\beta}\,dt'J\,dz'\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}\Big|\widehat{n}\cdot\int\limits_0^t\widetilde{u}_{ppp^1p^2}\,dt'\Big|^2J\,dz'\\
&\quad+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{\widehat{S}}\sum_{i=1}^2\Big(\frac{1}{2}\overline{n}\cdot\int\limits_0^t\widetilde{u}_{ppp^ip^i}\,dt'+2((\widehat{H}(\cdot,0)+2/R_{\mathrm{e}})\zeta)_{,pp}\Big)^2J\,dz'\\
&\quad-4\sigma\frac{d}{dt}\int\limits_{\widehat{S}}(\widehat{H}(\cdot,0)+2/R_{\mathrm{e}})^2_{,pp}J\,dz',
\end{aligned}
$$

going back to the variables ξ in (3.42), next summing up the resulting inequality and (3.41) over all neighbourhoods of the partition of unity and finally going back to the variables x we get

$$
\begin{aligned}
(3.43)\qquad & \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_{xxx}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xxx}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xxx}^2\right)dx\\
&+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\frac{1}{2}\widetilde{\delta}^{\alpha\beta}\overline{n}\cdot\int\limits_0^t v_{p^\gamma p^\delta p^\eta p^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t v_{p^\gamma p^\delta p^\eta p^\beta}\,dt'\,ds\\
&+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\left|\overline{n}\cdot\int\limits_0^t v_{ppp^1p^2}\,dt'\right|^2 ds\\
&+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}\sum_{i=1}^2\left(\frac{1}{2}\overline{n}\cdot\int\limits_0^t v_{ppp^ip^i}\,dt'+2(H(\cdot,0)+2/R_{\rm e})_{,pp}\right)^2 ds\\
&\le\varepsilon\bigg(\|v_{xxxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2+\Big\|\int\limits_0^t v\,dt'\Big\|_{4,S_t}^2\\
&\quad+\|H(\cdot,0)+2/R_{\rm e}\|_{2,S^1}^2\|R(\cdot,t)-R(\cdot,0)\|_{4,S^1}^2\bigg)\\
&\quad+C_1\bigg(|v|_{3,2,\Omega_t}^2+\Big\|\int\limits_0^t v\,dt'\Big\|_{0,\Omega_t}^2+\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\varrho_{\sigma x}\|_{1,\Omega_t}^2\\
&\quad+\|\vartheta\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{2,\Omega_t}^2+\|\vartheta_{0t}\|_{2,\Omega_t}^2\\
&\quad+\|R(\cdot,t)-R(\cdot,0)\|_{0,S^1}^2+\|f\|_{2,\Omega_t}^2+\|r\|_{2,\Omega_t}^2+\|\theta_1\|_{4,\Omega_t}^2\bigg)\\
&\quad+C_2\bigg[X_8(1+X_8^2)Y_8+\|v\|_{3,\Omega_t}^2\Big\|\int\limits_0^t v\,dt'\Big\|_{4,S_t}^2\\
&\quad+\|R(\cdot,t)-R(\cdot,0)\|_{3,S^1}^2\Big\|\int\limits_0^t v\,dt'\Big\|_{4,S_t}^2\\
&\quad+\|R(\cdot,t)-R(\cdot,0)\|_{4+1/2,S^1}^2\Big\|\int\limits_0^t v\,dt'\Big\|_{3,S_t}^2\\
&\quad+\bigg|\frac{d}{dt}\int\limits_{S_t}(H(\cdot,0)+2/R_{\rm e})_{,pp}^2\,ds\bigg|\bigg].
\end{aligned}
$$

In view of the interpolation inequality (2.2) we have

$$
(3.44)\qquad \bigg|\frac{d}{dt}\int\limits_{S_t}(H(\cdot,0)+2/R_{\rm e})_{,pp}^2\,ds\bigg|\le\varepsilon\|v\|_{3,\Omega_t}^2+C_2\|H(\cdot,0)+2/R_{\rm e}\|_{2,S^1}^4.
$$

