The Christensen measurable solutions of a generalization of the Gołąb–Schinzel functional equation

by Janusz Brzdek (Rzeszów)

Abstract. Let $K$ denote the set of all reals or complex numbers. Let $X$ be a topological linear separable $F$-space over $K$. The following generalization of the result of C. G. Popa [16] is proved.

Theorem. Let $n$ be a positive integer. If a Christensen measurable function $f : X \to K$ satisfies the functional equation

$$f(x + f(x)^n y) = f(x) f(y),$$

then it is continuous or the set $\{x \in X : f(x) \neq 0\}$ is a Christensen zero set.

1. Introduction. The functional equation

$$f(x + f(x)y) = f(x) f(y)$$

is well known and has been studied by many authors (see e.g. [1], [2], [4], [5], [11]–[13], [15], [16], [19]). It is called the Gołąb–Schinzel functional equation. C. G. Popa [16] has proved that every Lebesgue measurable solution $f : \mathbb{R} \to \mathbb{R}$ of (1) is either continuous or equal to zero almost everywhere. We are going to show that the same is true for each Christensen measurable solution of the functional equation

$$f(x + f(x)^n y) = f(x) f(y)$$

mapping a real (complex) linear topological separable $F$-space into the set of all reals (complex numbers), where $n$ is a positive integer.

Equation (2) is a natural generalization of (1). It is also a particular case ($k = 0$, $t = 1$) of the functional equation

$$f(f(y)^k x + f(x)^t y) = tf(x) f(y)$$

considered in various cases e.g. in [3], [4], [7], [18]. It is also worth mentioning that there is a strict connection between the solutions of equation (2) in the

1991 Mathematics Subject Classification: Primary 39B52.

Key words and phrases: Gołąb–Schinzel functional equation, Christensen measurability, $F$-space.
class of functions \( f : \mathbb{R} \to \mathbb{R} \) and a class of subgroups of the Lie group \( L^1_{\mathbb{R}^n+1} \) (cf. [5], [6]).

Throughout this paper \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) will denote the sets of all positive integers, all integers, reals, and complex numbers, respectively. \( X \) stands for a linear space over a field \( K \in \{ \mathbb{R}, \mathbb{C} \} \), unless explicitly stated otherwise. \( m \) and \( m_1 \) are Lebesgue and inner Lebesgue measures in \( K \), respectively.

2. Preliminary lemmas. Let us start with the following

**Lemma 1.** A function \( f : X \to K, f \neq 0 \) (i.e. \( f^{-1}(\{0\}) \neq X \)), is a solution of equation (2) iff there exist an additive subgroup \( A \) of \( X \), a multiplicative subgroup \( W \) of \( K \), and a function \( w : W \to X \) such that

\[
\begin{align*}
(3) & \quad a^n A = A \quad \text{for } a \in W; \\
(4) & \quad w(ab) - a^n w(b) - w(a) \in A \quad \text{for } a, b \in W; \\
(5) & \quad w(a) \in A \quad \text{iff } a = 1; \\
(6) & \quad f(x) = \begin{cases} 
  a & \text{if } x \in w(a) + A \text{ and } a \in W, \\
  0 & \text{otherwise,}
\end{cases} \quad \text{for } x \in X.
\end{align*}
\]

Furthermore, \( W = f(X) \setminus \{0\} \) and \( A = f^{-1}(\{1\}) \).

The proof does not differ essentially from the proof of Theorem 1 of [13] (cf. also [19] and [12], pp. 275–277). Therefore we omit it.

The subsequent corollary follows from Lemma 1.

**Corollary 1.** If a function \( f : X \to K, f \neq 0 \), satisfies equation (2), \( A = f^{-1}(\{1\}) \), and \( W = f(X) \setminus \{0\} \), then:

(i) \( A \) is an additive group;
(ii) \( W \) is a multiplicative group;
(iii) \( A \setminus \{0\} \) is the set of periods of \( f \);
(iv) if \( x, y \in X \) and \( f(x) = f(y) \neq 0 \), then \( x - y \in A \);
(v) \( a^n A = A \) for \( a \in W \).

