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Slicing of generalized surfaces with curvature measures and diameter's estimate

by Silvano Delladio (Povo, Trento)

Abstract. We prove generalizations of Meusnier's theorem and Fenchel's inequality for a class of generalized surfaces with curvature measures. Moreover, we apply them to obtain a diameter estimate.

1. Introduction. The spaces of generalized Gauss graphs defined in [1] are natural candidates to be good ambient spaces for setting problems of calculus of variations involving surfaces, for example the problem of minimizing functionals depending on the area and on the curvatures of the argument surface.

Taking the point of view of the direct method of the calculus of variations, one is then interested in estimates for generalized Gauss graphs which may yield compactness of minimizing sequences.

A related question, which is also of independent interest, is to find appropriate generalizations of classical differential geometric results related to curvatures. Let us consider an estimate from above of the diameter of a compact surface by means of the L^1-norm of the second fundamental form (compare [8]). For a regular two-dimensional surface embedded in $\mathbb{R}^3$ a possible way to get such an estimate rests on a couple of classical geometric results: Meusnier's theorem and Fenchel's inequality. In fact, from these two results one can deduce an estimate (called a *slice estimate*) for the slices of the Gauss graph obtained by slicing with planes orthogonal to a fixed direction. Then the final estimate easily follows from the Morse–Sard theorem.

In this paper we prove suitable generalizations of Meusnier's theorem for two-dimensional generalized Gauss graphs and Fenchel's inequality for one-dimensional generalized Gauss graphs. Then we are able to prove a suitable generalization of the slice estimate. Unfortunately, in the general case, the

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classical Morse–Sard theorem does not hold. Nevertheless, we conclude the proof of the diameter estimate by means of a suitable transversality result, implying the "good" behaviour of generalized Gauss graphs with respect to slicing with planes orthogonal to a fixed direction.

In Section 2 we state some notation and recall from [1] just as much as is necessary throughout the paper. In Sections 3 and 4 we prove respectively the Meusnier-type and Fenchel-type results (Theorems 3.4 and 4.1), while, in Section 5, we prove the slice estimate (Theorem 5.9). The original idea giving rise to the diameter estimate for classical surfaces is explained in Remark 5.10 (slice estimate) and Remark 5.12 (diameter estimate). The proof of the generalized estimate (Theorem 5.19), obtained by means of the "transversality result" (Proposition 5.13), concludes Section 5.

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2. General notation and preliminaries. The standard notation of geometric measure theory will be adopted. For example, if U is an open subset of a euclidean space, we let $\mathcal{D}^n(U)$ denote the set of smooth n-forms with compact support in U, equipped with the usual locally convex topology. The usual mass and the normal mass of currents will be denoted by $\mathbf{M}$ and $\mathbf{N}$ respectively. The rectifiable current carried (or *supported*) by $\mathcal{R}$, oriented by ξ and with multiplicity θ will be denoted by $[\![\mathcal{R},\xi,\theta]\!]$.

Throughout this paper we will deal with a generalized notion of Gauss graph immersed in the euclidean space $\mathbb{R}^{n+1}_x\times\mathbb{R}^{n+1}_{\widetilde{y}}$. Let $e_1,\ldots,e_{n+1}$ and $\widetilde{e}_1,\ldots,\widetilde{e}_{n+1}$ be the standard bases of $\mathbb{R}^{n+1}_x$ and $\mathbb{R}^{n+1}_{\widetilde{y}}$ respectively and denote by $\widetilde{z}\in\mathbb{R}^{n+1}_{\widetilde{y}}$ the image of $z\in\mathbb{R}^{n+1}_x$ through the trivial isomorphism $\mathbb{R}^{n+1}_x\ni e_j\mapsto\widetilde{e}_j\in\mathbb{R}^{n+1}_{\widetilde{y}}$, i.e. $\widetilde{z}=\sum_{j=1}^{n+1}z^j\,\widetilde{e}_j$ if $z=\sum_{j=1}^{n+1}z^j\,e_j$. The notation for the generic point of $\mathbb{R}^{n+1}_x\times\mathbb{R}^{n+1}_{\widetilde{y}}$ will be (x,y) or, indifferently, $x+\widetilde{y}$. The one-dimensional linear space generated by a vector $\mathbf{u}\in\mathbb{R}^3_x$ will be denoted by $[\mathbf{u}]$. Given $\xi\in\Lambda^n(\mathbb{R}^{n+1}_x\times\mathbb{R}^{n+1}_{\widetilde{y}})$, ξ_k will denote the kth *stratum* of ξ, i.e.

$$\xi_k=\sum_{\substack{\alpha\in I(n+1,k)\\ \beta\in I(n+1,n-k)}}\xi^{\alpha\beta}\,e_\alpha\wedge\widetilde{e}_\beta,$$

where $I(n+1,j)=\{(\sigma_1,\ldots,\sigma_j)\mid 1\le\sigma_1<\ldots<\sigma_j\le n+1\}$ and

$$e_\alpha=e_{\alpha_1}\wedge\ldots\wedge e_{\alpha_k},\quad \widetilde{e}_\beta=\widetilde{e}_{\beta_1}\wedge\ldots\wedge\widetilde{e}_{\beta_{n-k}},\quad \xi^{\alpha\beta}=\langle\xi,e_\alpha\wedge\widetilde{e}_\beta\rangle.$$

In order to describe the process of slicing our surfaces orthogonally to a fixed unit vector $\mathbf{v}$ in $\mathbb{R}^3_x$, we introduce the couple of *slicing maps* $f:\mathbb{R}^3_x\to\mathbb{R}$

and $\widehat{f}:\mathbb{R}^3_x\times\mathbb{R}^3_{\widetilde{y}}\to\mathbb{R}$ defined by

$$f(x)=x\cdot\mathbf{v}\quad\text{and}\quad\widehat{f}=f\circ p,$$

where $p:\mathbb{R}^3_x\times\mathbb{R}^3_{\widetilde{y}}\to\mathbb{R}^3_x$ is the usual projection. If E is a subset of $\mathbb{R}^3_x$ (resp. $\mathbb{R}^3_x\times\mathbb{R}^3_{\widetilde{y}}$) then the slice $E\cap f^{-1}(t)$ (resp. $E\cap\widehat{f}^{-1}(t)$) will be denoted by E_t.

Also, we will need the map $\varUpsilon:\mathbb{R}^3_x\setminus[\mathbf{v}]\to S^2_x$ defined by

$$\varUpsilon(x)=\frac{x-(x\cdot\mathbf{v})\mathbf{v}}{|x-(x\cdot\mathbf{v})\mathbf{v}|}.$$

The essential reason which makes this map useful is the following: the push-forward, by means of $\varUpsilon$, of the slice G_t of a regular two-dimensional Gauss graph G is the Gauss graph of pG_t. We note that $\operatorname{range}(\varUpsilon)=S^2_x\cap[\mathbf{v}]^\perp$.

We recall some preliminaries from [1].

Definition 2.1 ([1, Definition 2.7]). Let $\varOmega$ be an open subset of $\mathbb{R}^{n+1}_x$. Moreover, let φ and φ^* denote the canonical 1-form and its adjoint, respectively, i.e.

$$\varphi(x,y)=\sum_{j=1}^{n+1}y^j\,dx^j\quad\text{and}\quad\varphi^*(x,y)=\star\varphi(x,y)=\sum_{j=1}^{n+1}\operatorname{sign}(j,\bar{j}\,)y^j\,dx^{\bar{j}}\,.$$

Then we define $\operatorname{curv}_n(\varOmega)$ as the set of n-dimensional rectifiable currents $\varXi=[\![G,\eta,\varrho]\!]$ in $\mathbb{R}^{n+1}_x\times\mathbb{R}^{n+1}_{\widetilde{y}}$ such that:

(i) $\varXi$ is supported in $\varOmega\times S^n_{\widetilde{y}}$, i.e. $G\subset\varOmega\times S^n_{\widetilde{y}}$, and $\varXi(g\varphi^*)=\int_G g|\eta_0|\varrho\,d\mathcal{H}^n$ for all $g\in C_{\mathrm{c}}(\varOmega\times\mathbb{R}^{n+1}_{\widetilde{y}})$,

(ii) $\partial\varXi$ is rectifiable supported in $\varOmega\times S^n_{\widetilde{y}}$ and $\partial\varXi(\varphi\wedge\omega)=0$ for all $\omega\in\mathcal{D}^{n-2}(\varOmega\times\mathbb{R}^{n+1}_{\widetilde{y}})$.