Using (3.38) yields

$$(3.45)\quad \Big\|\int_0^t v\,dt'\Big\|^2_{4,S_t} \le \varepsilon(\|v_{xxxx}\|^2_{0,\Omega_t} + \|\varrho_{\sigma xxx}\|^2_{0,\Omega_t} + \|\vartheta_{0xxx}\|^2_{0,\Omega_t})$$
$$+ C_1\Big(\|v\|^2_{0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t} + \Big\|\int_0^t v\,dt'\Big\|^2_{0,\Omega_t}$$
$$+ \|H(\cdot,0) + 2/R_{\rm e}\|^2_{2,S^1} + \|R(\cdot,t) - R(\cdot,0)\|^2_{4,S^1}\Big)$$
$$+ C_2(\|v\|^2_{3,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t} + \|\varrho_{\sigma x}\|^2_{2,\Omega_t})\Big\|\int_0^t v\,dt'\Big\|^2_{4,\Omega_t}$$
$$\times\Big(1 + \Big\|\int_0^t v\,dt'\Big\|^2_{3,\Omega_t}\Big).$$

From (3.43)–(3.45) we obtain (3.40). ∎

The proofs of Lemmas 3.10–3.12 (formulated below) are similar to the proofs of Lemmas 3.10–3.12 of [16]. To estimate the boundary terms associated with the boundary condition $(3.1)_4$ we use the arguments from Lemmas 4.10–4.12 of [19].

LEMMA 3.10. *Let v, ϱ, ϑ_0 be a sufficiently smooth solution of problem* (3.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v^2_{xxt} + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xxt} + \frac{\varrho c_v}{\theta}\vartheta^2_{0xxt}\Big)\,dx$$
$$+ \frac{\sigma}{2}\frac{d}{dt}\int_{S_t} g^{\alpha\beta} v_{p^\gamma p^\delta p^\alpha}\cdot\overline{n} v_{p^\gamma p^\delta p^\beta}\cdot\overline{n}\,ds$$
$$+ c_0(\|v_{xxt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma xxt}\|^2_{0,\Omega_t} + \|\vartheta_{0xxxt}\|^2_{0,\Omega_t})$$
$$\le \varepsilon(\|v_{xxtt}\|^2_{0,\Omega_t} + \|v_{xxxx}\|^2_{0,\Omega_t} + \|\vartheta_{0xxxx}\|^2_{0,\Omega_t}$$
$$+ \|\vartheta_{0xxtt}\|^2_{0,\Omega_t} + \|H(\cdot,0) + 2/R_{\rm e}\|^2_{0,S^1})$$
$$+ C_1(|v|^2_{3,1,\Omega_t} + \|\varrho_{\sigma x}\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}$$
$$+ \|\vartheta_{0x}\|^2_{2,\Omega_t} + \|\vartheta_{0t}\|^2_{2,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t} + |f|^2_{2,1,\Omega_t}$$
$$+ |r|^2_{2,1,\Omega_t} + \|\theta_{1t}\|^2_{3,\Omega_t} + \|\theta_1\|^2_{3,\Omega_t}) + C_2 X_9(1 + X_9^2)Y_9,$$

where the summation over repeated indices $(\alpha,\beta,\gamma,\delta = 1,2)$ and coordinates x is assumed and

$$X_9=|v|^2_{3,2,\Omega_t}+|\varrho_\sigma|^2_{3,1,\Omega_t}+|\vartheta_0|^2_{3,1,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{3,\Omega_{t'}}\,dt',$$

$$Y_9=|v|^2_{4,3,\Omega_t}+|\varrho_\sigma|^2_{3,1,\Omega_t}+\|\vartheta_{0x}\|^2_{3,\Omega_t}+\|\vartheta_{0t}\|^2_{3,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}$$
$$+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'.$$