**Lemma 2.** Let \( f : K \to K \) be a microperiodic function (i.e. the set of periods of \( f \) is dense in \( K \)) satisfying equation (2). Suppose that there exists \( a \in K \) such that \( |f(a)| \notin \{0,1\} \). Then \( m_1(f^{-1}(K_j)) = 0 \) for \( j \in \mathbb{N} \), where \( K_j = \{ a \in K : 1/j \leq |a| \leq j \} \).

**Proof.** For an indirect proof suppose that there is \( k \in \mathbb{N} \) with \( m_1(f^{-1}(K_k)) > 0 \). Then, in view of Corollary 1(ii), there exists \( b \in K \) with \( |f(b)| > (k+1)^2 \). Put \( D = b + f(b)^n f^{-1}(K_k) \). It is easily seen that \( m_1(D) > 0 \) and, by (2), \( |f(a)| > k \) for \( a \in D \). Thus \( D \cap f^{-1}(K_k) = \emptyset \). On the other hand, according to a theorem of H. Steinhaus (see e.g. [14], Theorem 3.7.1), \( \text{int}(D - f^{-1}(K_k)) \neq \emptyset \). Consequently, there exists \( c \in D - f^{-1}(K_k) \) such
that $c \neq 0$ and $f(a + c) = f(a)$ for $a \in K$, which means that $f^{-1}(K_k) \cap D = (f^{-1}(K_k) + c) \cap D \neq \emptyset$, a contradiction.

Given $b \in \mathbb{C} \setminus \{0\}$ and $j \in \mathbb{N}$ let us put
\begin{equation}
C_j(b) = \{a \in \mathbb{C} \setminus \{0\} : (j - 1)\frac{2\pi}{3} \leq \text{Arg } b^{-1}a < j\frac{2\pi}{3}\},
\end{equation}
where $\text{Arg } c \in [0, 2\pi)$ denotes the argument of $c \in \mathbb{C} \setminus \{0\}$. It is easy to see that $\mathbb{C} \setminus \{0\} = \bigcup\{C_j(b) : j = 1, 2, 3\}$.

**Lemma 3.** Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a microperiodic solution of (2) such that the set $f(\mathbb{C})$ is infinite and $|a| = 1$ for $a \in f(\mathbb{C}) \setminus \{0\}$. Then $m_i(f^{-1}(C_j(b))) = 0$ for every $j = 1, 2, 3$, $b \in \mathbb{C} \setminus \{0\}$.

**Proof.** For an indirect proof suppose that there exist $b \in \mathbb{C} \setminus \{0\}$ and $k \in \{1, 2, 3\}$ with $m_i(f^{-1}(C_k(b))) > 0$. Since $f(\mathbb{C})$ is infinite, in view of Corollary 1(ii), $f(\mathbb{C}) \setminus \{0\}$ is dense in the set $J = \{a \in \mathbb{C} : |a| = 1\}$. Thus there is $d \in \mathbb{C}$ such that $f(d) \neq 0$ and $(f(d)C_k(b)) \cap C_k(b) = \emptyset$. Define $D = d + f(d)^n f^{-1}(C_k(b))$. Then, in virtue of (2), $f(D) = f(d)C_k(b)$. Hence $D \cap f^{-1}(C_k(b)) = \emptyset$. On the other hand, $m_i(D) > 0$, which, according to the theorem of Steinhaus, means that $\text{int}(D - f^{-1}(C_k(b))) \neq \emptyset$. Consequently, there exists a period $c \in D - f^{-1}(C_k(b))$ of $f$, from which we derive that $f^{-1}(C_k(b)) \cap D = (f^{-1}(C_k(b)) + c) \cap D \neq \emptyset$, a contradiction.

**Lemma 4.** If a function $f : K \rightarrow K$, $f \neq 1$, satisfies equation (2), then $m_i(f^{-1}(\{a\})) = 0$ for each $a \in f(K) \setminus \{0\}$.