The next proposition makes clearer, from a geometrical point of view, the hypothesis (i) in Definition 2.1.

Proposition 2.2 ([1, Remark 2.3]). *If $\varXi=[\![G,\eta,\varrho]\!]$ is supported in $\varOmega\times S^n_{\widetilde{y}}$, then the condition*

$$\varXi(g\varphi^*)=\int_G g|\eta_0|\,\varrho\,d\mathcal{H}^n\qquad\textit{for all } g\in C_{\mathrm{c}}(\varOmega\times\mathbb{R}^{n+1}_{\widetilde{y}})$$

is equivalent to

$$\begin{aligned}\varXi(\varphi\wedge\omega)&=0\qquad\textit{for all }\omega\in\mathcal{D}^{n-1}(\varOmega\times\mathbb{R}^{n+1}_{\widetilde{y}})\qquad\textit{and}\\ \varXi(g\varphi^*)&\ge 0\qquad\textit{for all } g\in C_{\mathrm{c}}(\varOmega\times\mathbb{R}^{n+1}_{\widetilde{y}})\textit{ with } g\ge 0.\end{aligned}$$

The following theorem gives us some information about the structure of the currents belonging to $\operatorname{curv}_n(\varOmega)$.

THEOREM 2.3 ([1, Theorem 2.9]). *Let* $\Xi = [\![G, \eta, \varrho]\!] \in \mathrm{curv}_n(\Omega)$. *Then*

(i) $v \cdot y = 0$ *for* $\mathcal{H}^n$*-a.e.* $(x, y) \in G$ *and for all* $v \in T_{(x,y)}G$,

(ii) $p_{|G}^{-1} \subset \{(x, \zeta(x)), (x, -\zeta(x))\}$ *for* $\mathcal{H}^n$*-a.e.* $x \in M = pG$, *where* $p : \mathbb{R}_x^{n+1} \times \mathbb{R}_{\widetilde{y}}^{n+1} \to \mathbb{R}_x^{n+1}$ *is the usual projection and* $\zeta : M \to S^n$ *is an* $\mathcal{H}^n$*-measurable map such that* $\zeta(x) \in (T_x M)^\perp$ $\mathcal{H}^n$*-a.e. on* M.

3. A Meusnier-type result. If M is an n-dimensional C^2 surface embedded in $\mathbb{R}_x^{n+1}$, oriented by a continuous normal vector field $\nu : M \to S_x^n \subset \mathbb{R}_x^{n+1}$, then we will denote by $\mathbf{II}_x$ the second fundamental form of M at x, while Φ will be the Gauss-graph map, i.e.

$$\Phi : M \to M \times S_{\widetilde{y}}^n, \qquad x \mapsto (x, \nu(x)).$$

The graph of ν, $\Phi(M)$, will be denoted by G. The tangent planes to G at (x, y) and to M at x will be denoted by $T(x, y)$ and $T_0(x, y)$ respectively (note that $T_0(x, y) = p(T(x, y))$. Moreover, let

$$\tau(x) = \star\nu(x) \quad \text{for all } x \in M$$

and

$$\xi(x, y) = \Lambda^n d\Phi_x(\tau(x)) \quad \text{for all } (x, y) \in G.$$

Then an orientation of G is given by $\eta = \xi/|\xi|$.

Also, let us recall (see, for example, [6]) that, for each $x \in M$, there exists an orthonormal basis $\tau_1(x), \ldots, \tau_n(x)$ of $T_x M$ and a set of numbers $\kappa_1(x), \ldots, \kappa_n(x)$, called respectively *principal directions of curvature* and *principal curvatures* of M at x, such that

$$d\Phi_x(\tau_i(x)) = \tau_i(x) + \kappa_i(x)\widetilde{\tau_i(x)}.$$

From now on, we will restrict ourselves to the case of two-dimensional surfaces in $\mathbb{R}_x^3$, although something in what follows could be easily stated even for higher-dimensional surfaces. Moreover, for brevity, we will often omit in formulas the obvious arguments x, (x, y) and Φ.

Remark 3.1 (how to recover $\mathbf{II}$ from η). As

$$\xi = (\tau_1 + \kappa_1\widetilde{\tau}_1) \wedge (\tau_2 + \kappa_2\widetilde{\tau}_2) = \underbrace{\tau_1 \wedge \tau_2}_{\xi_0} + \underbrace{\kappa_2\tau_1 \wedge \widetilde{\tau}_2 - \kappa_1\tau_2 \wedge \widetilde{\tau}_1}_{\xi_1} + \underbrace{\kappa_1\kappa_2\widetilde{\tau}_1\widetilde{\tau}_2}_{\xi_2}$$

it is not difficult to verify that, for every tangent vector $\mathbf{u}$,

$$\mathbf{II}(\mathbf{u}) = (\xi_1, (\tau \llcorner \mathbf{u}) \wedge \widetilde{\mathbf{u}}), \quad \text{i.e.} \quad |\eta_0|^2\mathbf{II}(\mathbf{u}) = (\eta_1, (\eta_0 \llcorner \mathbf{u}) \wedge \widetilde{\mathbf{u}}).$$

Remark 3.2 (Meusnier's formula in terms of η and Υ). Let $Q_0 = (x_0, \nu(x_0)) \in G$ be a regular point for the slicing function f and let $t_0 = f(Q_0)$. Then G_{t_0} has to be a regular curve, namely of class C^2, in a neighbourhood of Q_0, and $\mathbf{v}^{T_0}$ (i.e. the projection of $\mathbf{v}$ on T_0) cannot vanish along this regular arc since $\mathbf{v}^{T_0} = \nabla^M f$. It follows that, in a neighbourhood of

x_0, $\Upsilon \circ \nu_{|M_{t_0}}$ is the Gauss map of M_{t_0} considered immersed in the plane $P_{t_0} = (\mathbb{R}^3_x)_{t_0}$. Let

$$\Gamma = (\Gamma_0, \Gamma_1) : [-\varepsilon, \varepsilon] \to G_{t_0}$$

be a C^2 parametrization by arc length of a piece of G_{t_0} such that $\Gamma(0) = Q_0$ and let us denote by κ and $\mathbf{n}$ respectively the scalar curvature and the normal vector of M_{t_0} (in a neighbourhood of x_0). Recalling that $\Gamma_1 = \nu \circ \Gamma_0$, we can easily recover the scalar curvature κ from Υ:

$$|\kappa|\,|\dot{\Gamma}_0| = |d(\Upsilon \circ \nu)(\dot{\Gamma}_0)| = |d\Upsilon(d\nu(\dot{\Gamma}_0))| = |d\Upsilon(d(\nu \circ \Gamma_0))| = |d\Upsilon(\dot{\Gamma}_1)|.$$

By Remark 3.1 and recalling that $|\mathbf{v}^{T_0}| = |\nu \cdot \mathbf{n}|$, we can write the Meusnier formula (see, for example, [2])

$$|\mathbf{II}(\dot{\Gamma}_0)| = |\dot{\Gamma}_0|^2 |\kappa\, \nu \cdot \mathbf{n}|$$

as follows:

$$|(\eta_1, (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0)| = |d\Upsilon(\dot{\Gamma}_1)|\,|\dot{\Gamma}_0|\,|\mathbf{v}^{T_0}|\,|\eta_0|^2.$$

Finally, we remark that the transversality condition

$$\mathbf{v}^{T_0} = \nabla^M f \neq 0 \qquad \text{along } M_t$$

holds for a.e. $t \in \mathbb{R}$, as follows from the Morse–Sard theorem (see [4]).

