ANNALES
POLONICI MATHEMATICI
LXIV.3 (1996)

The Christensen measurable solutions of a generalization of the Gołąb–Schinzel functional equation

by Janusz Brzdęk (Rzeszów)

Abstract. Let K denote the set of all reals or complex numbers. Let X be a topological linear separable F-space over K. The following generalization of the result of C. G. Popa [16] is proved.

Theorem. *Let n be a positive integer. If a Christensen measurable function $f : X \to K$ satisfies the functional equation*

$$f(x + f(x)^n y) = f(x)f(y),$$

then it is continuous or the set $\{x \in X : f(x) \neq 0\}$ is a Christensen zero set.

1. Introduction. The functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{1}$$

is well known and has been studied by many authors (see e.g. [1], [2], [4], [5], [11]–[13], [15], [16], [19]). It is called the Gołąb–Schinzel functional equation. C. G. Popa [16] has proved that every Lebesgue measurable solution $f : \mathbb{R} \to \mathbb{R}$ of (1) is either continuous or equal to zero almost everywhere. We are going to show that the same is true for each Christensen measurable solution of the functional equation

$$f(x + f(x)^n y) = f(x)f(y) \tag{2}$$

mapping a real (complex) linear topological separable F-space into the set of all reals (complex numbers), where n is a positive integer.

Equation (2) is a natural generalization of (1). It is also a particular case ($k = 0$, $t = 1$) of the functional equation

$$f(f(y)^k x + f(x)^n y) = tf(x)f(y)$$

considered in various cases e.g. in [3], [4], [7], [18]. It is also worth mentioning that there is a strict connection between the solutions of equation (2) in the

1991 *Mathematics Subject Classification*: Primary 39B52.
Key words and phrases: Gołąb–Schinzel functional equation, Christensen measurability, F-space.

class of functions $f : \mathbb{R} \to \mathbb{R}$ and a class of subgroups of the Lie group L^1_{n+1} (cf. [5], [6]).

Throughout this paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the sets of all positive integers, all integers, reals, and complex numbers, respectively. X stands for a linear space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$, unless explicitly stated otherwise. m and m_{i} are Lebesgue and inner Lebesgue measures in K, respectively.

2. Preliminary lemmas. Let us start with the following

LEMMA 1. *A function $f : X \to K$, $f \neq 0$ (i.e. $f^{-1}(\{0\}) \neq X$), is a solution of equation* (2) *iff there exist an additive subgroup A of X, a multiplicative subgroup W of K, and a function $w : W \to X$ such that*

(3) $a^n A = A$ *for* $a \in W$;

(4) $w(ab) - a^n w(b) - w(a) \in A$ *for* $a, b \in W$;

(5) $w(a) \in A$ *iff* $a = 1$;

(6) $$f(x) = \begin{cases} a & \text{if } x \in w(a) + A \text{ and } a \in W, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for } x \in X.$$

Furthermore, $W = f(X) \setminus \{0\}$ and $A = f^{-1}(\{1\})$.

The proof does not differ essentially from the proof of Theorem 1 of [13] (cf. also [19] and [12], pp. 275–277). Therefore we omit it.

The subsequent corollary follows from Lemma 1.

COROLLARY 1. *If a function $f : X \to K$, $f \neq 0$, satisfies equation* (2), *$A = f^{-1}(\{1\})$, and $W = f(X) \setminus \{0\}$, then*:

(i) *A is an additive group*;
(ii) *W is a multiplicative group*;
(iii) *$A \setminus \{0\}$ is the set of periods of f*;
(iv) *if $x, y \in X$ and $f(x) = f(y) \neq 0$, then $x - y \in A$*;
(v) *$a^n A = A$ for $a \in W$.*

LEMMA 2. *Let $f : K \to K$ be a microperiodic function* (*i.e. the set of periods of f is dense in K*) *satisfying equation* (2). *Suppose that there exists $a \in K$ such that $|f(a)| \notin \{0, 1\}$. Then $m_{\mathrm{i}}(f^{-1}(K_j)) = 0$ for $j \in \mathbb{N}$, where $K_j = \{a \in K : 1/j \le |a| \le j\}$.*

Proof. For an indirect proof suppose that there is $k \in \mathbb{N}$ with $m_{\mathrm{i}}(f^{-1}(K_k)) > 0$. Then, in view of Corollary 1(ii), there exists $b \in K$ with $|f(b)| > (k+1)^2$. Put $D = b + f(b)^n f^{-1}(K_k)$. It is easily seen that $m_{\mathrm{i}}(D) > 0$ and, by (2), $|f(a)| > k$ for $a \in D$. Thus $D \cap f^{-1}(K_k) = \emptyset$. On the other hand, according to a theorem of H. Steinhaus (see e.g. [14], Theorem 3.7.1), $\mathrm{int}(D - f^{-1}(K_k)) \neq \emptyset$. Consequently, there exists $c \in D - f^{-1}(K_k)$ such that $c \neq 0$ and $f(a+c) = f(a)$ for $a \in K$, which means that $f^{-1}(K_k) \cap D = (f^{-1}(K_k) + c) \cap D \neq \emptyset$, a contradiction.

