Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator

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Abstract. We find a bounded solution of the non-homogeneous Monge–Ampère equation under very weak assumptions on its right hand side.

Introduction. In this paper we are interested in solving, under possibly weak assumptions on the measure $d\mu$, the following Dirichlet problem for the complex Monge–Ampère equation in a given strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$:

\[ u \in \text{PSH} \cap L^\infty(\Omega), \]
\[ (dd^cu)^n = d\mu, \]
\[ \lim_{z^i \to z} u(z^i) = \phi(z), \quad z \in \partial \Omega, \quad \phi \in C(\partial \Omega), \]

where $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$ and so $dd^c = 2\pi i \partial \bar{\partial}$. It has been shown by E. Bedford and B. A. Taylor [BT1] that the wedge product $(dd^cu)^n = dd^cu \wedge \ldots \wedge dd^cu$ is well defined for plurisubharmonic (psh), locally bounded functions $u$, and that (*) is solvable for measures having continuous densities with respect to the Lebesgue measure (here denoted by $d\lambda$). The equation has attracted attention of a number of authors; we refer to [B] for a more detailed account. In particular, it is known that continuous solutions exist if $d\mu = f\,d\lambda$, where $f \in L^2(\Omega, d\lambda)$ (U. Cegrell–L. Persson [CP]), but for $f \in L^1(\Omega, d\lambda)$ this is not necessarily true [CS]. In Theorem 3 below we show that if $f \in L^p(\Omega, d\lambda)$, $p > 1$, then there exists a continuous solution of (*). This is the answer to the question posed in [CS] and [P] (see also [B], [BL]). For the case of rotation invariant measures in a ball a solution was given in [P].

The result can be extended from $L^p$, $p > 1$, to some Orlicz spaces as shown

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in Theorem 4. To prove it we use an a priori estimate for the $\|u\|_{L^\infty}$ norm of a solution of $(\ast)$ if $d\mu$ satisfies a certain integral condition (Theorem 1). E. Bedford [B] conjectured that some such estimate is possible. It is shown that the integral condition cannot be substantially weakened. Combining Theorem 1 with the results of [KO] we solve the Dirichlet problem $(\ast)$ for a large family of measures $d\mu$.

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**Preliminaries.** Here we present some notions and results which are used in the paper. The background material can be found in [B], [K], [S]. $\Omega$ will denote throughout a strictly pseudoconvex domain in $\mathbb{C}^n$. For a compact subset $K \subset \Omega$ we define the relative extremal function and the relative capacity [BT2] (see also [B], [K]) by the formulas

$$u_K(z) = \sup \{ u(z) : u \in \text{PSH} \cap L^\infty, \ u < 0 \text{ in } \Omega, \ u \leq -1 \text{ on } K \},$$
$$\text{cap}(K, \Omega) = \sup \left\{ (dd^c u)^n : u \in \text{PSH}(\Omega), \ -1 \leq u < 0 \right\}.$$

By [BT2],

$$\text{cap}(K, \Omega) = \int_K (dd^c u_K^*)^n = \int_\Omega (dd^c u_K^*)^n,$$

where $u_K^* := \lim_{z' \to z} u_K(z)$. If $u_K^* = u_K$ we say that $K$ is regular. For an open subset $U \subset \Omega$ the relative capacity is defined by

$$\text{cap}(U, \Omega) = \sup \{ \text{cap}(K, \Omega) : K \subset U, \ K \text{ compact} \}.$$

Another extremal function (of logarithmic growth) and an associated capacity were introduced by J. Siciak (see [S], [AT], [B], [K]):

$$L_K(z) = \sup \{ u(z) : u \in \text{PSH}(\mathbb{C}^n), \ u(z) < \log(1 + |z|) + O(1), \ u \leq 0 \text{ on } K \},$$
$$T_R(K) := \exp\left( -\sup\{ L_K^*(|z|) : |z| \leq R \} \right)$$

for a compact set $K \subset \mathbb{C}^n$ and a given $R > 0$. We extend the definition of $T_R$ to open sets in the same way as the definition of $\text{cap}$ above.