Before stating the Meusnier-type theorem, we give the following simple lemma.

Lemma 3.3. *Let T be a two-dimensional linear subspace of $\mathbb{R}^3_x \times \mathbb{R}^3_y$ and $T_0 = pT$. Then*

(i) *given $\mathbf{v} \in \mathbb{R}^3_x$, one has $\mathbf{v}^{T_0} = 0$ if and only if $\mathbf{v}^T = 0$,*
(ii) *given $\mathbf{w} \in T$, one has $\mathbf{w}_0 \cdot \mathbf{u}^{T_0} = \mathbf{w} \cdot \mathbf{u}^T$ for all $\mathbf{u} \in \mathbb{R}^3_x$.*

Proof. (i) trivially follows from $T_0 = pT$ since $\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}_0$ for every $\mathbf{w} \in \mathbb{R}^3_x \times \mathbb{R}^3_y$ (and thus, in particular, for every $\mathbf{w} \in T$).

As far as (ii) is concerned, we note that $\mathbf{w} \cdot \mathbf{u}^T = \mathbf{w} \cdot \mathbf{u} = \mathbf{w}_0 \cdot \mathbf{u}$. Moreover, $\mathbf{w}_0 = p\mathbf{w} \in pT = T_0$, whence $\mathbf{w}_0 \cdot \mathbf{u} = \mathbf{w}_0 \cdot \mathbf{u}^{T_0}$. ■

Now we are ready to prove the main theorem of this section. As we will see in Section 3, the hypotheses will be satisfied by a parametrization Γ of almost every slice of a generalized Gauss graph.

Theorem 3.4 (Meusnier-type). *Let $\Gamma = (\Gamma_0, \Gamma_1) : [-\varepsilon, \varepsilon] \to \mathbb{R}^3_x \times S^2$ be a Lipschitz map differentiable at 0 and such that*

(i) $|\dot{\Gamma}(0)| = 1$,
(ii) $\Gamma_1(0) \cdot \dot{\Gamma}_0(0) = 0$.

Moreover, let η and $\mathbf{v}$ be respectively a unit simple two-vector in $\mathbb{R}^3_x \times S^2$ and a unit vector in $\mathbb{R}^3_x$ such that (with $T_0 = pT$, where T is the two-dimensional linear subspace determined by η)

(iii) $\eta_0 \neq 0$,
(iv) $\mathbf{v}^{T_0} \neq 0$,
(v) $\dot{\Gamma}(0) \wedge \eta = 0$,
(vi) $\Gamma_1(0)$ *is orthogonal to* T_0,
(vii) $\dot{\Gamma}_0(0) \cdot \mathbf{v}^{T_0} = 0$.

Then $\Gamma_1(0) \neq \pm\mathbf{v}$ *and we have the Meusnier formula*

$$|(\eta_1, (\eta_0 \llcorner \dot{\Gamma}_0(0)) \wedge \widetilde{\dot{\Gamma}}_0(0))| = |d(\Upsilon)_{\Gamma_1(0)}(\dot{\Gamma}_1(0))|\,|\dot{\Gamma}_0(0)|\,|\mathbf{v}^{T_0}|\,|\eta_0|^2.$$

P r o o f. From (iii) and (vi) it immediately follows that

$$\mathbf{v}^{T_0} = \mathbf{v} - (\mathbf{v} \cdot \Gamma_1(0))\Gamma_1(0). \tag{3.1}$$

Then (iv) implies that $\Gamma_1(0) \neq \pm\mathbf{v}$, whence the right side of the formula is well defined.

In proving the formula, we can suppose $\dot{\Gamma}_0(0) \neq 0$ (otherwise the formula holds trivially). Moreover, for brevity we shall write simply $d\Upsilon$ instead of $d(\Upsilon)_{\Gamma_1(0)}$ and we will omit the argument of Γ (and of its derivative) understanding that it is 0, while it will be specified in the other cases.

By (i), (v), (vii) and Lemma 3.3(ii) (choosing $\mathbf{w} = \dot{\Gamma}$) one has

$$|\mathbf{v}^T|\eta = \dot{\Gamma} \wedge \mathbf{v}^T = (\dot{\Gamma}_0 + \widetilde{\dot{\Gamma}}_1) \wedge (\mathbf{v}_0^T + \mathbf{v}^T - \mathbf{v}_0^T). \tag{3.2}$$

In particular,

$$|\mathbf{v}^T|\eta_1 = \dot{\Gamma}_0 \wedge (\mathbf{v}^T - \mathbf{v}_0^T) - \mathbf{v}_0^T \wedge \widetilde{\dot{\Gamma}}_1$$

and thus

$$\begin{aligned} |\mathbf{v}^T|(\eta_1, (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0) &= (\dot{\Gamma}_0 \wedge (\mathbf{v}^T - \mathbf{v}_0^T), (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0) \\ &\quad - (\mathbf{v}_0^T \wedge \widetilde{\dot{\Gamma}}_1, (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0) \\ &= \underbrace{(\dot{\Gamma}_0, \eta_0 \llcorner \dot{\Gamma}_0)}_{=0}(\mathbf{v}^T - \mathbf{v}_0^T, \widetilde{\dot{\Gamma}}_0) \\ &\quad - (\mathbf{v}_0^T, \eta_0 \llcorner \dot{\Gamma}_0)(\dot{\Gamma}_1, \dot{\Gamma}_0). \end{aligned}$$

As $|\mathbf{v}^T|\eta_0 = \dot{\Gamma}_0 \wedge \mathbf{v}_0^T$ (by (3.2)), we obtain

$$\begin{aligned} |\mathbf{v}^T|^2(\eta_1, (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0) &= -(\mathbf{v}_0^T, |\mathbf{v}^T|\eta_0 \llcorner \dot{\Gamma}_0)(\dot{\Gamma}_1, \dot{\Gamma}_0) \\ &= -(\mathbf{v}_0^T, (\dot{\Gamma}_0 \wedge \mathbf{v}_0^T) \llcorner \dot{\Gamma}_0)(\dot{\Gamma}_1, \dot{\Gamma}_0) \\ &= -|\dot{\Gamma}_0 \wedge \mathbf{v}_0^T|^2(\dot{\Gamma}_1, \dot{\Gamma}_0) = -|\mathbf{v}^T|^2|\eta_0|^2(\dot{\Gamma}_1, \dot{\Gamma}_0), \end{aligned}$$

i.e.

$$(\eta_1, (\eta_0 \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0) = -|\eta_0|^2(\dot{\Gamma}_1, \dot{\Gamma}_0) \tag{3.3}$$

since $\mathbf{v}^T \neq 0$ (as $\mathbf{v}^{T_0} \neq 0$ and by recalling Lemma 3.3(i)).

The proof will be complete once we show that the right side in the Meusnier formula can be reduced to the right side of (3.3).