Given $b \in \mathbb{C} \setminus \{0\}$ and $j \in \mathbb{N}$ let us put

$$C_j(b) = \{a \in \mathbb{C} \setminus \{0\} : (j-1)\tfrac{2}{3}\pi \le \operatorname{Arg} b^{-1}a < j\tfrac{2}{3}\pi\}, \tag{7}$$

where $\operatorname{Arg} c \in [0, 2\pi)$ denotes the argument of $c \in \mathbb{C} \setminus \{0\}$. It is easy to see that $\mathbb{C} \setminus \{0\} = \bigcup\{C_j(b) : j = 1, 2, 3\}$.

LEMMA 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a microperiodic solution of* (2) *such that the set $f(\mathbb{C})$ is infinite and $|a| = 1$ for $a \in f(\mathbb{C}) \setminus \{0\}$. Then $m_{\mathrm{i}}(f^{-1}(C_j(b))) = 0$ for every $j = 1, 2, 3$, $b \in \mathbb{C} \setminus \{0\}$.*

Proof. For an indirect proof suppose that there exist $b \in \mathbb{C} \setminus \{0\}$ and $k \in \{1, 2, 3\}$ with $m_{\mathrm{i}}(f^{-1}(C_k(b))) > 0$. Since $f(\mathbb{C})$ is infinite, in view of Corollary 1(ii), $f(\mathbb{C}) \setminus \{0\}$ is dense in the set $J = \{a \in \mathbb{C} : |a| = 1\}$. Thus there is $d \in \mathbb{C}$ such that $f(d) \neq 0$ and $(f(d)C_k(b)) \cap C_k(b) = \emptyset$. Define $D = d + f(d)^n f^{-1}(C_k(b))$. Then, in virtue of (2), $f(D) = f(d)C_k(b)$. Hence $D \cap f^{-1}(C_k(b)) = \emptyset$. On the other hand, $m_{\mathrm{i}}(D) > 0$, which, according to the theorem of Steinhaus, means that $\operatorname{int}(D - f^{-1}(C_k(b))) \neq \emptyset$. Consequently, there exists a period $c \in D - f^{-1}(C_k(b))$ of f, from which we derive that $f^{-1}(C_k(b)) \cap D = (f^{-1}(C_k(b)) + c) \cap D \neq \emptyset$, a contradiction.

LEMMA 4. *If a function $f : K \to K$, $f \neq 1$, satisfies equation* (2), *then $m_{\mathrm{i}}(f^{-1}(\{a\})) = 0$ for each $a \in f(K) \setminus \{0\}$.*

Proof. For an indirect proof suppose that there is $a \in f(K) \setminus \{0\}$ with $m_{\mathrm{i}}(f^{-1}(\{a\})) > 0$. Fix $b \in f^{-1}(\{a\})$ and put $D = f^{-1}(\{a\}) - b$. Then, on account of Corollary 1(iv), $D \subset A := f^{-1}(\{1\})$. Thus $m_{\mathrm{i}}(A) > 0$. Consequently, by the theorem of Steinhaus and Corollary 1(i), $A = K$, a contradiction.

LEMMA 5. *Let $f : X \to K$ be a function satisfying equation* (2), *$W = f(X) \setminus \{0\}$, and $A = f^{-1}(\{1\})$. Suppose that there is $a_0 \in W$ such that $a_0^n \neq 1$ and $(a_0^n - 1)^{-1}A \subset A$. Then*

$$a^n \neq 1 \quad \text{for each } a \in W \setminus \{1\} \tag{8}$$

and there exists $x_0 \in X \setminus \bigcup\{(a^n - 1)^{-1}A : a \in W \setminus \{1\}\}$ such that

$$f(x) = \begin{cases} a & \text{if } x \in (a^n - 1)x_0 + A \text{ and } a \in W, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } x \in X. \tag{9}$$

Proof. In view of Lemma 1 there is a function $w : W \to X$ such that (4)–(6) hold. Let $x_0 = (a_0^n - 1)^{-1}w(a_0)$. Since, by (4),

$$w(ab) - a^n w(b) - w(a),\ w(ba) - b^n w(a) - w(b) \in A \quad \text{for } a, b \in W,$$

Corollary 1(i) implies that $a^n w(b) + w(a) - b^n w(a) - w(b) \in A$ for $a, b \in W$. Thus, for each $b \in W$, $-(b^n - 1)x_0 + w(b) \in A$. Consequently, according to

(5), (6), and Corollary 1(i), conditions (8) and (9) hold and $(a^n - 1)x_0 \notin A$ for $a \in W \setminus \{1\}$, which completes the proof.