Important inequalities between $\text{cap}$ and $T$ were proved by H. Alexander and B. A. Taylor [AT]. If $B := B(0, R)$ and $K \subset B(0, r), r < R$, is compact, then

$$\exp(-A(r)\text{cap}(K, B)^{-1}) \leq T_R(K) \leq \exp(-2\pi\text{cap}(K, B)^{-1/n}).$$

The main tool in pluripotential theory is the following Comparison Principle of Bedford and Taylor [BT2]:
Comparison Principle. If \( u, v \in \text{PSH} \cap L^\infty(\Omega) \) and \( \liminf_{z \to \partial \Omega}(u(z) - v(z)) \geq 0 \), then

\[
\int_{\{u < v\}} (dd^c u)^n \leq \int_{\{u < v\}} (dd^c v)^n.
\]

Due to the same authors and presented here in a simplified version, sufficient for our applications, is

Convergence Theorem [BT2]. If \( u_j \in \text{PSH} \cap L^\infty(\Omega) \), \( j = 1, 2, \ldots \), and \( u_j \rightharpoonup u \) a.e. in \( \Omega \) or \( u_j \uparrow u \) with \( u \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega) \) then

\[
(dd^c u_j)^n \to (dd^c u)^n
\]
in the sense of currents.

An a priori estimate. We begin with proving an a priori estimate for the \( L^\infty \) norm of a solution to the Dirichlet problem (*) when \( d\mu \) is assumed to satisfy a certain integral condition.

Theorem 1. Let \( \Omega \) be a strictly pseudoconvex domain in \( \mathbb{C}^n \) and let \( \mu \) be a Borel measure in \( \Omega \) such that \( \int_{\Omega} d\mu \leq 1 \). Consider an increasing function \( h : \mathbb{R} \to (1, \infty) \) satisfying

\[
\int_1^\infty (yh^{1/n}(y))^{-1} dy < \infty.
\]

If \( \mu \) satisfies the integral condition

\[
(*) \quad \int_{\Omega} |v|^n h(|v|) d\mu \leq A
\]

whenever

\[
v \in \text{PSH}(\Omega) \cap C(\overline{\Omega}), \quad v = 0 \text{ on } \partial \Omega, \quad \int_{\Omega} (dd^c v)^n \leq 1,
\]

then the norm \( \|u\|_{L^\infty} \) of a solution of the Dirichlet problem (*) is bounded by a constant \( B = B(h, A) \) which does not depend on \( \mu \).

Proof. It is no restriction to assume that \( \phi = 0 \) in (*): the general case will follow by the Comparison Principle [BT2]. Let \( u \) be a solution of (*).

For \( s < 0 \) denote by \( U_s \) the open set \( \{u < s\} \) and put

\[
a(s) := \text{cap}(U_s, \Omega) = \text{cap}(U_s), \quad b(s) := \mu(U_s).
\]

Our proof rests on the following two propositions.

Proposition 1. \( b(s) \leq Aa(s)h^{-1}([a(s)]^{-1/n}) \).

Proposition 2. \( t^n a(s) \leq b(s + t) \) if \( t > 0 \) and \( s + t < 0 \).
Proof of Proposition 1. Consider \( v = (ra(s))^{-1/n}u_K \), where \( K \subset U_s \) is a compact regular set with \( \text{cap}(K) = ra(s) \) \((r < 1)\). Then \( \int (dd^cv)^n = 1 \) and so the integral condition (***) applies, giving

\[
A \geq \int_{\Omega} |v|^n h(|v|) \, d\mu \geq \int_{K} |v|^n h(|v|) \, d\mu = (ra(s))^{-1} h([ra(s)]^{-1/n}) \mu(K),
\]

which is just the desired estimate as \( r \to 1 \) (and so \( \mu(K) \to b(s) \)).

Proof of Proposition 2. We apply the Comparison Principle \([BT2]\) to the pair of functions \( u_K \) and \( v := (rt)^{-1}(u - s - t) \), where \( K, r \) are defined as above. Note that \( K \subset \{ v < u_K \} \subset U_{s+t} \). Hence

\[
ra(s) = \int_{\{v < u_K\}} (dd^cu_K)^n \leq \int_{\{v < u_K\}} (dd^cu)^n \leq (rt)^{-n} \mu(U_{s+t}) = (rt)^{-n}b(s + t).
\]

The proposition follows if we let \( r \to 1 \).