Let us start by computing $d\Upsilon(\dot\Gamma_1)$. Recalling (3.1) and (iv) again, it is easy to check that

$$\begin{aligned} d\Upsilon(\dot\Gamma_1) &= \frac{d}{ds}\Big|_{s=0} \Upsilon\circ\Gamma_1(s) \\ &= \frac{\dot\Gamma_1 - (\dot\Gamma_1\cdot\mathbf{v})\mathbf{v}}{(1-(\Gamma_1\cdot\mathbf{v})^2)^{1/2}} + \frac{(\Gamma_1\cdot\mathbf{v})(\dot\Gamma_1\cdot\mathbf{v})(\Gamma_1-(\Gamma_1\cdot\mathbf{v})\mathbf{v})}{(1-(\Gamma_1\cdot\mathbf{v})^2)^{3/2}}, \end{aligned}$$

whence

$$\begin{aligned} |d\Upsilon(\dot\Gamma_1)|^2 &= \frac{|\dot\Gamma_1|^2-(\dot\Gamma_1\cdot\mathbf{v})^2}{1-(\Gamma_1\cdot\mathbf{v})^2} + \frac{(\Gamma_1\cdot\mathbf{v})^2(\dot\Gamma_1\cdot\mathbf{v})^2}{(1-(\Gamma_1\cdot\mathbf{v})^2)^2} \\ &\quad + 2\frac{(\Gamma_1\cdot\mathbf{v})(\dot\Gamma_1\cdot\mathbf{v})(\Gamma_1\cdot\dot\Gamma_1-(\Gamma_1\cdot\mathbf{v})(\dot\Gamma_1\cdot\mathbf{v}))}{(1-(\Gamma_1\cdot\mathbf{v})^2)^2} \\ &= \frac{|\dot\Gamma_1|^2-|\dot\Gamma_1|^2(\Gamma_1\cdot\mathbf{v})^2-(\dot\Gamma_1\cdot\mathbf{v})^2}{(1-(\Gamma_1\cdot\mathbf{v})^2)^2} \end{aligned}$$

since $\Gamma_1\cdot\dot\Gamma_1=0$ (as $\Gamma_1(s)\cdot\Gamma_1(s)=1$ for all s), i.e.

$$|\mathbf{v}^{T_0}|^4\,|d\Upsilon(\dot\Gamma_1)|^2 = |\dot\Gamma_1|^2|\mathbf{v}^{T_0}|^2-(\dot\Gamma_1\cdot\mathbf{v})^2.$$

We now have to prove the following formula:

$$|\dot\Gamma_0|^2(|\dot\Gamma_1|^2\,|\mathbf{v}^{T_0}|^2-(\dot\Gamma_1\cdot\mathbf{v})^2) = (\dot\Gamma_1\cdot\dot\Gamma_0)^2|\mathbf{v}^{T_0}|^2. \tag{3.4}$$

We can suppose $\dot\Gamma_1\neq 0$, since otherwise (3.4) is trivial. Let β be the angle between $\dot\Gamma_0$ and $\dot\Gamma_1$ and let ε be a vector chosen in such a way that ε, $\dot\Gamma_0/|\dot\Gamma_0|$ and $\mathbf{v}$ form an orthonormal basis of $\mathbb{R}^3_x$ (this is possible since, by (v) and (vii), one has $\dot\Gamma_0\cdot\mathbf{v}=\dot\Gamma_0\cdot(\mathbf{v}-\mathbf{v}^{T_0})+\dot\Gamma_0\cdot\mathbf{v}^{T_0}=\dot\Gamma_0\cdot\mathbf{v}^{T_0}=0$).

Then, again from (ii), it follows that $\Gamma_1=(\Gamma_1\cdot\varepsilon)\varepsilon+(\Gamma_1\cdot\mathbf{v})\mathbf{v}$, whence there must exist α such that

$$\Gamma_1\cdot\varepsilon=\cos\alpha \quad\text{and}\quad \Gamma_1\cdot\mathbf{v}=\sin\alpha.$$

Moreover, we note that:

(a) $|\mathbf{v}^{T_0}|^2=\cos^2\alpha$, because of (3.1);

(b) the vector

$$\mathbf{u}=\dot\Gamma_1-\left(\dot\Gamma_1\cdot\frac{\dot\Gamma_0}{|\dot\Gamma_0|}\right)\frac{\dot\Gamma_0}{|\dot\Gamma_0|}$$

belongs to the plane spanned by ε, v and it is orthogonal to Γ_1. In particular,

$|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\varGamma_1\cdot\varepsilon| = |\mathbf{u}|\,|\cos\alpha|$, so that

$$(\dot\varGamma_1\cdot\mathbf{v})^2 = |\mathbf{u}\cdot\mathbf{v}|^2 = |\mathbf{u}|^2\cos^2\alpha = \left\{|\dot\varGamma_1|^2 - \left(\dot\varGamma_1\cdot\frac{\dot\varGamma_0}{|\dot\varGamma_0|}\right)^2\right\}\cos^2\alpha$$
$$= |\dot\varGamma_1|^2\sin^2\beta\cos^2\alpha.$$

Now it is trivial to check (3.4). ■

4. A Fenchel-type result

THEOREM 4.1 (Fenchel-type). *Let $\Sigma\in\mathrm{curv}_1(\mathbb{R}^2_x)$ be such that $\Sigma\neq 0$ and $\partial\Sigma = 0$. Then $\mathbf{M}(\Sigma_1)\ge 2\pi$.*

Proof. Because it is always possible to find an indecomposable nontrivial component Σ^* of Σ without boundary and as $\mathbf{M}(\Sigma_1)\ge\mathbf{M}((\Sigma^*)_1)$, we can assume, without loss of generality, that Σ itself is indecomposable. Then, by [3; 4.2.25], there exists an injective Lipschitz map $\varLambda = (\varLambda_0,\widetilde{\varLambda}_1)$: $[0,\mathbf{M}(\Sigma)]\to\mathbb{R}^2_x\times S^1$ such that $\varLambda_\#[0,\mathbf{M}(\Sigma)] = \Sigma$ and $|\dot\varLambda| = 1$ a.e. in $[0,\mathbf{M}(\Sigma)]$.

In particular (for $\Sigma = [\![R,\upsilon,\theta]\!]$) it follows that $R = \varLambda([0,\mathbf{M}(\Sigma)])$, and

$$\dot\varLambda = \upsilon\circ\varLambda \qquad \text{a.e. in } [0,\mathbf{M}(\Sigma)]. \tag{4.1}$$

We note that the statement trivially follows whenever $\dot\varLambda_0\equiv 0$. Indeed, we then have $\varLambda_0 = \text{constant} = \overline{x}$ and so $R = \{\overline{x}\}\times S^1$, whence $\mathbf{M}(\Sigma_1) = \mathcal{H}^1(S^1) = 2\pi$.

Thus, from now on, we can assume that

$$\dot\varLambda_0 \not\equiv 0. \tag{4.2}$$

We need the following lemma that we shall prove later.

LEMMA 4.2. *Let $\gamma : [0,l]\to\mathbb{R}^2$ be an integrable map such that*

(i) $\int_0^l \gamma(s)\,ds = 0$,

(ii) $\mathrm{image}(\gamma)\subset S_\alpha = \{(\varrho\cos\theta,\varrho\sin\theta)\mid \varrho\ge 0,\ \theta\in[\alpha,\alpha+\pi)\}$ *for some* $\alpha\in[0,2\pi)$.

Then γ is identically zero.

Now we apply the lemma with $\gamma = \dot\varLambda_0$ and $l = \mathbf{M}(\Sigma)$ to conclude (by (4.2)) that there is no α in $[0,2\pi)$ such that $\dot\varLambda_0(s)\in S_\alpha$ for every s in $[0,\mathbf{M}(\Sigma)]$.

But $\dot\varLambda_0(s) = |\dot\varLambda_0|\star\varLambda_1(s)$ for a.e. s, just by definition of $\mathrm{curv}_1(\mathbb{R}^2_x)$, so that the previous statement is equivalent to the following:

(4.3) there is no α in $[0,2\pi)$ such that $\varLambda_1(s)\in S_\alpha$ for all s in $[0,\mathbf{M}(\Sigma)]$.