LEMMA 6. *Let Y be a linear space over a subfield F of the field K. Let $f : Y \to K \setminus \{0\}$ be a solution of equation* (2) *such that $f(x)^n \in F$ for each $x \in Y$. Then $f = 1$.*

Proof. Suppose that there is $x \in Y$ with $f(x)^n \neq 1$ and put $z = (1 - f(x)^n)^{-1}x$. Then $x + f(x)^n z = z$ and, in view of (2),

$$f(x)f(z) = f(x + f(x)^n z) = f(z) \neq 0,$$

from which we derive $f(x) = 1$, a contradiction.

Hence $f(x)^n = 1$ for each $x \in Y$. Thus $f(x+y) = f(x)f(y)$ for $x, y \in Y$ and consequently, for each $x \in Y$,

$$f(x) = f\left(n\frac{1}{n}x\right) = f\left(\frac{1}{n}x\right)^n = 1.$$

This completes the proof.

LEMMA 7. *If a function $f : X \to K$, $f \neq 0$, satisfies equation* (2), *then $f(f(x)^{-n}(z - x)) = f(z)f(x)^{-1}$ for $x, z \in X$ with $f(x) \neq 0$.*

Proof. Fix $x \in X$ with $f(x) \neq 0$. Setting $z = f(x)^n y + x$ in (2), we get $f(z) = f(x)f(f(x)^{-n}(z - x))$ for $z \in X$, which yields the assertion.

LEMMA 8. *Let B be an additive subgroup of a real linear space Y and let V be an infinite multiplicative subgroup of $\mathbb{R}$ such that*

$$ax \in B \quad \text{for } x \in B,\ a \in V. \tag{10}$$

Then the set $B_x = \{a \in \mathbb{R} : ax \in B\}$ is dense in $\mathbb{R}$ for each $x \in B$.

Proof. Note that, for each $c \in \mathbb{R}$, $c > 0$, there is $b \in V$ with $|b| < c$. Since, for each $x \in B$, B_x is an additive group and, by (10), $V \subset B_x$, we obtain the statement.

LEMMA 9. *Let B be an additive subgroup of a complex linear space and let V be an infinite multiplicative subgroup of $\mathbb{C}$ such that $V \not\subset \mathbb{R}$ and* (10) *holds. Then the set $B_x = \{a \in \mathbb{C} : ax \in B\}$ is dense in $\mathbb{C}$ for each $x \in B$.*

Proof. Let $x \in B$ and $J = \{a \in \mathbb{C} : |a| = 1\}$. Note that $V \subset B_x$. If $V \subset J$, then V is dense in J. Thus B_x is dense in $\mathbb{C}$, because it is an additive group. On the contrary, if there is $a \in V \setminus (R \cup J)$, then, for each $c \in \mathbb{R}$, $c > 0$, there exists $k \in \mathbb{Z}$ with $|a^k| < c$ and $|a^{k+1}| < c$. Since a^k and a^{k+1} are linearly independent over $\mathbb{R}$, the additive group generated by V is dense in $\mathbb{C}$, which completes the proof.

LEMMA 10 (cf. [16], Théorème 1). *If $D_1, D_2 \subset K$ and $m_{\mathrm{i}}(D_j) > 0$, $j = 1, 2$, then $\operatorname{int}(D_1 \cdot D_2) \neq \emptyset$.*

P r o o f. First consider the case where $K=\mathbb{R}$. There exist closed sets $F_i \subset D_i$ such that $m(F_i) > 0$ for $i = 1, 2$. Put $F_i^k = F_i \cap ([-k, -1/k] \cup [1/k, k])$ for $k \in \mathbb{N}$, $i = 1, 2$. It is easily seen that there are $p, q \in \mathbb{N}$ with $m(F_1^p) > 0$ and $m(F_2^q) > 0$. Let $F_1^+ = F_1^p \cap (0, \infty)$, $F_1^- = F_1^p \cap (-\infty, 0)$, $F_2^+ = F_2^q \cap (0, \infty)$, and $F_2^- = F_2^q \cap (-\infty, 0)$. Define

$$F_i^0 = \begin{cases} F_i^+ & \text{if } m(F_i^+) > 0, \\ F_i^- & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, 2.$$

Observe that $m(F_i^0) > 0$ for $i = 1, 2$. Thus (see e.g. [17], Theorem 8.26), $m(\ln F_i^0) > 0$ for $i = 1, 2$. Hence, in virtue of the theorem of Steinhaus, $\operatorname{int}(\ln(F_1^0 \cdot F_2^0)) = \operatorname{int}(\ln F_1^0 + \ln F_2^0) \neq \emptyset$, which means that $\operatorname{int}(D_1 \cdot D_2) \neq \emptyset$.