End of the proof of Theorem 1. Fix \( s_0 \) so that \( a = a(s_0) \neq 0 \). We need to find a lower bound for \( s_0 \). To this end we first define an increasing sequence \( s_0, s_1, \ldots, s_N \) by

\[
s_j := \sup\{ s : a(s) \leq \lim_{t \to s_{j-1}^+} ea(t) \}.
\]

Then

\[
\lim_{t \to s_j^-} a(t) \leq \lim_{t \to s_{j-1}^+} ea(t) \quad \text{and} \quad a(s_j) \geq ea(s_{j-2}).
\]

We continue this process till

(1) \( 1 \leq a(s_N) \).

For fixed \( s \) and \( s' \) such that \( a(s) \leq ea(s') \) and \( t := s - s' \) we have by the above two propositions

\[
a(s') \leq t^{-n}b(s) \leq At^{-n}a(s)h^{-1}([a(s)]^{-1/n}) = Aet^{-n}a(s')h^{-1}([a(s)]^{-1/n}).
\]

Hence

\[
t \leq (Ae)^{1/n} h_1(a(s))
\]

where \( h_1(x) := h^{-1/n}(x^{-1/n}) \). Letting \( s \to s_{j+1}^- \) and \( s' \to s_j^+ \) we thus get

\[
t_j := s_{j+1}^- - s_j \leq (Ae)^{1/n} h_1(a(s_{j+1})).
\]

Since the function \( h_2(x) := h_1(e^x) = h^{-1/n}(e^{-x/n}) \) is increasing we can further estimate
\[
\sum_{j=0}^{N-1} t_j \leq (Ae)^{1/n} \sum_{j=0}^{N-1} h_2(\log a(s_{j+1})) \\
\leq (Ae)^{1/n} \left( \sum_{j=0}^{N-2} h_2(x) dx + 2h_2(\log a(s_N)) \right) \\
\leq 2(Ae)^{1/n} \left( \int_{-\infty}^{0} h_2(x) dx + h_2(\infty) \right).
\]

By our hypothesis on \( h \), we have \( h_2(\infty) \leq 1 \) and
\[
\int_{-\infty}^{0} h_2(x) dx = \int_{-\infty}^{\infty} h_{-1/n}(e^{-x/n}) dx \\
= n \int_{1}^{\infty} h_{-1/n}(y)y^{-1} dy =: nc(h) < \infty.
\]

These remarks combined with (2) give
\[
s_N - s_0 = \sum_{j=0}^{N-1} t_j \leq 2(Ae)^{1/n}(nc(h) + 1) =: c.
\]

This means that for \( s' \geq s_0 + c \) we have \( a(s') > 1 \) (see (1)). So fixing \( s' = s_0 + c + 1 \) we conclude that \( s' \geq 0 \) because otherwise, by applying Proposition 2, we would get a contradiction with the assumptions:
\[
\mu(U_{s'}) > 1.
\]

Thus \( s_0 \geq -c - 1 =: B \). The proof is complete.

Remark. The hypothesis that \( \mu \) satisfies (**) can be replaced by
\[
\mu(K) \leq A \text{cap}(K) h^{-1}((\text{cap}(K))^{-1/n})
\]
for any \( K \subset \Omega \) compact and regular. The above proof still works.

It turns out that the integral condition (**) is not far from being sharp. From [BL, Corollary 2.2] (see also [D, Th. 2.2]) it follows that any bounded solution of (\( * \)) satisfies (**\( \star \)) with \( h \equiv 1 \) and \( A = n!\|u\|_{L^\infty} \mu \). However, if we let \( h \equiv 1 \) then (**\( \star \)) ceases to be a sufficient condition for boundedness of \( u \) (when \( n > 1 \)). This can be seen by considering radial psh functions in a ball \( B = B(0, R) \). In that case we have a characterization of bounded solutions of (\( * \)) given in [P] (see also [M]). A radial psh function \( u \) is bounded if and only if
\[
\int_{0}^{R} r^{-1} F^{1/n}(r) dr < \infty,
\]
where \( F(r) = \int_{B(0, r)} (dd^c u)^n \).
It is easy to see that for the rotation invariant measure $d\mu = (dd^c u)^n$ the integral in $(\ast\ast)$ assumes its maximal value for $v(z) = (2\pi)^{-n} \log |z|$. Suppose that

\[(4) \quad (2\pi)^n \int_B |v|^n \, d\mu = \int_0^R |\log r|^n F'(r) \, dr < \infty.\]

Via integration by parts this is equivalent to

\[\int_0^R |\log r|^{n-1} r^{-1} F(r) \, dr < \infty.\]

Write $F(r) = |\log r|^{-n} g^{-1}(r)$. Then (4) takes the form

\[\int_0^R [[|\log r| r g(r)]^{-1} \, dr < \infty,\]

whereas (3) now says

\[\int_0^R [[|\log r| r g^{1/n}(r)]^{-1} \, dr < \infty.\]

Taking $g$ such that the former inequality is satisfied but the latter is not, e.g. $g(r) = (\log |\log r|)^n$, we arrive at the desired conclusion.