By (4.3) together with the compactness and connectedness of $\Lambda_1([0,\mathbf{M}(\Sigma)])$ implied by the continuity of Λ_1, we obtain

$$\mathcal{H}^1(\Lambda_1([0,\mathbf{M}(\Sigma)])) \geq \pi.$$

Then we can find s_1, s_2 in $[0,\mathbf{M}(\Sigma)]$ (with $s_1 < s_2$) such that $\Lambda_1(s_1) = -\Lambda_1(s_2)$. It follows that

$$\int_{s_1}^{s_2} |\dot{\Lambda}_1(s)|\, ds \geq \pi$$

and, since $\Lambda_1(0) = \Lambda_1(\mathbf{M}(\Sigma))$, also that

$$\int_{s_2}^{\mathbf{M}(\Sigma)} |\dot{\Lambda}_1(s)|\, ds + \int_0^{s_1} |\dot{\Lambda}_1(s)|\, ds \geq \pi.$$

Now the conclusion immediately follows by recalling that

$$\mathbf{M}(\Sigma_1) = \int_0^{\mathbf{M}(\Sigma)} |\dot{\Lambda}_1(s)|\, ds$$

by (4.1). ∎

P r o o f o f L e m m a 4.2. It is enough to prove the assertion for $\alpha = 0$. In this case $\gamma_2 \geq 0$ and as $\int_0^l \gamma_2(s)\, ds = 0$ it follows that γ_2 is identically zero. Then

$$\mathrm{image}(\gamma) \subset S_0 \cap \mathbb{R}_x \times 0 = \{(x,0) \mid x \geq 0\},$$

i.e. $\gamma_1 \geq 0$ and then, as $\int_0^l \gamma_1(s)\, ds = 0$, also γ_1 has to be identically zero. ∎

5. Estimating the diameter

LEMMA 5.1. *Let η, y be respectively a simple two-vector in $\mathbb{R}^3_x \times \mathbb{R}^3_y$ and a unit vector in $\mathbb{R}^3_x$ such that*

$$(\star y, \eta_0) = |\eta_0|,$$

where $\star$ is the Hodge operator in $\mathbb{R}^3_x$ with respect to the canonical basis e_1, e_2, e_3. Then, for any unit vector $\mathbf{v}$ in $\mathbb{R}^3_x$, one has

$$(\eta \llcorner \mathbf{v}^T)_0 = \eta_0 \llcorner \mathbf{v}^{T_0} = -|\eta_0|\, |y - (y\cdot \mathbf{v})\mathbf{v}| \bullet \Upsilon(y),$$

where T is the two-dimensional linear space related to η, $T_0 = pT$ and $\bullet$ denotes the Hodge operator in $[\mathbf{v}]^\perp \cong \mathbb{R}^2$ with respect to an ordered orthonormal basis e_1', e_2' such that e_1', e_2', $\mathbf{v}$ is canonically oriented.

P r o o f. Without loss of generality, we can assume $\mathbf{v} = e_3$ and $e_1' = e_1$, $e_2' = e_2$.

As $e_3 - e_3^{T_0}$ is orthogonal to the linear space oriented by η_0, one has $\eta_0 \llcorner e_3^{T_0} = \eta_0 \llcorner e_3$. Analogously, $\eta \llcorner e_3^T = \eta \llcorner e_3$ and then also $(\eta \llcorner e_3^T)_0 =$

$(\eta \llcorner e_3)_0 = \eta_0 \llcorner e_3$. Moreover, by hypothesis, $\eta_0 = |\eta_0| \star y$. It follows that

$$(\eta \llcorner e_3^T)_0 = \eta_0 \llcorner e_3^{T_0} = \eta_0 \llcorner e_3 = |\eta_0|(\star y) \llcorner e_3$$
$$= |\eta_0|(-y_1 e_2 + y_2 e_1) = -|\eta_0|(y_1^2 + y_2^2)^{1/2} \bullet \varUpsilon(y). \ \blacksquare$$

LEMMA 5.2. *Let* $\mathbf{v}$ *and* X *be respectively a unit vector and a linear subspace of* $\mathbb{R}^3_x$ *such that* $X^\perp \cap S^2 \setminus \{\pm\mathbf{v}\}$ *is not empty. Then* $\varUpsilon_{|X^\perp \cap S^2 \setminus \{\pm\mathbf{v}\}}$ *is injective if and only if one of the following conditions holds*:

(i) $\dim X = 2$,
(ii) $\dim X = 1$ *and* $\mathbf{v}^X \neq 0$.

Proof. Let $\mathbf{w}$ be any vector in $S^2 \cap [\mathbf{v}]^\perp$ and consider the open half-plane

$$H_{\mathbf{w}} = \{s\mathbf{v} + t\mathbf{w} \mid s, t \in \mathbb{R} \text{ and } t > 0\}.$$

Then the assertion is a straightforward consequence of the following easy statement:

$$\varUpsilon_{|H_{\mathbf{w}} \cap S^2} = \text{constant} = \mathbf{w}. \ \blacksquare$$

Let $\varXi = [\![G, \eta, \varrho]\!] \in \operatorname{curv}_2(\mathbb{R}^3_x)$ be such that $\partial \varXi = 0$ and consider the function $\widehat{f} : \mathbb{R}^3_x \times \mathbb{R}^3_{\widetilde{y}} \to \mathbb{R}$ defined as in Section 2: $\widehat{f}(x, y) = x \cdot \mathbf{v}$. The following remarks will be useful to prove the next theorem.

Remark 5.3. From the general slicing theory (see [3], [5], [7]), we know that $\varXi_t = \langle \varXi, \widehat{f}, t\rangle$ is a null-boundary one-dimensional rectifiable current for a.e. $t \in \mathbb{R}$. More precisely,

(5.1) the tangent plane T to G exists and $\mathbf{v}^T \neq 0$ $\mathcal{H}^1$-a.e. along $G_t = \widehat{f}^{-1}(t) \cap G$

for a.e. $t \in \mathbb{R}$, and $\varXi_t = [\![G_t, \upsilon_t, \theta_t]\!]$, where $\upsilon_t = \eta \llcorner (\mathbf{v}^T/|\mathbf{v}^T|)$ and $\theta_t = \varrho_{|G_t}$. It follows that

$$|\mathbf{v}^T|\eta = \upsilon_t \wedge \mathbf{v}^T \quad \mathcal{H}^1\text{-a.e. along } G_t \text{ for a.e. } t \in \mathbb{R}. \tag{5.2}$$

Remark 5.4. As $\mathbf{v}^T = 0$ whenever $\eta = \eta_2$, (5.1) implies that $\eta \neq \eta_2$ $\mathcal{H}^1$-a.e. along G_t. In particular, if (5.1) holds true, then also

$$\eta_1 \neq 0 \quad \mathcal{H}^1\text{-a.e. along } G_t \cap \{\eta_0 = 0\}. \tag{5.3}$$

Remark 5.5. Let $\mathbf{w}_i = \mathbf{u}_i + \widetilde{\mathbf{v}}_i$ $(i = 1, 2)$ be a couple of $\mathcal{H}^2 \llcorner G$-measurable orthonormal vector fields such that $\eta = \mathbf{w}_1 \wedge \mathbf{w}_2$ $\mathcal{H}^2 \llcorner G$-a.e., i.e.

$$\eta_0 = \mathbf{u}_1 \wedge \mathbf{u}_2, \quad \eta_1 = \mathbf{u}_1 \wedge \widetilde{\mathbf{v}}_2 - \mathbf{u}_2 \wedge \widetilde{\mathbf{v}}_1 \quad \text{and} \quad \eta_2 = \widetilde{\mathbf{v}}_1 \wedge \widetilde{\mathbf{v}}_2 \quad \mathcal{H}^2 \llcorner G\text{-a.e.} \tag{5.4}$$

It follows that

$$\text{(5.4) holds } \mathcal{H}^1\text{-a.e. along } G_t \text{ for a.e. } t \in \mathbb{R}. \tag{5.5}$$

Remark 5.6. Let $\star$ denote the Hodge operator in $\mathbb{R}^3_x$ with respect to the canonical basis e_1, e_2, e_3. Then

$$(\star y, \eta_0) = |\eta_0| \quad \mathcal{H}^1\text{-a.e. along } G_t \text{ for a.e. } t \in \mathbb{R}, \tag{5.6}$$

since $(\star y, \eta_0) = |\eta_0|\ \mathcal{H}^2 \llcorner G$-a.e. (by Definition 2.1).