Now assume that $K = \mathbb{C}$. Let $F_i \subset D_i$ be a closed set such that $m(F_i) > 0$ for $i = 1, 2$. Put $C_k = \{a \in \mathbb{C} : 1/k \le |a| \le k\}$ and $F_i^k = F_i \cap C_k$ for $k \in \mathbb{N}$, $i = 1, 2$. It is easily seen that there are $p, q \in \mathbb{N}$ with $m(F_1^p) > 0$ and $m(F_2^q) > 0$. Define functions $h_1 : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C}$, $h_2 : (0, 2\pi) \times (0, \infty) \to \mathbb{C}$, and $h_3 : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2$ by the formulas: $h_1(a, b) = a + ib$, $h_2(a, b) = b(\cos a + i \sin a)$, $h_3(a, b) = (a, \ln b)$. Let $F_1^0 = h_1^{-1}(F_1^p)$ and $F_2^0 = h_1^{-1}(F_2^q)$. Then F_i^0 is a Borel set and $m(F_i^0) > 0$ for $i = 1, 2$ (m denotes also the Lebesgue measure in $\mathbb{R}^2$). Note that $h = h_3 \circ h_2^{-1} \circ h_1$ is a diffeomorphism onto the set $h(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$. Thus $h(F_i^0)$ is a Borel set and $m(h(F_i^0)) > 0$ for $i = 1, 2$ (see e.g. [17], Theorem 8.26(c)). Hence, by the theorem of Steinhaus, $\operatorname{int}(h(F_1^0) + h(F_2^0)) \neq \emptyset$. Since $h(F_1^0) + h(F_2^0) = h_3 \circ h_2^{-1}(h_1(F_1^0) h_1(F_2^0))$, we have $\operatorname{int}(h_1(F_1^0) \cdot h_1(F_2^0)) \neq \emptyset$, which implies the assertion.

3. Christensen measurability. Throughout this part we assume that X is a separable F-space as a topological linear space over K. We shall use the notation and terminology from [8]–[10] concerning Christensen measurability. Now, we only recall necessary definitions and facts.

Let M be the σ-algebra of all universally measurable subsets of X; i.e. M is the intersection of all completions of the Borel σ-algebra of X with respect to probability Borel measures. In the following a *measure* is a countable additive Borel measure extended to M.

DEFINITION 1. A set $B \in M$ is a *Haar zero set* iff there exists a probability measure u on X such that $u(B + x) = 0$ for each $x \in X$.

DEFINITION 2. A set $P \subset X$ is a *Christensen zero set* iff it is a subset of a Haar zero set.

DEFINITION 3. A set $D \subset X$ is *Christensen measurable* iff $D = B \cup P$, where $B \in M$ and P is a Christensen zero set.

Let us define

$$\mathcal{C}_0 = \{B \subset X : B \text{ is Christensen zero set}\},$$
$$\mathcal{C} = \{B \subset X : B \text{ is Christensen measurable}\}.$$

Lemma 11 (see [9], Theorem 1). *Every countable union of Christensen zero sets is a Christensen zero set.*

Lemma 12 (see [9], Theorem 2). *If* $B \in \mathcal{C} \setminus \mathcal{C}_0$, *then* $0 \in \operatorname{int}(B - B)$.

Definition 4. A function $f : X \to K$ is said to be *Christensen measurable* iff $f^{-1}(U) \in \mathcal{C}$ for each open set $U \subset K$.

Lemma 13 (see [10], Theorem 1). *Let* $f : X \to K$ *be a Christensen measurable linear functional. Then* f *is continuous.*

Put $L_k = \{a \in K : k - 1 \le |a| < k\}$ and $a_k = m(L_k)$ for $k \in \mathbb{N}$. Given a Borel set $D \subset X$ and $x \in X$ denote $u_x(D) = m_p(k_x^{-1}(D))$, where $k_x : K \to X$, $k_x(a) = ax$, and, for each Borel set $B \subset K$, $m_p(B) = \sum_{k=1}^{\infty} 2^{-k} a_k^{-1} m(B \cap L_k)$. Since k_x is continuous, u_x is a well defined Borel measure and $u_x(X) = 1$ for each $x \in X \setminus \{0\}$.