**Proof.** For an indirect proof suppose that there is $a \in f(K) \setminus \{0\}$ with $m_i(f^{-1}(\{a\})) > 0$. Fix $b \in f^{-1}(\{a\})$ and put $D = f^{-1}(\{a\}) - b$. Then, on account of Corollary 1(iv), $D \subset A := f^{-1}(\{1\})$. Thus $m_i(A) > 0$. Consequently, by the theorem of Steinhaus and Corollary 1(i), $A = K$, a contradiction.

**Lemma 5.** Let $f : X \rightarrow K$ be a function satisfying equation (2), $W = f(X) \setminus \{0\}$, and $A = f^{-1}(\{1\})$. Suppose that there is $a_0 \in W$ such that $a_0^n \neq 1$ and $(a_0^n - 1)A \subset A$. Then
\begin{equation}
a^n \neq 1 \quad \text{for each } a \in W \setminus \{1\}
\end{equation}
and there exists $x_0 \in X \setminus \bigcup\{(a^n - 1)^{-1}A : a \in W \setminus \{1\}\}$ such that
\begin{equation}
f(x) = \begin{cases} a & \text{if } x \in (a^n - 1)x_0 + A \text{ and } a \in W, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for } x \in X.
\end{equation}

**Proof.** In view of Lemma 1 there is a function $w : W \rightarrow X$ such that (4)–(6) hold. Let $x_0 = (a_0^n - 1)^{-1}w(a_0)$. Since, by (4),
\[
w(ab) - a^n w(b) - w(a), w(ba) - b^n w(a) - w(b) \in A \quad \text{for } a, b \in W,
\]
Corollary 1(i) implies that $a^n w(b) + w(a) - b^n w(a) - w(b) \in A$ for $a, b \in W$. Thus, for each $b \in W$, $-(b^n - 1)x_0 + w(b) \in A$. Consequently, according to
(5), (6), and Corollary 1(i), conditions (8) and (9) hold and \((a^n - 1)x_0 \notin A\) for \(a \in W \setminus \{1\}\), which completes the proof.

**Lemma 6.** Let \(Y\) be a linear space over a subfield \(F\) of the field \(K\). Let \(f : Y \to K \setminus \{0\}\) be a solution of equation (2) such that \(f(x)^n \in F\) for each \(x \in Y\). Then \(f = 1\).

**Proof.** Suppose that there is \(x \in Y\) with \(f(x)^n \neq 1\) and put \(z = (1 - f(x)^n)^{-1}x\). Then \(x + f(x)^n z = z\) and, in view of (2),

\[
f(x)f(z) = f(x + f(x)^n z) = f(z) \neq 0,
\]

from which we derive \(f(x) = 1\), a contradiction.

Hence \(f(x)^n = 1\) for each \(x \in Y\). Thus \(f(x + y) = f(x)f(y)\) for \(x, y \in Y\) and consequently, for each \(x \in Y\),

\[
f(x) = f \left( \frac{1}{n} x \right) = f \left( \frac{1}{n} \right)^n = 1.
\]

This completes the proof.

**Lemma 7.** If a function \(f : X \to K, f \neq 0\), satisfies equation (2), then \(f(f(x)^{-n}(z - x)) = f(z)f(x)^{-1}\) for \(x, z \in X\) with \(f(x) \neq 0\).

**Proof.** Fix \(x \in X\) with \(f(x) \neq 0\). Setting \(z = f(x)^n y + x\) in (2), we get \(f(z) = f(x)f(f(x)^{-n}(z - x))\) for \(z \in X\), which yields the assertion.

**Lemma 8.** Let \(B\) be an additive subgroup of a real linear space \(Y\) and let \(V\) be an infinite multiplicative subgroup of \(\mathbb{R}\) such that

\[
(10) \quad ax \in B \quad \text{for} \quad x \in B, \quad a \in V.
\]

Then the set \(B_x = \{a \in \mathbb{R} : ax \in B\}\) is dense in \(\mathbb{R}\) for each \(x \in B\).