Remark 5.7. One can always find two disjoint rectifiable sets G^1 and G^2 such that $G^1 \cup G^2 = G$ and $p_i = p_{|G^i}$ is injective ($i = 1, 2$) (see Theorem 2.3(ii)). From $\mathcal{H}^2(p\{\eta_0 = 0\}) = 0$ it follows that

$$\int_{G^i \cap \{\eta_0 \neq 0\}} |\eta_1|\, d\mathcal{H}^2 = \int_{pG^i} \frac{|\eta_1 \circ p_i^{-1}|}{|\eta_0 \circ p_i^{-1}|}\, d\mathcal{H}^2$$

and

$$\eta_0 \circ p_i^{-1} \neq 0 \quad \mathcal{H}^1\text{-a.e. along } (pG^i)_t \text{ for a.e. } t \in \mathbb{R}.$$

By Remark 5.3 and Lemma 3.3(i), these easily imply that

$$\int_{G_t^i \cap \{\eta_0 \neq 0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 = \int_{pG_t^i} \frac{|\eta_1 \circ p_i^{-1}|}{|\eta_0 \circ p_i^{-1}||\mathbf{v}^{T_0} \circ p_i^{-1}|}\, d\mathcal{H}^1 \quad \text{for a.e. } t \in \mathbb{R}, \tag{5.7}$$

where $G_t^i = G_t \cap G^i$.

Remark 5.8. For a.e. $t \in \mathbb{R}$ one can find an indecomposable null-boundary component of Ξ_t which will be denoted by $\Xi_t^* = [\![G_t^*, \upsilon_t^*, \theta_t^*]\!]$ (let us note that $\upsilon_t^* = \upsilon_{t|G_t^*}$ and $\theta_t^* = \theta_{t|G_t^*} = \varrho_{|G_t^*}$). We stress the obvious statement that Ξ_t^* can be chosen to be non-trivial if Ξ_t is. By [3; 4.2.25], there exists a map $\Gamma_t^* = ((\Gamma_t^*)_0, (\widetilde{\Gamma}_t^*)_1) : [0, \mathcal{H}^1(G_t^*)] \to \mathbb{R}^3_x \times S^2$ such that

$$\Gamma_t^* \text{ is a Lipschitz parametrization of } G_t^* \text{ and } |\dot{\Gamma}_t^*| = 1 \text{ a.e.} \tag{5.8}$$

Recalling Theorem 2.3(i), we find immediately that (omitting for simplicity the symbols t and $*$)

$$T_0 \subset [\Gamma_1]^\perp \quad \text{a.e. in } [0, \mathcal{H}^1(G_t^*)] \text{ for a.e. } t \in \mathbb{R} \tag{5.9}$$

and, moreover, we can easily apply Theorem 3.4 to find that

$$\begin{aligned} &|(\eta_1 \circ \Gamma, ((\eta_0 \circ \Gamma) \llcorner \dot{\Gamma}_0) \wedge \widetilde{\dot{\Gamma}}_0)| \\ &= |d\Upsilon(\dot{\Gamma}_1)||\dot{\Gamma}_0||\mathbf{v}^{T_0} \circ \Gamma||\eta_0 \circ \Gamma|^2 \quad \text{a.e. in } [0, \mathcal{H}^1(G_t^*)] \text{ for a.e. } t \in \mathbb{R}. \end{aligned} \tag{5.10}$$

Now, let us denote by J the null-measure subset of $\mathbb{R}$ outside of which all the properties pointed out by the foregoing remarks hold. Moreover, let $\psi : \mathbb{R}^3_x \times (\mathbb{R}^3_{\tilde{y}} \setminus [\tilde{\mathbf{v}}]) \to \mathbb{R}^3_x \times [\tilde{\mathbf{v}}]^\perp$ be defined by $\psi = \mathbf{1} \oplus \Upsilon$, i.e. $\psi(x, y) = (x, \Upsilon(y))$.

Theorem 5.9. *Let $t \in \mathbb{R} \setminus J$ by such that*

(i) *Ξ_t is non-trivial*,
(ii) $\mathbf{v} \notin \text{image}(\Gamma_1)$.

Then

$$\int_{G_t^*} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \geq \|(\psi_\# \Xi_t^*)_1\| \geq 2\pi.$$

P r o o f. Without loss of generality we can suppose Ξ_t to be indecomposable, i.e. $\Xi_t^* = \Xi_t$. By (ii), $\Sigma = -\psi_\# \Xi_t$ is a well defined rectifiable current. We first show that

$$\partial \Sigma = 0 \quad \text{and} \quad \Sigma \in \mathrm{curv}_1([\mathbf{v}]^\perp). \tag{5.11}$$

The first equality immediately follows from $\partial \Xi_t = 0$ taking into account (ii). The second is proved as follows.

Lemma 5.1 and (5.6) imply that

$$(\bullet\Upsilon(y), (\eta \llcorner \mathbf{v}^T)_0) = -|(\eta \llcorner \mathbf{v}^T)_0| \quad \mathcal{H}^1\text{-a.e. along } G_t.$$

By the transversality condition (5.1), we can restate this as

$$(\bullet\Upsilon(y), (\upsilon_t)_0) = -|(\upsilon_t)_0| \quad \mathcal{H}^1\text{-a.e. along } G_t. \tag{5.12}$$

Taking into account (5.1) together with Lemma 3.3(i) and using then Lemma 5.2 (with $X = T_0$), we can assume that $\psi_{|G_t}$ is injective. It follows that

$$\mathcal{H}^1(\psi\{(x,y) \in G_t \mid d\psi(\upsilon_t(x,y)) = 0\}) = 0$$

and

$$\Sigma = \left[\!\!\left[\psi(G_t), -\frac{d\psi(\upsilon_t \circ \psi^{-1})}{|d\psi(\upsilon_t \circ \psi^{-1})|}, 1 \right]\!\!\right].$$

Then, if g is any function with compact support and $\varphi^\bullet$ denotes the Hodge transform of the canonical one-form in $[\mathbf{v}]^\perp$, one has

$$\Sigma(g\varphi^\bullet) = -\int_{\psi(G_t)} g \left\langle \frac{d\psi(\upsilon_t \circ \psi^{-1})}{|d\psi(\upsilon_t \circ \psi^{-1})|}, \varphi^\bullet \right\rangle d\mathcal{H}^1$$

and therefore, since $\langle d\psi(\upsilon_t \circ \psi^{-1}), \varphi^\bullet\rangle = \langle (\upsilon_t \circ \psi^{-1})_0, \varphi^\bullet \rangle = -|(\upsilon_t \circ \psi^{-1})_0|$ by (5.12), we obtain

$$\Sigma(g\varphi^\bullet) = \int_{\psi(G_t)} g \frac{|(\upsilon_t \circ \psi^{-1})_0|}{|d\psi(\upsilon_t \circ \psi^{-1})|}\, d\mathcal{H}^1 = \int_{\psi(G_t)} g \left| \left(\frac{d\psi(\upsilon_t \circ \psi^{-1})}{|d\psi(\upsilon_t \circ \psi^{-1})|} \right)_0 \right| d\mathcal{H}^1,$$

which is just the integral condition in the definition of $\mathrm{curv}_1([\mathbf{v}]^\perp)$. This concludes the proof of (5.11).