Lemma 14. *Let* $D \in \mathcal{C} \setminus \mathcal{C}_0$ *and* $x \in X \setminus \{0\}$. *Then there exist a Borel set* $D_x \subset D$ *and* $y_x \in X$ *such that*

$$m(k_x^{-1}(y_x + D_x)) > 0. \tag{11}$$

Proof. There exist $B \in M$ and $P \in \mathcal{C}_0$ with $D = B \cup P$. In view of Lemma 11, $B \notin \mathcal{C}_0$. Thus there is $y \in X$ such that $\overline{u}(B + y) > 0$, where $\overline{u}$ denotes the extension of u_x to M. Put $u_0(T) = \overline{u}(T + y)$ for each $T \in M$. Then u_0 is a probability measure. Hence there are a Borel set $B_x \subset B$ and a set $B_0 \subset B$ such that $u_0(B_0) = 0$ and $B = B_x \cup B_0$. Furthermore $u_x(B_x + y) = \overline{u}(B_x + y) = u_0(B_x) = u_0(B_x \cup B_0) = u_0(B) = \overline{u}(B + y) > 0$. Consequently, $m_p(k_x^{-1}(B_x + y)) > 0$, which implies (11). This ends the proof.

Lemma 15. *Let* $L \subset K \setminus \{0\}$ *and* $x \in X \setminus \{0\}$. *Let* $f : X \to K$ *be a function satisfying equation* (2). *Suppose that* $f^{-1}(L) \in \mathcal{C} \setminus \mathcal{C}_0$. *Then there exists* $z \in X$ *such that* $f(z) \ne 0$ *and* $m_{\mathrm{i}}(f_x^{-1}(f(z)^{-1}L)) > 0$, *where* $f_x : K \to K$, $f_x(a) = f(ax)$.

Proof. It follows from Lemma 14 that there are a Borel set $D_x \subset D := f^{-1}(L)$ and $y_x \in X$ such that (11) holds. Put $B = (Kx - y_x) \cap D_x$. Then, according to the definition of k_x and (11), $B \ne \emptyset$. Fix $z \in B$. It is easily seen that $f(z) \ne 0$ and there exists $b \in K$ with $z = bx - y_x$. Thus

$$B - z = ((Kx - y_x) \cap D_x) - bx + y_x = (Kx \cap (D_x + y_x)) - bx,$$

which means that $k_x^{-1}(B - z) = k_x^{-1}(D_x + y_x) - b$. Hence, in view of (11),

$$m(f(z)^{-n}(k_x^{-1}(B - z))) > 0. \tag{12}$$

Note that, by Lemma 7,

$$\begin{aligned} f_x(f(z)^{-n}(k_x^{-1}(B-z))) &= f(f(z)^{-n}(k_x^{-1}(B-z))x) \subset f(f(z)^{-n}(B-z)) \\ &= f(z)^{-1}f(B) \subset f(z)^{-1}L. \end{aligned}$$

Consequently, $f(z)^{-n}(k_x^{-1}(B-z)) \subset f_x^{-1}(f(z)^{-1}L)$, from which we derive by (12), that $m_{\mathrm{i}}(f_x^{-1}(f(z)^{-1}L)) > 0$. This completes the proof.

LEMMA 16. *Let $f : X \to K$ be a Christensen measurable function satisfying equation* (2) *such that the set $W = f(X)\setminus\{0\}$ is infinite. Suppose that the set $S_f = \{x \in X : f(x) \neq 0\}$ is not a Christensen zero set. Then the set $A = f^{-1}(\{1\})$ is a proper linear subspace of X over the field*

$$F = \begin{cases} \mathbb{R} & \textit{if } \ f(x)^n \in \mathbb{R} \textit{ for each } \ x \in X, \\ \mathbb{C} & \textit{otherwise.} \end{cases} \tag{13}$$

Proof. Since $A \neq X$, it suffices to show that A is a linear subspace of X over F.

For an indirect proof suppose that $A \neq A_0$, where A_0 denotes the linear subspace of X (over F) spanned by A. Let $f_0 = f|_{A_0}$. It is easy to check that f_0 is a solution of (2) and $f_0 \neq 1$. Thus, in view of Lemma 6, $f_0^{-1}(\{0\}) \neq \emptyset$, from which we derive that there are $a_0 \in F\setminus\{0\}$ and $y \in A\setminus\{0\}$ such that $f(a_0y) = 0$. Note that the functions $f_1 : X \to F$, $f_1(x) = f(x)^n$, and $f_y : F \to F$, $f_y(a) = f_1(ay)$, also satisfy (2) for $n = 1$. Since $f_y(a_0) = f(a_0y)^n = 0$, we have $f_y \neq 1$. Furthermore, $W_n \subset F$, $\{a \in F : ay \in A\} \subset f_y^{-1}(\{1\})$, and, by Corollary 1(v), $aA = A$ for $a \in W_n$, where $W_n = \{a^n : a \in W\}$. Hence, by Lemma 8, Lemma 9, and Corollary 1(i)–(iii), f_y is microperiodic.