**Proof.** Note that, for each \(c \in \mathbb{R}, c > 0\), there is \(b \in V\) with \(|b| < c\).

Since, for each \(x \in B\), \(B_x\) is an additive group and, by (10), \(V \subset B_x\), we obtain the statement.

**Lemma 9.** Let \(B\) be an additive subgroup of a complex linear space and let \(V\) be an infinite multiplicative subgroup of \(\mathbb{C}\) such that \(V \not\subset \mathbb{R}\) and (10) holds. Then the set \(B_x = \{a \in \mathbb{C} : ax \in B\}\) is dense in \(\mathbb{C}\) for each \(x \in B\).

**Proof.** Let \(x \in B\) and \(J = \{a \in \mathbb{C} : |a| = 1\}\). Note that \(V \subset B_x\). If \(V \subset J\), then \(V\) is dense in \(J\). Thus \(B_x\) is dense in \(\mathbb{C}\), because it is an additive group. On the contrary, if there is \(a \in V \setminus (R \cup J)\), then, for each \(c \in \mathbb{R}, c > 0\), there exists \(k \in \mathbb{Z}\) with \(|a|^k < c\) and \(|a^{k+1}| < c\). Since \(a^k\) and \(a^{k+1}\) are linearly independent over \(\mathbb{R}\), the additive group generated by \(V\) is dense in \(\mathbb{C}\), which completes the proof.

**Lemma 10** (cf. [16], Théorème 1). If \(D_1, D_2 \subset K\) and \(m_1(D_j) > 0, j = 1, 2\), then \(\text{int}(D_1 \cdot D_2) \neq \emptyset\).
Proof. First consider the case where $K = \mathbb{R}$. There exist closed sets $F_i \subset D_i$ such that $m(F_i) > 0$ for $i = 1, 2$. Put $F_i^k = F_i \cap ([-k, -1/k] \cup [1/k, k])$ for $k \in \mathbb{N}$, $i = 1, 2$. It is easily seen that there are $p, q \in \mathbb{N}$ with $m(F_i^p) > 0$ and $m(F_i^q) > 0$. Let $F_i^+ = F_i^p \cap (0, \infty)$, $F_i^- = F_i^p \cap (-\infty, 0)$, $F_i^2 = F_i^q \cap (0, \infty)$, and $F_i^3 = F_i^q \cap (-\infty, 0)$. Define

$$F_i^0 = \begin{cases} F_i^+ & \text{if } m(F_i^+) > 0, \\ F_i^- & \text{otherwise,} \end{cases}$$

for $i = 1, 2$.

Observe that $m(F_i^0) > 0$ for $i = 1, 2$. Thus (see e.g. [17], Theorem 8.26), $m(\ln(F_i^0)) > 0$ for $i = 1, 2$. Hence, in virtue of the theorem of Steinhaus, $\int(\ln(F_i^0) \cdot F_i^0) = \int(\ln(F_i^0) + \ln F_i^0) \neq 0$, which means that $\int(D_1 \cdot D_2) \neq 0$.