Now, consider the decomposition

$$\int_{G_t} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 = \underbrace{\int_{G_t \cap \{\eta_0 \neq 0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1}_{I_1} + \underbrace{\int_{G_t \cap \{\eta_0 = 0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1}_{I_2}. \tag{5.13}$$

For the first integral, we note that, by (5.7),

$$\begin{aligned} I_1 &= \int\limits_{G_t^1\cap\{\eta_0\neq 0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 + \int\limits_{G_t^2\cap\{\eta_0\neq 0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \\ &= \int\limits_{pG_t^1} \frac{|\eta_1\circ p_1^{-1}|}{|\eta_0\circ p_1^{-1}||\mathbf{v}^{T_0}|}\, d\mathcal{H}^1 + \int\limits_{pG_t^2} \frac{|\eta_1\circ p_2^{-1}|}{|\eta_0\circ p_2^{-1}||\mathbf{v}^{T_0}|}\, d\mathcal{H}^1 \\ &= \int\limits_{[0,\mathcal{H}^1(G_t)]} \frac{|\eta_1\circ\varGamma||\dot{\varGamma}_0|}{|\eta_0\circ\varGamma||\mathbf{v}^{T_0}\circ\varGamma|}\, ds. \end{aligned}$$

Hence, by (5.10), we obtain

$$I_1 \geq \int\limits_{[0,\mathcal{H}^1(G_t)]\setminus\{\dot{\varGamma}_0=0\}} |d\varUpsilon(\dot{\varGamma}_1)|\, ds. \tag{5.14}$$

Now we have to tackle I_2. As

$$I_2 = \int\limits_{G_t\cap\{\eta_0=0\}} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \geq \int\limits_{\{\dot{\varGamma}_0=0\}} \frac{|\eta_1\circ\varGamma|}{|\mathbf{v}^T\circ\varGamma|}\, ds \tag{5.15}$$

the conclusion will easily follow by the Fenchel-type theorem, once we prove that

$$\int\limits_{\{\dot{\varGamma}_0=0\}} \frac{|\eta_1\circ\varGamma|}{|\mathbf{v}^T\circ\varGamma|}\, ds \geq \int\limits_{\{\dot{\varGamma}_0=0\}} |d\varUpsilon(\dot{\varGamma}_1)|\, ds. \tag{5.16}$$

Indeed, (5.14)–(5.16) imply that

$$\int\limits_{G_t} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^2 = I_1 + I_2 \geq \int\limits_{[0,\mathcal{H}^1(G_t)]} |d\varUpsilon(\dot{\varGamma}_1)|\, ds$$

and the right hand integral is not less than 2π by Theorem 4.1, taking into account (i).

To prove (5.16) we note that, by (5.8), $|\dot{\varGamma}_1| = |\dot{\varGamma}| = 1$ almost everywhere in $\{\dot{\varGamma}_0 = 0\}$. Then, also by recalling (5.2), we obtain

$$\int\limits_{\{\dot{\varGamma}_0=0\}} \frac{|\eta_1\circ\varGamma|}{|\mathbf{v}^T\circ\varGamma|}\, ds = \int\limits_{\{\dot{\varGamma}_0=0\}} \frac{|\widetilde{\dot{\varGamma}}_1 \wedge (\mathbf{v}_0^T\circ\varGamma)|}{|\mathbf{v}^T\circ\varGamma|^2}\, ds = \int\limits_{\{\dot{\varGamma}_0=0\}} \frac{|\mathbf{v}_0^T\circ\varGamma|}{|\mathbf{v}^T\circ\varGamma|^2}\, ds,$$

whence the assertion will follow by showing that

$$|\mathbf{v}_0^T\circ\varGamma| \geq |\mathbf{v}^T\circ\varGamma|^2 |d\varUpsilon(\dot{\varGamma}_1)| \quad \text{a.e. in } \{\dot{\varGamma}_0 = 0\}. \tag{5.17}$$

Recalling (5.1), Remark (5.4) and (5.5) we can assume that

$$\mathbf{u}_1 \neq 0 \quad \text{and} \quad \mathbf{u}_2 = c\mathbf{u}_1 \quad \text{a.e. along } Z_t := \varGamma(\{\dot{\varGamma}_0 = 0\}) \tag{5.18}$$

(by renaming the vector fields if need be), where c is an $\mathcal{H}^1 \llcorner Z_t$-measurable function. Then the equations

$$\mathbf{v}^T = (\mathbf{v}\cdot\mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v}\cdot\mathbf{w}_2)\mathbf{w}_2 = (\mathbf{v}\cdot\mathbf{u}_1)((1+c^2)\mathbf{u}_1 + \widetilde{\mathbf{v}}_1 + c\widetilde{\mathbf{v}}_2) \tag{5.19}$$

and

$$|\mathbf{v}_i|^2 = 1 - |\mathbf{u}_i|^2 \quad \text{for } i = 1,2,$$
$$\mathbf{v}_1\cdot\mathbf{v}_2 = -c|\mathbf{u}_1|^2 \quad (\text{since } \mathbf{w}_1\cdot\mathbf{w}_2 = 0)$$

hold a.e. along Z_t, whence, with a short computation, it follows that

$$|\mathbf{v}^T|^2 = (1+c^2)(\mathbf{v}\cdot\mathbf{u}_1)^2 \quad \text{a.e. along } Z_t. \tag{5.20}$$

By (5.20), (5.19) and recalling that $|\mathbf{v}^{T_0}| = |\mathbf{v}\cdot\mathbf{u}_1|/|\mathbf{u}_1|$, we can restate (5.17) as follows:

$$|d\varUpsilon(\dot{\varGamma}_1)| \le \frac{1}{|\mathbf{v}^{T_0}\circ\varGamma_0|}, \quad \text{a.e. in } \{\dot{\varGamma}_0 = 0\}. \tag{5.21}$$

To prove (5.21), we use the formula

$$|d\varUpsilon(\dot{\varGamma}_1)|^2 = \frac{1-(\varGamma_1\cdot\mathbf{v})^2 - (\dot{\varGamma}_1\cdot\mathbf{v})^2}{(1-(\varGamma_1\cdot\mathbf{v})^2)^2} \quad \text{a.e. in } \{\dot{\varGamma}_0 = 0\}, \tag{5.22}$$

which one can get by the same calculation made in proving Theorem 3.4 (here we recall again that $\varUpsilon$ and $d\varUpsilon$ are well defined because of hypothesis (ii) and take into account (5.8)). Then

$$|d\varUpsilon(\dot{\varGamma}_1)| \le \frac{1}{(1-(\varGamma_1\cdot\mathbf{v})^2)^{1/2}} \quad \text{a.e. in } \{\dot{\varGamma}_0 = 0\},$$

whence (5.21) immediately follows upon noting that, by (5.9),

$$|\mathbf{v}^{T_0}\circ\varGamma_0| \le |\mathbf{v}^{[\varGamma_1]^\perp}\circ\varGamma_0| = (1-(\varGamma_1\cdot\mathbf{v})^2)^{1/2}.$$

Remark 5.10. If G is the Gauss graph of a compact C^2 surface M embedded in $\mathbb{R}^3_x$, then the proof of the statement becomes easier (and it has been that case which provided us with the path followed to prove the general case). Indeed, under this assumption, $\eta_0 \ne 0$ on G and hence

$$\int_{G_t} \frac{|\eta_1|}{|\mathbf{v}^T|}\,d\mathcal{H}^1 = I_1 = \int_{[0,\mathcal{H}^1(G_t)]} \frac{|\eta_1\circ\varGamma||\dot{\varGamma}_0|}{|\eta_0\circ\varGamma||\mathbf{v}^{T_0}\circ\varGamma|}\,ds = \int_{[0,\mathcal{H}^1(G_t)]} \frac{\|\mathbf{II}_{\varGamma_0}\|\,|\dot{\varGamma}_0|}{|\mathbf{v}^{T_0}\circ\varGamma|}\,ds$$
$$\ge \int_{[0,\mathcal{H}^1(G_t)]} \frac{|\mathbf{II}_{\varGamma_0}(\dot{\varGamma}_0/|\dot{\varGamma}_0|)||\dot{\varGamma}_0|}{|\mathbf{v}^{T_0}\circ\varGamma|}\,ds.$$

As $|\mathbf{v}^{T_0}| = |\nu\cdot\mathbf{n}|$, where ν is the unit normal vector to M in $\mathbb{R}^3_x$ and $\mathbf{n}$ is the unit normal vector to M_t in the slicing plane $f^{-1}(t)$, from Meusnier's theorem we deduce that

$$\int_{G_t} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 = \int_{[0,\mathcal{H}^1(G_t)]} |\kappa \circ \Gamma_0|\, |\dot{\Gamma}_0|\, ds,$$

where κ denotes the scalar curvature of M_t. Hence the assertion follows from Fenchel's theorem.