First consider the case where there is $b \in F$ with $|f_y(b)| \notin \{0,1\}$. Let $F_j = \{a \in F : 1/j \leq |a| \leq j\}$ for $j \in \mathbb{N}$. Since $S_f = \bigcup\{f_1^{-1}(F_j) : j \in \mathbb{N}\}$, according to Lemma 11 there exists $p \in \mathbb{N}$ such that $f_1^{-1}(F_p) \notin \mathcal{C}_0$. Thus, by Lemma 15 (with $n = 1$), $m_{\mathrm{i}}(f_y^{-1}(f_1(z)^{-1}F_p)) > 0$ for some $z \in S_f$. Note that there is $k \in \mathbb{N}$ with $f_1(z)^{-1}F_p \subset F_k$. Hence $m_{\mathrm{i}}(f_y^{-1}(F_k)) > 0$, which contradicts Lemma 2.

Now, assume that the set $W_y := f_y(F)\setminus\{0\}$ is finite. Then W_y is a multiplicative cyclic subgroup of F (cf. Corollary 1(ii)) and $|a| = 1$ for each $a \in W_y$. There exists $c \in F$ such that $W_y = \{c^k : k \in \mathbb{N}\}$. Put $k_0 = \min\{k \in \mathbb{N} : c^k = 1\}$ and define

$$T_j = \begin{cases} c^j(0,\infty) & \text{if } F = \mathbb{R}, \\ \{a \in \mathbb{C}\setminus\{0\} : 2\pi k_0^{-1}(j-1) \leq \operatorname{Arg} a < 2\pi k_0^{-1} j\} & \text{if } F = \mathbb{C}, \end{cases}$$

for $j \in \mathbb{N}$, $j \leq k_0$. Observe that $S_f = \bigcup\{f_1^{-1}(T_j) : j \in \mathbb{N}, j \leq k_0\}$. Thus there is a positive integer $k \leq k_0$ such that $f_1^{-1}(T_k) \notin \mathcal{C}_0$. It results from Lemma 15 that there exists $z \in S_f$ with $m_{\mathrm{i}}(f_y^{-1}(f_1(z)^{-1}T_k)) > 0$. Moreover, there is exactly one positive integer $p \leq k_0$ such that $c^p \in f_1(z)^{-1}T_k$. Consequently, $m_{\mathrm{i}}(f_y^{-1}(\{c^p\})) > 0$, contrary to Lemma 4.

It remains to study the case where $F = \mathbb{C}$, W_y is infinite, and $|a| = 1$ for each $a \in W_y$. Since $S_f = \bigcup\{f_1^{-1}(C_j(1)) : j = 1,2,3\}$, where $C_j(b)$, for $b \in \mathbb{C}\setminus\{0\}$, is given by (7), we have $f_1^{-1}(C_k(1)) \notin \mathcal{C}_0$ for some $k \in \{1,2,3\}$. Thus, on account of Lemma 15, there is $z \in S_f$ with $m_{\mathrm{i}}(f_y^{-1}(f_1(z)^{-1}C_k(1))) > 0$. Clearly, $f_1(z)^{-1}C_k(1) = C_k(f_1(z)^{-1})$. Hence $m_{\mathrm{i}}(f_y^{-1}(C_k(f_1(z)^{-1}))) > 0$, contrary to Lemma 3. This completes the proof.

4. The main result. Now, we have all tools to prove the announced theorem.

THEOREM. *Suppose that X is a linear topological separable F-space over K. Let $f : X \to K$ be a Christensen measurable solution of equation* (2). *Then either f is continuous or the set $S_f = \{x \in X : f(x) \neq 0\}$ is a Christensen zero set.*

Furthermore, if f is continuous and satisfies (2), *then*

$$f(X) \subset \mathbb{R} \quad \text{or} \quad n = 1 \tag{14}$$

and the following two statements hold:

(i) *if $f(X) \subset \mathbb{R}$, then there exists a continuous $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$ such that, for n odd, either*

$$f(x) = \sqrt[n]{g(x)+1} \quad \text{for } x \in X \tag{15}$$

or

$$f(x) = \sqrt[n]{\sup(g(x)+1, 0)} \quad \text{for } x \in X, \tag{16}$$

and for n even, f is of the form (16);