Now assume that $K = \mathbb{C}$. Let $F_i \subset D_i$ be a closed set such that $m(F_i) > 0$ for $i = 1, 2$. Put $C_k = \{a \in \mathbb{C} : 1/k \leq |a| \leq k\}$ and $F_i^k = F_i \cap C_k$ for $k \in \mathbb{N}$, $i = 1, 2$. It is easily seen that there are $p, q \in \mathbb{N}$ with $m(F_i^p) > 0$ and $m(F_i^q) > 0$. Define functions $h_1 : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C}$, $h_2 : (0, 2\pi) \times (0, \infty) \to \mathbb{C}$, and $h_3 : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2$ by the formulas: $h_1(a, b) = a + ib$, $h_2(a, b) = b(\cos a + i \sin a)$, $h_3(a, b) = (a, \ln b)$. Let $F_i^0 = h_1^{-1}(F_i^p)$ and $F_i^0 = h_3^{-1}(F_i^q)$. Then $F_i^0$ is a Borel set and $m(F_i^0) > 0$ for $i = 1, 2$ ($m$ denotes also the Lebesgue measure in $\mathbb{R}^2$). Note that $h = h_3 \circ h_2^{-1} \circ h_1$ is a diffeomorphism onto the set $h(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$. Thus $h(F_i^0)$ is a Borel set and $m(h(F_i^0)) > 0$ for $i = 1, 2$ (see e.g. [17], Theorem 8.26(c)). Hence, by the theorem of Steinhaus, $\int(h(F_i^0) + h(F_i^0)) \neq 0$. Since $h(F_i^0) + h(F_i^0) = h_3 \circ h_2^{-1}(h_1(F_i^0)h_1(F_i^0))$, we have $\int(h_1(F_i^0) \cdot h_1(F_i^0)) \neq 0$, which implies the assertion.

3. Christensen measurability. Throughout this part we assume that $X$ is a separable $F$-space as a topological linear space over $K$. We shall use the notation and terminology from [8]–[10] concerning Christensen measurability. Now, we only recall necessary definitions and facts.

Let $M$ be the $\sigma$-algebra of all universally measurable subsets of $X$; i.e. $M$ is the intersection of all completions of the Borel $\sigma$-algebra of $X$ with respect to probability Borel measures. In the following a measure is a countable additive Borel measure extended to $M$.

Definition 1. A set $B \in M$ is a Haar zero set iff there exists a probability measure $u$ on $X$ such that $u(B + x) = 0$ for each $x \in X$.

Definition 2. A set $P \subset X$ is a Christensen zero set iff it is a subset of a Haar zero set.

Definition 3. A set $D \subset X$ is Christensen measurable iff $D = B \cup P$, where $B \in M$ and $P$ is a Christensen zero set.
Let us define
\[ C_0 = \{ B \subset X : B \text{ is Christensen zero set} \}, \]
\[ C = \{ B \subset X : B \text{ is Christensen measurable} \}. \]

**Lemma 11** (see [9], Theorem 1). *Every countable union of Christensen zero sets is a Christensen zero set.*

**Lemma 12** (see [9], Theorem 2). *If \( B \in \mathcal{C} \setminus C_0 \), then \( 0 \in \text{int}(B - B) \).*

**Definition 4.** A function \( f : X \to K \) is said to be Christensen measurable iff \( f^{-1}(U) \in \mathcal{C} \) for each open set \( U \subset K \).

**Lemma 13** (see [10], Theorem 1). *Let \( f : X \to K \) be a Christensen measurable linear functional. Then \( f \) is continuous.*

Put \( L_k = \{ a \in K : k - 1 \leq |a| < k \} \) and \( a_k = m(L_k) \) for \( k \in \mathbb{N} \). Given a Borel set \( D \subset X \) and \( x \in X \) denote \( u_x(D) = m_p(k_x^{-1}(D)) \), where \( k_x : K \to X \), \( k_x(a) = ax \), and, for each Borel set \( B \subset K \), \( m_p(B) = \sum_{k=1}^{\infty} 2^{-k} a_k^{-1} m(B \cap L_k) \). Since \( k_x \) is continuous, \( u_x \) is a well defined Borel measure and \( u_x(X) = 1 \) for each \( x \in X \setminus \{0\} \).

**Lemma 14.** *Let \( D \in \mathcal{C} \setminus C_0 \) and \( x \in X \setminus \{0\} \). Then there exist a Borel set \( D_x \subset D \) and \( y_x \in X \) such that*
\[ m(k_x^{-1}(y_x + D_x)) > 0. \]  

**Proof.** There exist \( B \in M \) and \( P \in C_0 \) with \( D = B \cup P \). In view of Lemma 11, \( B \not\in C_0 \). Thus there is \( y \in X \) such that \( \pi(B + y) > 0 \), where \( \pi \) denotes the extension of \( u_x \) to \( M \). Put \( u_0(T) = \pi(T + y) \) for each \( T \in M \). Then \( u_0 \) is a probability measure. Hence there are a Borel set \( B_x \subset B \) and a set \( B_0 \subset B \) such that \( u_0(B_0) = 0 \) and \( B = B_x \cup B_0 \). Furthermore \( u_x(B_x + y) = \pi(B_x + y) = u_0(B_x) = u_0(B_x \cup B_0) = u_0(B) = \pi(B + y) > 0 \). Consequently, \( m_p(k_x^{-1}(B_x + y)) > 0 \), which implies (11). This ends the proof.