Lemma 3.11. *Let* v,ϱ,ϑ_0 *be a sufficiently smooth solution of problem* (3.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v^2_{xtt}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xtt}+\frac{\varrho c_v}{\theta}\vartheta^2_{0xtt}\right)dx+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}g^{\alpha\beta}\overline{n}\cdot v_{tpp^\alpha}\overline{n}\cdot v_{tpp^\beta}\,ds$$
$$+c_0(\|v_{ttx}\|^2_{1,\Omega_t}+\|\varrho_{\sigma tt}\|^2_{1,\Omega_t}+\|\vartheta_{0ttxx}\|^2_{0,\Omega_t})$$
$$\le\varepsilon(\|v_{xxxt}\|^2_{0,\Omega_t}+\|v_{xttt}\|^2_{0,\Omega_t}+\|\vartheta_{0xxxt}\|^2_{0,\Omega_t}$$
$$+\|\vartheta_{0xttt}\|^2_{0,\Omega_t}+\|H(\cdot,0)+2/R_{\rm e}\|^2_{2,S^1})$$
$$+C_1(|v|^2_{3,0,\Omega_t}+\|\varrho_{\sigma x}\|^2_{1,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}$$
$$+\|\vartheta_{0x}\|^2_{2,\Omega_t}+|\vartheta_{0t}|^2_{2,0,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}+|f|^2_{2,0,\Omega_t}$$
$$+|r|^2_{2,0,\Omega_t}+\|\theta_{1tt}\|^2_{2,\Omega_t}+\|\theta_{1t}\|^2_{2,\Omega_t}+\|\theta_1\|^2_{2,\Omega_t})$$
$$+C_2X_{10}(1+X^2_{10})Y_{10},$$

where

$$X_{10}=|v|^2_{3,0,\Omega_t}+|\varrho_\sigma|^2_{3,0,\Omega_t}+|\vartheta_0|^2_{3,0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{3,\Omega_{t'}}\,dt',$$

$$Y_{10}=|v|^2_{4,1,\Omega_t}+|\varrho_\sigma|^2_{3,1,\Omega_t}+\|\vartheta_{0x}\|^2_{3,\Omega_t}+|\vartheta_{0t}|^2_{3,1,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t}$$
$$+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}+\int\limits_0^t\|v\|^2_{4,\Omega_{t'}}\,dt'.\ ■$$

Lemma 3.12. *Let* v,ϱ,ϑ_0 *be a sufficiently smooth solution of problem* (3.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v^2_{ttt}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma ttt}+\frac{\varrho c_v}{\theta}\vartheta^2_{0ttt}\right)dx+\frac{\sigma}{2}\frac{d}{dt}\int\limits_{S_t}g^{\alpha\beta}\overline{n}\cdot v_{tts^\alpha}\overline{n}\cdot v_{tts^\beta}\,ds$$
$$+c_0(\|v_{ttt}\|^2_{1,\Omega_t}+\|\varrho_{\sigma ttt}\|^2_{0,\Omega_t}+\|\vartheta_{0ttt}\|^2_{1,\Omega_t})$$
$$\le C_1(\|v_{tt}\|^2_{1,\Omega_t}+\|\vartheta_{0tt}\|^2_{1,\Omega_t}+\|f_{ttt}\|^2_{0,\Omega_t}+|f|^2_{2,0,\Omega_t}$$
$$+\|r_{ttt}\|^2_{0,\Omega_t}+|r|^2_{2,0,\Omega_t}+|\theta_1|^2_{4,1,\Omega_t})+C_2X_{11}(1+X^3_{11})Y_{11},$$

where

$$X_{11}=|v|^2_{3,0,\Omega_t}+|\varrho_\sigma|^2_{3,0,\Omega_t}+|\vartheta_0|^2_{3,0,\Omega_t},$$
$$Y_{11}=|v|^2_{4,1,\Omega_t}+|\varrho_\sigma|^2_{3,0,\Omega_t}+|\vartheta_0|^2_{4,1,\Omega_t}.\ ■$$