Now we state a corollary of Theorem 5.9. Let

$$R := \{t \in \mathbb{R} \setminus J \mid \Xi_t \text{ is non-trivial and } \mathbf{v} \notin \operatorname{image}(\Gamma_1)\}$$

and note that $R \setminus f(G)$ is a null-measure set.

Corollary 5.11. *We have*

$$m(R) \le \frac{1}{2\pi} \int_G |\eta_1|\, d\mathcal{H}^2,$$

where m denotes the Lebesgue measure in $\mathbb{R}$.

Proof. Indeed, from the slicing theorem (see [3], [5], [7]), one obtains

$$\int_G |\eta_1|\, d\mathcal{H}^2 = \int_{f(G)} \left(\int_{G_t} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \right) dt \ge \int_{f(G)} \left(\int_{G_t^*} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \right) dt$$
$$\ge \int_R \left(\int_{G_t^*} \frac{|\eta_1|}{|\mathbf{v}^T|}\, d\mathcal{H}^1 \right) dt.$$

The conclusion follows from Theorem 5.9.

Remark 5.12. Let G be still as in Remark 5.10. Then, by Morse–Sard's theorem (see [4]), the set of critical values of $f_{|G}$ is a null-measure subset of $\mathbb{R}$. In other words, G_t is a regular level surface (of class C^2) for a.e. $t \in \mathbb{R}$. It follows that $m(f(G)) = m(R)$. Then Corollary 5.11 implies that

$$m(f(G)) \le \frac{1}{2\pi} \int_G |\eta_1|\, d\mathcal{H}^2.$$

In particular, if G is connected one also has (by the arbitrariness of $\mathbf{v}$)

$$\operatorname{diam}(pG) \le \frac{1}{2\pi} \int_G |\eta_1|\, d\mathcal{H}^2.$$

We conclude this section by proving the diameter estimate in the non-regular case. First, we state a simple transversality result which will play the same role as the Morse–Sard theorem in Remark 5.12. Then we make some remarks and definitions useful to end the proof of the estimate.

Proposition 5.13. *Let Ξ have a finite mass. Then there exists a set Q of full measure in S_x^2 such that if $\mathbf{v} \in Q$ then*

$$\mathbf{v} \notin \operatorname{image}(\Gamma_1) \quad \textit{for a.e. } t \in \mathbb{R}.$$

Proof. If $\mathbf{v} \in S_x^2$, let $A_{\mathbf{v}} = q^{-1}(\widetilde{\mathbf{v}}) \cap G$. Then the area formula implies that $\# A_{\mathbf{v}} < \infty$ for a.e. $\mathbf{v} \in S_x^2$. The conclusion follows since $\mathbf{v} \notin \mathrm{image}(\Gamma_1)$ provided Ξ_t is defined and $t \notin \widehat{f}(A_{\mathbf{v}})$.

Now we recall that, if $\mathbf{v} \in S_x^2$, then the one-dimensional linear subspace of $\mathbb{R}_x^3$ generated by $\mathbf{v}$ is denoted by $[\mathbf{v}]$. Also, let us introduce the set

$$\sigma_{\mathbf{v}} = \{x \in [\mathbf{v}] \mid \Xi_{x \cdot \mathbf{v}} \neq 0\}.$$

Remark 5.14. Without loss of generality, the set $\sigma_{\mathbf{v}}$ can be assumed to be equivalent to the set $\pi_{\mathbf{v}}$ obtained by projecting pG orthogonally on $[\mathbf{v}]$, i.e.

$$\mathcal{H}^1(\pi_{\mathbf{v}} \bigtriangleup \sigma_{\mathbf{v}}) = 0.$$

Also, observe that the map $d : S_x^2 \to \overline{\mathbb{R}}$ defined by

$$d(\mathbf{v}) = \mathcal{H}^1(\sigma_{\mathbf{v}}) = \mathcal{H}^1(\pi_{\mathbf{v}})$$

is continuous.

Definition 5.15. Ξ is said to be *segment-projecting at* $\mathbf{v} \in S_x^2$ if $\sigma_{\mathbf{v}}$ is equivalent to a segment, i.e. if there exists a connected set $I_{\mathbf{v}} \subset [\mathbf{v}]$ such that $\mathcal{H}^1(I_{\mathbf{v}} \bigtriangleup \sigma_{\mathbf{v}}) = 0$. We say that Ξ is *segment-projecting* if it is segment-projecting at every $\mathbf{v} \in S_x^2$.

Example 5.16. If Ξ is indecomposable, then it is segment-projecting. Indeed, let $\mathbf{v}$ be given in S_x^2; then, by [7; Lemma 28.5], there exists a measure zero set $Z \subset \mathbb{R}$ such that

$$\langle \Xi, \widehat{f}, t \rangle = \partial[\Xi \llcorner \{\widehat{f} < t\}] \quad \text{for every } t \in \mathbb{R} \setminus Z. \tag{5.23}$$

Let t_1 and t_2 be any elements in $I = \{t \in \mathbb{R} \mid \langle \Xi, \widehat{f}, t \rangle \neq 0\} = f(\sigma_{\mathbf{v}})$. Then it is enough to prove that

$$(t_1, t_2) \setminus Z \subset I.$$

Otherwise there would exist $t^* \in (t_1, t_2) \setminus Z$ such that $\langle \Xi, \widehat{f}, t^* \rangle = 0$. Hence, recalling that $\partial \Xi = 0$ and (5.23) holds, we would have

$$\partial[\Xi \llcorner \{\widehat{f} < t^*\}] = \partial[\Xi \llcorner \{\widehat{f} \geq t^*\}] = 0.$$

Then Ξ would admit the non-trivial decomposition

$$\Xi = \Xi \llcorner \{\widehat{f} < t^*\} \ + \ \Xi \llcorner \{\widehat{f} \geq t^*\}$$

with $\mathbf{N}(\Xi) = \mathbf{N}(\Xi \llcorner \{\widehat{f} < t^*\}) + \mathbf{N}(\Xi \llcorner \{\widehat{f} \geq t^*\})$, but this is absurd because Ξ is indecomposable. The conclusion follows by the arbitrariness of $\mathbf{v}$.

Definition 5.17. Let Ξ be segment-projecting. Then we define the *x-diameter* of Ξ as

$$x\text{-diam } \Xi = \sup_{\mathbf{v} \in S_x^2} d(\mathbf{v}).$$

Remark 5.18. By of the continuity of d, we have

$$x\text{-diam } \Xi = \sup_{\mathbf{v}\in Q} d(\mathbf{v})$$

whenever Q is a dense subset of S_x^2. In particular, that is true if Q is $\mathcal{H}^2$-measurable and $\mathcal{H}^2(Q) = 4\pi$.

Finally, we are able to prove the generalized diameter estimate.

THEOREM 5.19. *Let $\Xi = [\![G, \eta, \varrho]\!] \in \operatorname{curv}_2(\mathbb{R}_x^3)$ be of finite mass, without boundary and segment-projecting almost everywhere. Then*

$$x\text{-diam } \Xi \le \frac{1}{2\pi} \int_G |\eta_1| \, d\mathcal{H}^2.$$

Proof. Corollary 5.11 and Proposition 5.13 imply that

$$d(\mathbf{v}) = \mathcal{H}^1(\sigma_{\mathbf{v}}) \le \frac{1}{2\pi} \int_G |\eta_1| \, d\mathcal{H}^2$$

for all $\mathbf{v}$ in a full measure set $Q \subset S_x^2$. The conclusion follows from Remark 5.18. ∎

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Dipartimento di Matemetica
Università di Trento
38050 Povo (Trento), Italy
E-mail: delladio@science.unitn.it

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