**Lemma 15.** *Let \( L \subset K \setminus \{0\} \) and \( x \in X \setminus \{0\} \). Let \( f : X \to K \) be a function satisfying equation (2). Suppose that \( f^{-1}(L) \in \mathcal{C} \setminus C_0 \). Then there exists \( z \in X \) such that \( f(z) \neq 0 \) and \( m_0(f_x^{-1}(f(z)^{-1}L)) > 0 \), where \( f_x : K \to K \), \( f_x(a) = f(ax) \).

**Proof.** It follows from Lemma 14 that there are a Borel set \( D_x \subset D := f^{-1}(L) \) and \( y_x \in X \) such that (11) holds. Put \( B = (Kx - y_x) \cap D_x \). Then, according to the definition of \( k_x \) and (11), \( B \neq \emptyset \). Fix \( z \in B \). It is easily seen that \( f(z) \neq 0 \) and there exists \( b \in K \) with \( z = bx - y_x \). Thus
\[ B - z = ((Kx - y_x) \cap D_x) - bx + y_x = (Kx \cap (D_x + y_x)) - bx, \]
which means that \( k_x^{-1}(B - z) = k_x^{-1}(D_x + y_x) - b \). Hence, in view of (11),
\[ m(f(z)^{-n}(k_x^{-1}(B - z))) > 0. \]
Note that, by Lemma 7,
\[
x^{-n}(f^{-1}(B - z)) = f(z)^{-n}(f^{-1}(B - z)) = f(z)^{-1}f(B) \subset f(z)^{-1}L.
\]
Consequently, \( f^{-n}(f^{-1}(B - z)) \subset f(z)^{-1}(f(z)^{-1}L) \), from which we derive by (12), that \( m_1(f^{-1}(f(z)^{-1}L)) > 0 \). This completes the proof.

**Lemma 16.** Let \( f : X \to K \) be a Christensen measurable function satisfying equation (2) such that the set \( W = f(X) \setminus \{0\} \) is infinite. Suppose that the set \( S_f = \{ x \in X : f(x) \neq 0 \} \) is not a Christensen zero set. Then the set \( A = f^{-1}(\{1\}) \) is a proper linear subspace of \( X \) over the field
\[
F = \begin{cases}
\mathbb{R} & \text{if } f(x)^n \in \mathbb{R} \text{ for each } x \in X, \\
\mathbb{C} & \text{otherwise}.
\end{cases}
\]
**Proof.** Since \( A \neq X \), it suffices to show that \( A \) is a linear subspace of \( X \) over \( F \).

For an indirect proof suppose that \( A \neq A_0 \), where \( A_0 \) denotes the linear subspace of \( X \) over \( F \) spanned by \( A \). Let \( f_0 = f|_{A_0} \). It is easy to check that \( f_0 \) is a solution of (2) and \( f_0 \neq 1 \). Thus, in view of Lemma 6, \( f_0^{-1}(\{0\}) \neq \emptyset \), from which we derive that there are \( a_0 \in F \setminus \{0\} \) and \( y \in A \setminus \{0\} \) such that \( f(a_0 y) = 0 \). Note that the functions \( f_1 : X \to F, f_1(x) = f(x) \), and \( f_y : F \to F, f_y(a) = f_1(ay) \), also satisfy (2) for \( n = 1 \). Since \( f_y(a_0) = f_1(a_0) = 0 \), we have \( f_y \neq 1 \). Furthermore, \( W_n \subset F, \{ a \in F : ay \in A \} \subset f_y^{-1}(\{1\}) \), and, by Corollary 1(v), \( aA = A \) for \( a \in W_n \), where \( W_n = \{ a^n : a \in W \} \). Hence, by Lemma 8, Lemma 9, and Corollary 1(i)-(iii), \( f_y \) is microperiodic.