(ii) *if $f(X) \not\subset \mathbb{R}$ and $n = 1$, then there exists a continuous $\mathbb{C}$-linear functional $g : X \to \mathbb{C}$, $g \neq 0$, such that $f(x) = g(x) + 1$, $x \in X$.*

Proof. Note that if $f \neq 0$ is continuous, then $\operatorname{int} S_f \neq \emptyset$, which means that $S_f \notin \mathcal{C}_0$. Therefore suppose that $S_f \in \mathcal{C}\setminus\mathcal{C}_0$. Put $W = f(X)\setminus\{0\}$ and $A = f^{-1}(\{1\})$.

First, consider the case where W is finite. Then, in view of Lemma 1, there is a function $w : W \to X$ with $S_f = \bigcup\{w(a) + A : a \in W\}$. Thus, by Lemma 11, $A \notin \mathcal{C}_0$. Hence Lemma 12 and Corollary 1(i) imply that $\operatorname{int} A \neq \emptyset$, from which we derive $A = X$. Consequently, (15) or (16) holds with $g = 0$.

Now, assume that W is infinite. Since, in the case where $K = \mathbb{C}$, X is also a real topological linear F-space (with the same topology), without loss of generality we may assume that

$$\text{if } K = \mathbb{C}, \text{ then } f(X) \not\subset \mathbb{R}. \tag{17}$$

It results from Lemma 16 that A is a proper linear subspace of X over the field F given by (13). Thus, by Lemma 5, condition (8) is valid and there exists $x_0 \in X \setminus A$ such that f is of the form (9). Hence

$$S_f = A + (W_n - 1)x_0, \tag{18}$$

where $W_n = \{a^n : a \in W\}$. Furthermore, in view of Lemma 12, $0 \in \mathrm{int}(S_f - S_f)$, whence

$$A + Fx_0 = X. \tag{19}$$

On account of (19) and Lemma 14 there exist a Borel set $B \subset S_f$ and $a \in F$, $x \in A$ with $m(k_0^{-1}(ax_0 + x + B)) > 0$, where $k_0 : F \to X$, $k_0(a) = ax_0$. On the other hand, from (18), we obtain $ax_0 + x + S_f = A + (W_n - 1 + a)x_0$. Thus $k_0^{-1}(ax_0 + x + S_f) = W_n - 1 + a$. Since $k_0^{-1}(ax_0 + x + B) = a + k_0^{-1}(x + B)$, we have $k_0^{-1}(x + B) \subset W_n - 1$ and $m(k_0^{-1}(x + B)) = m(k_0^{-1}(ax_0 + x + B)) > 0$, from which we derive that $m_{\mathrm{i}}(W_n) = m_{\mathrm{i}}(W_n - 1) > 0$ (in F). Hence and from Lemma 10 and Corollary 1(ii) we get $\mathrm{int}\, W_n \neq \emptyset$ (in F), whence

$$(0, \infty) \subset W_n \quad \text{and} \quad 1 \in \mathrm{int}\, W_n \text{ (in } F). \tag{20}$$

We shall prove that (8), (17), and (20) imply $F = K$.

For an indirect proof suppose that $K = \mathbb{C}$ and $F = \mathbb{R}$. Then there is $a \in W \setminus \mathbb{R}$ with $a^n \in \mathbb{R}$. Observe that, by (20) and Corollary 1(ii), $a \cdot |a|^{-1} \in W \setminus \mathbb{R}$, whence, by (8), $-1 = (a \cdot |a|^{-1})^n \in W$ and $(-1)^n \neq 1$. This means that n is odd. Consequently, $a^{n+1} \cdot |a|^{-n-1} = -a \cdot |a|^{-1} \notin \mathbb{R}$ and $(a^{n+1} \cdot |a|^{-n-1})^n = (-1)^{n+1} = 1$, which contradicts (8).

In this way we have proved that $F = K$. Thus, by (19), A is a hyperplane of X (i.e. $\mathrm{codim}\, A = 1$) and, according to Corollary 1 and (20),

$$\text{for } K = \mathbb{C}, \quad W = \mathbb{C} \setminus \{0\}, \tag{21}$$

$$\text{for } K = \mathbb{R}, \quad W = (0, \infty) \text{ or } W = \mathbb{R} \setminus \{0\}, \tag{22}$$

whence (8) yields condition (14).