Coupling Theorem 1 above with Theorem 1 from [KO] we obtain a fairly general class of measures for which the Dirichlet problem $(\ast)$ is solvable. For the definition of a measure locally dominated by capacity which we need in the statement of the next theorem we refer to [KO]. Essentially we require from such a measure (say $\mu$) that there exists $c > 0$ such that given two concentric balls $B_1 := B(a, r) \subset B_2 := B(a, 2r) \subset \Omega$ and a compact subset $E \subset B_1$, the following estimate holds:

\[\mu(E) \leq c\mu(B_1) \text{cap}(E, B_2).\]

(The actual definition is a bit less restrictive.)

**Theorem 2.** If a measure $\mu$ in $\Omega$ is locally dominated by capacity and satisfies the condition $(\ast\ast)$ from Theorem 1 with $h$ such that

\[h(ax) \leq bh(x), \quad x > 0,\]

for some $a > 1$ and $b > 1$, then there exists a solution of $(\ast)$.

**Proof.** For a while we assume that $\mu$ has compact support in $\Omega$. Define a regularizing sequence of measures $\mu_t$ by fixing a radial non-negative function $\omega \in C_0^\infty(B)$ with $\int \omega \, d\lambda = 1$ (here $B$ is the unit ball in $\mathbb{C}^n$) and setting

\[\mu_t = \omega_t \ast \mu, \quad \text{where} \quad \omega_t(z) = t^{-2n} \omega(z/t), \quad t > 0.\]
By Theorem 1 and Remark following it, it is enough to find \( t_0 > 0 \) and \( A > 0 \) such that for any compact set \( K \subset \Omega \),

\[
\mu_t(K) \leq A \text{cap}(K, \Omega) h^{-1}((\text{cap}(K, \Omega))^{-1/n}), \quad t < t_0.
\]

**Proposition 3.** If \( E \Subset \Omega \) is regular then for any \( d > 1 \) there exists \( t_0 \) such that

\[
\text{cap}(K_y, \Omega) \leq d \text{cap}(K, \Omega), \quad |y| < t_0,
\]

where \( K \subset E \) is regular and \( K_y := \{ x : x - y \in K \} \).

**Proof.** For \( K \subset E \) define \( w_y := u_{K_y}(x + y) \), where \( u_{K_y} \) is the extremal function of \( K_y \). For any \( c \) such that \( 0 < c < 1/2 \) define \( \Omega_c = \{ u_E < -c \} \). By continuity of \( u_E \) one can fix \( t_0 > 0 \) such that if \( |y| \leq t_0 \) and \( x \in \Omega_{c/2} \) then \( x + y \in \Omega \). Therefore

\[
g(x) := \begin{cases} 
\max(w_y - c, (1 + 2c)u_E(x)), & x \in \Omega_{c/2}, \\
(1 + 2c)u_E(x), & x \notin \Omega_{c/2}, 
\end{cases}
\]

is a well defined plurisubharmonic function in \( \Omega \). Since \( K \subset E \) and \( w_y = -1 \) on \( K \) one concludes that \( g = w_y - c \) in a neighbourhood of \( K \). Hence

\[
\text{cap}(K, \Omega) \geq (1 + 2c)^{-n} \int_{K}(dd^c g)^n = (1 + 2c)^{-n} \int_{K}(dd^c w_y)^n = (1 + 2c)^{-n} \text{cap}(K_y, \Omega).
\]

Thus the proposition is proved.