First consider the case where there is \( b \in F \) with \( f_y(b) \notin \{0, 1\} \). Let \( F_j = \{ a \in F : 1/j \leq |a| \leq j \} \) for \( j \in \mathbb{N} \). Since \( S_f = \bigcup \{ f^{-1}(F_j) : j \in \mathbb{N} \} \), according to Lemma 11 there exists \( p \in \mathbb{N} \) such that \( f^{-1}_p(F_p) \notin C_0 \). Thus, by Lemma 15 (with \( n = 1 \)), \( m_1(f_y^{-1}(f_1(1)F_p)) > 0 \) for some \( z \in S_f \). Note that there is \( k \in \mathbb{N} \) with \( f_1(z)^{-1}F_p \subset F_k \). Hence \( m_1(f_y^{-1}(F_k)) > 0 \), which contradicts Lemma 2.

Now, assume that the set \( W_y := f_y(F) \setminus \{0\} \) is finite. Then \( W_y \) is a multiplicative cyclic subgroup of \( F \) (cf. Corollary 1(ii)) and \( |a| = 1 \) for each \( a \in W_y \). There exists \( c \in F \) such that \( W_y = \{ c^k : k \in \mathbb{N} \} \). Put \( k_0 = \min\{ k \in \mathbb{N} : c^k = 1 \} \) and define
\[
T_j = \begin{cases}
c^{(0, \infty)} & \text{if } F = \mathbb{R}, \\
\{ a \in \mathbb{C} : 2\pi k_0^{-1}(j - 1) \leq \operatorname{Arg} a < 2\pi k_0^{-1}j \} & \text{if } F = \mathbb{C},
\end{cases}
\]
for \( j \in \mathbb{N}, j \leq k_0 \). Observe that \( S_f = \bigcup \{ f_1^{-1}(T_j) : j \in \mathbb{N}, j \leq k_0 \} \). Thus there is a positive integer \( k \leq k_0 \) such that \( f_1^{-1}(T_k) \notin C_0 \). It results from Lemma 15 that there exists \( z \in S_f \) with \( m_1(f_y^{-1}(f_1(z)^{-1}T_k)) > 0 \). Moreover, there is exactly one positive integer \( p \leq k_0 \) such that \( c^p \in f_1(z)^{-1}T_k \). Consequently, \( m_1(f_y^{-1}(\{c^p\})) > 0 \), contrary to Lemma 4.
It remains to study the case where $F = \mathbb{C}$, $W_y$ is infinite, and $|a| = 1$ for each $a \in W_y$. Since $S_f = \bigcup \{ f_1^{-1}(C_j(1)) : j = 1, 2, 3 \}$, where $C_j(b)$, for $b \in \mathbb{C} \setminus \{0\}$, is given by (7), we have $f_1^{-1}(C_k(1)) \not\in C_0$ for some $k \in \{1, 2, 3\}$. Thus, on account of Lemma 15, there is $z \in S_f$ with $m(f^{-1}_y(f_1(z)^{-1}C_k(1))) > 0$. Clearly, $f_1^{-1}C_k(1) \not\in C_0$ for some $k \in \{1, 2, 3\}$.

4. The main result. Now, we have all tools to prove the announced theorem.

**Theorem.** Suppose that $X$ is a linear topological separable $F$-space over $K$. Let $f : X \to K$ be a Christensen measurable solution of equation (2). Then either $f$ is continuous or the set $S_f = \{ x \in X : f(x) \neq 0 \}$ is a Christensen zero set.