Define a linear functional $g : X \to K$ by

$$g(ax_0 + y) = a \quad \text{for } a \in K,\ y \in A. \tag{23}$$

It is easy to check that, on account of (9) and (18),

$$g(x) = f(x)^n - 1 \quad \text{for } x \in S_f, \tag{24}$$

which, in view of (8), (18), and (22), means that, in the case where $f(X) \subset \mathbb{R}$, both conclusions of (i) are valid. In the case where $n = 1$ and $f(X) \not\subset \mathbb{R}$, (21), (18), and (24) imply that $f(x) = g(x) + 1$, $x \in X$. Therefore, on account of Lemma 13, it remains to show that g is Christensen measurable.

If $n = 1$ and $f(X) \not\subset \mathbb{R}$, this is obvious, because f is Christensen measurable. On the other hand, if $f(X) \subset \mathbb{R}$, then $g(x) = f(x)^n - 1$ for $x \in g^{-1}((-1, \infty))$. Furthermore, for each set $U \subset \mathbb{R}$, $g^{-1}(U) = g^{-1}(U^+) \cup (-g^{-1}(-U^-)) \cup g^{-1}(U_0)$, where $U^+ = U \cap (0, \infty)$, $U^- = U \cap (-\infty, 0)$ and

$U_0 = U \cap \{0\}$. This implies that g is Christensen measurable, which ends the proof.

Remark. It is easy to check that each function $f : X \to K$ satisfying (14) and conditions (i), (ii) of the Theorem is a solution of equation (2).

Finally, since in the case where X is locally compact, $\mathcal{C}_0$ coincides with the set of all the Haar measure zero subsets of X (see [9], p. 256), from the Theorem we get the following

COROLLARY 2. *Let $k \in \mathbb{N}$ and let $f : K^k \to K$ be a Lebesgue measurable solution of equation* (2). *Then either f is continuous or the set S_f is of Lebesgue measure zero.*

Acknowledgements. I wish to thank Professor Karol Baron for calling my attention to the problem.

References

[1] J. Aczél and S. Gołąb, *Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphisms*, Aequationes Math. 4 (1970), 1–10.

[2] K. Baron, *On the continuous solutions of the Gołąb–Schinzel equation*, ibid. 38 (1989), 155–162.

[3] W. Benz, *The cardinality of the set of discontinuous solutions of a class of functional equations*, ibid. 32 (1987), 58–62.

[4] N. Brillouët et J. Dhombres, *Equations fonctionnelles et recherche de sous groupes*, ibid. 31 (1986), 253–293.

[5] J. Brzdęk, *Subgroups of the group Z_n and a generalization of the Gołąb–Schinzel functional equation*, ibid. 43 (1992), 59–71.

[6] —, *A generalization of the Gołąb–Schinzel functional equation*, ibid. 39 (1990), 268–269.

[7] —, *On the solutions of the functional equation $f(xf(y)^l + yf(x)^k) = tf(x)f(y)$*, Publ. Math. Debrecen 38 (1991), 175–183.

[8] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland Math. Stud. 10, North-Holland, 1974.

[9] —, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255–260.

[10] P. Fischer and Z. Słodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. 79 (1980), 449–453.

[11] S. Gołąb et A. Schinzel, *Sur l'équation fonctionnelle $f(x + yf(x)) = f(x)f(y)$*, Publ. Math. Debrecen 6 (1959), 113–125.

[12] D. Ilse, I. Lechmann und W. Schulz, *Gruppoide und Funktionalgleichungen*, Deutscher Verlag Wiss., Berlin, 1984.

[13] P. Javor, *On the general solution of the functional equation $f(x + yf(x)) = f(x)f(y)$*, Aequationes Math. 1 (1968), 235–238.

[14] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, PWN and Uniw. Śląski, Warszawa–Kraków–Katowice, 1985.

[15] P. Plaumann und S. Strambach, *Zweidimensionale Quasialgebren mit Nullteilern*, Aequationes Math. 15 (1977), 249–264.

[16] C. G. Popa, *Sur l'équation fonctionnelle* $f(x + yf(x)) = f(x)f(y)$, Ann. Polon. Math. 17 (1965), 193–198.
[17] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1974.
[18] M. Sablik and P. Urban, *On the solutions of the equation* $f(xf(y)^k + yf(x)^l) = f(x)f(y)$, Demonstratio Math. 18 (1985), 863–867.
[19] S. Wołodźko, *Solution générale de l'équation fonctionnelle* $f(x + yf(x)) = f(x)f(y)$, Aequationes Math. 2 (1968), 12–29.

Institute of Mathematics
Pedagogical University of Rzeszów
Rejtana 16A
35-310 Rzeszów, Poland

Reçu par la Rédaction le 17.12.1990
Révisé le 23.10.1995