To complete the proof of Theorem 2 let us fix a set \( E \) and a positive number \( t_0 \) such that the above proposition holds with \( E := \bigcup_{0 < t_0} \text{supp} \mu_t \Subset \Omega \) and \( d = a^n \). By the assumptions there exists \( A_0 > 0 \) such that

\[
\mu(K) \leq A_0 \text{cap}(K) h^{-1}((\text{cap}(K))^{-1/n}).
\]

Hence for \( t < t_0 \) we have by Proposition 3 and the extra assumption on \( h \),

\[
\mu_t(K) \leq \sup_{|y|<t} \mu(K_y) \leq A_0 \sup_{|y|<t} \text{cap}(K_y) h^{-1}((\text{cap}(K_y))^{-1/n}) \\
\leq A_0 d \text{cap}(K) h^{-1}((d \text{cap}(K))^{-1/n}) \\
\leq A_0 d^{1/n} \text{cap}(K) h^{-1}((\text{cap}(K))^{-1/n}).
\]

Setting \( A := A_0 a^n b^{1/n} \) we verify this way that \( \mu_t \) satisfies \((\iota)\) for \( t < t_0 \), with the constant \( A \) independent of \( t \). Thus by Theorem 1 the family of solutions of \((\ast)\) for \( \mu_t \), \( t < t_0 \), is uniformly bounded. So one can apply [KO, Th. 1] to get the conclusion.

To verify the statement for an arbitrary measure \( \mu \) note that by the above argument the solutions exist for \( \chi_j \, d\mu \), where \( \chi_j \) is a non-decreasing sequence of smooth cut-off functions with \( \chi_j \uparrow 1 \) in \( \Omega \). Moreover, the \( L^\infty \)
norms of those solutions are uniformly bounded by a constant depending only on $A$. Hence the result follows by applying the monotone convergence theorem of [BT2].

**Solutions for measures having densities in $L^p, p > 1$.** In Theorem 3 we are going to prove that for $d\mu = f \, d\lambda, f \in L^p(\Omega), p > 1$, the Dirichlet problem $(\ast)$ has a continuous solution. To this end we shall use the following

**Lemma 1.** Suppose $v \in \text{PSH}(\Omega) \cap C(\overline{\Omega}), v = 0$ on $\partial\Omega$ and $\int (dd^c v)^n = 1$. Then the Lebesgue measure $\lambda(U_s)$ of the set $\{v < s\}$ is bounded from above by $c \exp(-2\pi|s|), where $c$ does not depend on $v$.

**Proof.** The proof is a variation of the proof of Proposition 2 of [KO]. First we shall estimate $\text{cap}(U_s) = \text{cap}(U_s, \Omega)$ applying the Comparison Principle [BT2]. For $t > 1$ and a regular compact set $K \subset U_s$ we have

$$\text{cap}(K) = \int_K (dd^c u_K)^n = \int_{\{s^{-1}v < u_K\}} (dd^c u_K)^n \leq \frac{t^n s^{-n}}{\Omega} \int_{\{s^{-1}v \leq u_K\}} (dd^c u)^n \leq \frac{t^n}{C}.$$

Hence

$$\text{cap}(U_s) \leq |s|^{-n}. \quad (5)$$

Write $(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$ and set $U_s(z') := \{z_1 \in \mathbb{C} : (z_1, z') \in U_s\}$. Let $V_{z'}$ (resp. $V$) be the extremal function of logarithmic growth of $U_s(z')$ (resp. $U_s$). Then (see [TS])

$$\lambda(U_s(z')) \leq C_1 T_R(U_s(z')),$$

where $\lambda$ denotes the Lebesgue measure in $\mathbb{C}, C_1$ is an independent constant and

$$T_R(U_s(z')) := \exp(-\sup_{|z_1| < R} V_{z'}),$$

with $R$ chosen so that $\Omega \subset B(0, R)$. Thus

$$\lambda(U_s) = \int \lambda(U_s(z')) \, d\lambda(z') \leq C_1 \int T_R(U_s(z')) \, d\lambda(z') = C_1 \int \exp(-\sup_{|z_1| < R} V(z_1, z')) \, d\lambda(z'). \quad (6)$$

A simple argument using a result of Alexander [A] shows that the right hand side of (6) is dominated by

$$C_2 \exp(-\sup_{|z| < R} V(z)) = C_2 T_R(U_s)$$

(see [KO] for details). Finally, we apply an inequality between the capacities $\text{cap}$ and $T$ proved in [AT] to obtain

$$\lambda(U_s) \leq C_2 \exp[-2\pi(\text{cap}(U_s, B(0, R)))^{-1/n}] \leq C_2 \exp[-2\pi(\text{cap}(U_s, \Omega))^{-1/n}].$$
Hence by (5) we get

\[ \lambda(U_s) \leq C_2 \exp(-2\pi|s|), \]

which was to be proved.