Estimating $\|\vartheta\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}$ by $\|\vartheta_{0x}\|^2_{0,\Omega_t}+\|p_\sigma\|^2_{0,\Omega_t}$ (by using (3.20) and (3.21)) from the above lemmas for sufficiently small ε we obtain

THEOREM 3.13. *Let*

$$\begin{aligned}\phi(t)=&\int\limits_{\Omega_t}\varrho\sum_{0\le|\alpha|+i\le3}|D_x^\alpha\partial_t^i v|^2\,dx+\int\limits_{\Omega_t}\left(\frac{p_1}{\varrho}\varrho_\sigma^2+\overline{\varrho}^2_{\Omega_t}+\frac{p_2\varrho c_v}{p_\theta\theta}\vartheta_0^2\right)dx\\&+\int\limits_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\sum_{1\le|\alpha|+i\le3}|D_x^\alpha\partial_t^i\varrho_\sigma|^2dx+\int\limits_{\Omega_t}\frac{\varrho c_v}{\theta}\sum_{1\le|\alpha|+i\le3}|D_x^\alpha\partial_t^i\vartheta_0|^2\,dx\\&+\frac{\sigma}{2}\int\limits_{S_t}\widetilde{\delta}^{\alpha\beta}\sum_{|k|\le2}\overline{n}\cdot\int\limits_0^t\partial_p^k v_{p^\gamma p^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t\partial_p^k v_{p^\gamma p^\beta}\,dt'\,ds\\&+\frac{\sigma}{2}\int\limits_{S_t}\sum_{|k|\le2}\left|\overline{n}\cdot\int\limits_0^t\partial_p^k v_{p^1p^2}\,dt'\right|^2ds\\&+\frac{\sigma}{2}\int\limits_{S_t}\sum_{|k|\le2}\sum_{i=1}^2\left(\frac12\overline{n}\cdot\int\limits_0^t\partial_p^k v_{p^ip^i}\,dt'+2\partial_p^k(H(\cdot,0)+2/R_{\rm e})\right)^2ds\\&+\frac{\sigma}{2}\int\limits_{S_t}g^{\alpha\beta}\Big(\overline{n}\cdot\int\limits_0^t v_{s^\alpha}\,dt'\,\overline{n}\cdot\int\limits_0^t v_{s^\beta}\,dt'+\sum_{|k|\le2}D_t^k v_{s^\alpha}\cdot\overline{n}D_t^k v_{s^\beta}\cdot\overline{n}\\&+\sum_{|k|\le2}D_p^k v_{p^\alpha}\cdot\overline{n}D_p^k v_{s^\beta}\cdot\overline{n}\Big)\,ds\end{aligned}$$

($\widetilde{\delta}^{\alpha\beta}$ *is defined in Lemma* 3.4),

$$\begin{aligned}\Phi(t)=&|v|^2_{4,1,\Omega_t}+|\varrho_\sigma|^2_{3,0,\Omega_t}-\|\varrho_\sigma\|^2_{0,\Omega_t}+\|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t}\\&+|\vartheta_0|^2_{4,1,\Omega_t}-\|\vartheta_0\|^2_{0,\Omega_t}+\|\vartheta\|^2_{0,\Omega_t},\\\psi(t)=&\|v\|^2_{0,\Omega_t}+\|p_\sigma\|^2_{0,\Omega_t}+\|R(\cdot,t)-R(\cdot,0)\|^2_{0,S^1},\\F(t)=&\|f_{ttt}\|^2_{0,\Omega_t}+|f|^2_{2,0,\Omega_t}+\|r_{ttt}\|^2_{0,\Omega_t}+|r|^2_{2,0,\Omega_t}\\&+\|r\|_{0,\Omega_t}+|\theta_1|^2_{4,1,\Omega_t}+\|\theta_1\|_{1,\Omega_t}.\end{aligned}$$

Assume that $\nu>\frac13\mu$. *Then for sufficiently smooth solutions of problem* (3.1) *the following estimate holds*:

$$\begin{aligned}\frac{d\phi}{dt}+c_0\Phi\le{}&c_1P(X)X(1+X^3)(X+Y)+c_2F\\&+c_3\psi+c_4\|H(\cdot,0)+2/R_{\rm e}\|^4_{2,S^1}\\&+\varepsilon c_5(\|H(\cdot,0)+2/R_{\rm e}\|^2_{2,S^1}+\|R(\cdot,t)-R(\cdot,0)\|^2_{4,S^1})\\&+c_6\Big(\|R(\cdot,t)-R(\cdot,0)\|^2_{4+1/2,S^1}\Big\|\int\limits_0^t v\,dt'\Big\|^2_{3,S_t}\end{aligned}\tag{3.46}$$

$$+ \|R(\cdot,t) - R(\cdot,0)\|_{3,S^1}^2 \Big\| \int_0^t v\,dt' \Big\|_{4,S_t}^2 \Big),$$

where $0 < c_0 < 1$ *is a constant depending on* ϱ_*, ϱ^*, θ_*, θ^*, μ, ν *and* κ, c_i $(i=1,\dots,6)$ *depend on* ϱ_*, ϱ^*, θ_*, θ^*, T, $\int_0^T \|v\|_{3,\Omega_{t'}}^2\,dt'$, $\|S\|_{4+1/2}$, *constants from the imbedding lemma and the Korn inequalities* (see Section 5 of [18]), ε *is a small parameter and*

$$X = |v|_{3,0,\Omega_t}^2 + |\varrho_\sigma|_{3,0,\Omega_t}^2 + |\vartheta_0|_{3,0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \int_0^t \|v\|_{3,\Omega_{t'}}^2\,dt',$$

$$Y = |v|_{4,1,\Omega_t}^2 + |\varrho_{\sigma t}|_{2,0,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{2,\Omega_t}^2 + |\vartheta_{0t}|_{3,1,\Omega_t}^2 + \|\vartheta_{0x}\|_{3,\Omega_t}^2$$
$$+ \|\vartheta\|_{0,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2 + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'. \ \blacksquare$$

A slight modification of the proof of Theorem 3.13 yields

THEOREM 3.14. *Assume that* $\nu > \frac{1}{3}\mu$. *For sufficiently smooth solutions of problem* (3.1) *we have*

$$\frac{d\widetilde{\phi}}{dt} + c_0\Phi \le c_7 P\Big(\widetilde{\phi} + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)\Big(\widetilde{\phi} + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big) \tag{3.47}$$
$$\times \Big[1 + \Big(\widetilde{\phi} + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)^3\Big]\Big(\Phi + \widetilde{\phi} + \int_0^t \|v\|_{4,\Omega_{t'}}^2\,dt'\Big)$$
$$+ c_8 F + c_9\psi + c_{10}\|H(\cdot,0) + 2/R_{\rm e}\|_{2,S^1}^4$$
$$+ \varepsilon c_{11}(\|H(\cdot,0) + 2/R_{\rm e}\|_{2,S^1}^2) + \|R(\cdot,t) - R(\cdot,0)\|_{4,S^1}^2)$$
$$+ c_{12}\Big(\|R(\cdot,t) - R(\cdot,0)\|_{4-1/2,S^1}^2 \Big\| \int_0^t v\,dt' \Big\|_{3,S_t}^2$$
$$+ \|R(\cdot,t) - R(\cdot,0)\|_{3,S^1}^2 \Big\| \int_0^t v\,dt' \Big\|_{4,S_t}^2 \Big),$$

where ε *is a small parameter*, c_0 *and* c_i $(i = 7,\dots,12)$ *have the same properties as in Theorem* 3.13 *and* $\widetilde{\phi}(t) = |v|_{3,2,\Omega_t}^2 + |\varrho_\sigma|_{3,2,\Omega_t}^2 + |\vartheta_0|_{3,2,\Omega_t}^2 + \|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2$.

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