**Corollary.** If \( v \in \text{PSH}(\Omega) \cap C(\overline{\Omega}), \) \( v = 0 \) on \( \partial\Omega \) and \( \int_{\Omega} (dd^c v)^n \leq 1, \) then \( \|v\|_{L^p} \leq c(p). \)

**Proof.** By the lemma,

\[ \int |v|^p d\lambda \leq \int d\lambda + \sum_{s=1}^{\infty} \int \{v \geq -s\} |v|^p d\lambda \leq c \sum_{s=1}^{\infty} (s+1)^p e^{-2\pi s} =: c(p) < \infty. \]

Now we are in a position to prove

**Theorem 3.** If \( f \in L^p(\Omega, d\lambda), p > 1, f \geq 0 \) then the Dirichlet problem

\( (\ast) \)

has a continuous solution for \( d\mu = f d\lambda. \)

**Proof.** Set \( f_j := \min(f, j). \) Let \( u_j \) be the continuous solution of

\[ (dd^c u)^n = f_j d\lambda, \]

\[ \lim_{z' \to z} u(z') = \phi(z), \quad z \in \partial\Omega \]

(see [C], [CP]). Then by the convergence theorem of [BT2], \( u = \lim u_j \) is the desired solution provided \( u_j \) is uniformly bounded. This is the case if the integral condition \((\ast\ast)\) in Theorem 1 is satisfied for \( d\mu = f d\lambda \) and some suitable \( h. \) Let us verify this condition for \( h(x) = \max(1, x). \) By Hölder’s inequality we have

\[ \int |v|^q h(|v|) f d\lambda = \int_{\{v \geq -1\}} + \int_{\{v < -1\}} \leq \|f\|_{L^1} + \left( \int |v|^{(n+1)q} d\lambda \right)^{1/q} \|f\|_{L^p}, \]

where \( p^{-1} + q^{-1} = 1. \) Since by the Corollary above,

\[ \int |v|^{(n+1)q} d\lambda \leq c(q(n+1)), \]

one can apply Theorem 1 to conclude that \( u = \lim u_j \) is bounded.

Now, if \( u_{jk} \) solves \( (dd^c u)^n = |f_j - f_k| d\lambda, \) \( u = 0 \) on \( \partial\Omega, \) then by the Comparison Principle and the above argument,

\[ \|u_j - u_k\| \leq -u_{jk} \leq c_p \|f_j - f_k\|_{L^p}^{1/n}. \]

So \( u_j \) is uniformly convergent and \( u \) is continuous.

The last result readily extends to cover densities belonging to some Orlicz spaces. As an example (which can be refined yet) we give the following

**Theorem 4.** Let \( L^\varphi(\Omega, d\lambda) \) denote the Orlicz space corresponding to \( \varphi(t) = |t|(\log(1 + |t|))^n h(\log(1 + |t|)) \) with \( h \) satisfying the hypothesis of Theorem 1. If \( f \in L^\varphi(\Omega, d\lambda) \) then \((\ast)\) is solvable with \( d\mu = f d\lambda. \)
Proof. As in the preceding proof, it is enough to verify the condition (**). We apply Young’s inequality for the function $g(\log(1 + r)) = (\log(1 + r))^n h(\log(1 + r))$ and its inverse. Then

$$\int_0^{f(x)} g(|v(x)|) f(x) \, dx \leq \int_0^{g(|v(x)|)} g(\log(1 + r)) \, dr + \int_0^{|v(x)|} \exp(g^{-1}(t)) - 1 \, dt$$

$$\leq f(x) g(\log(1 + f(x))) + \int_0^{|v(x)|} e^s g'(s) \, ds$$

$$\leq \|f\|_{L^r} + g(|v(x)|) e^{|v(x)|}.$$

When integrated over $\Omega$, the right hand side remains bounded since by the lemma,

$$\int_\Omega g(|v(x)|) e^{|v(x)|} \, dx \leq c \sum_{s=1}^\infty e^{s(1-2\pi)} g(s+1) < \infty.$$

Thus the result follows from Theorem 1.

Example. If $\varphi(t) = 2(\log(1 + |t|)^m(\log(1 + |t|))^n, m > n$, then Theorem 4 applies. On the other hand, if $\varphi(t) = 2(\log(1 + |t|)^m, m < n$, it is no longer true; a suitable counterexample is given in [P].

References


Complex Monge–Ampère operator


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