Furthermore, if $f$ is continuous and satisfies (2), then

\begin{equation}
 f(X) \subset \mathbb{R} \quad \text{or} \quad n = 1
\end{equation}

and the following two statements hold:

(i) if $f(X) \subset \mathbb{R}$, then there exists a continuous $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$ such that, for $n$ odd, either

\begin{equation}
 f(x) = \sqrt{g(x) + 1} \quad \text{for} \ x \in X
\end{equation}

or

\begin{equation}
 f(x) = \sqrt{\sup(g(x) + 1, 0)} \quad \text{for} \ x \in X,
\end{equation}

and for $n$ even, $f$ is of the form (16);

(ii) if $f(X) \not\subset \mathbb{R}$ and $n = 1$, then there exists a continuous $\mathbb{C}$-linear functional $g : X \to \mathbb{C}$, $g \neq 0$, such that $f(x) = g(x) + 1$, $x \in X$

**Proof.** Note that if $f \neq 0$ is continuous, then $\text{int} \ S_f \neq \emptyset$, which means that $S_f \not\subset C_0$. Therefore suppose that $S_f \in C \setminus C_0$. Put $W = f(X) \setminus \{0\}$ and $A = f^{-1}(1).

First, consider the case where $W$ is finite. Then, in view of Lemma 1, there is a function $w : W \to X$ with $S_f = \bigcup \{ w(a) + A : a \in W \}$. Thus, by Lemma 11, $A \not\subset C_0$. Hence Lemma 12 and Corollary 1(i) imply that $\text{int} \ A \neq \emptyset$, from which we derive $A = X$. Consequently, (15) or (16) holds with $g = 0$.

Now, assume that $W$ is infinite. Since, in the case where $K = \mathbb{C}$, $X$ is also a real topological linear $F$-space (with the same topology), without loss of generality we may assume that

\begin{equation}
 \text{if } K = \mathbb{C}, \text{ then } f(X) \not\subset \mathbb{R}.
\end{equation}

It results from Lemma 16 that $A$ is a proper linear subspace of $X$ over the field $F$ given by (13). Thus, by Lemma 5, condition (8) is valid and there
exists \( x_0 \in X \setminus A \) such that \( f \) is of the form (9). Hence

\[(18) \quad S_f = A + (W_n - 1)x_0,\]

where \( W_n = \{ a^n : a \in W \} \). Furthermore, in view of Lemma 12, \( 0 \in \text{int}(S_f - S_f) \), whence

\[(19) \quad A + Fx_0 = X.\]

On account of (19) and Lemma 14 there exist a Borel set \( B \subset S_f \) and \( a \in F \), \( x \in A \) with \( m(k_0^{-1}(ax_0 + x + B)) > 0 \), where \( k_0 : F \to X \), \( k_0(a) = ax_0 \). On the other hand, from (18), we obtain \( ax_0 + x + S_f = A + (W_n - 1 + a)x_0 \). Thus \( k_0^{-1}(ax_0 + x + S_f) = W_n - 1 + a \). Since \( k_0^{-1}(ax_0 + x + B) = a + k_0^{-1}(x + B) \), we have \( k_0^{-1}(x + B) \subset W_n - 1 \) and \( m(k_0^{-1}(x + B)) = m(k_0^{-1}(ax_0 + x + B)) > 0 \), from which we derive that \( m(W_n) = m(W_n - 1) > 0 \) (in \( F \)). Hence and from Lemma 10 and Corollary 1(ii) we get \( \text{int} W_n \neq 0 \) (in \( F \)), whence

\[(20) \quad (0, \infty) \subset W_n \quad \text{and} \quad 1 \in \text{int} W_n \quad \text{(in} \ F).\]

We shall prove that (8), (17), and (20) imply \( F = K \).

For an indirect proof suppose that \( K = C \) and \( F = R \). Then there is \( a \in W \setminus R \) with \( a^n \in R \). Observe that, by (20) and Corollary 1(ii), \( a \cdot |a|^{-1} \in W \setminus R \), whence, by (8), \( -1 = (a \cdot |a|^{-1})^n \in W \) and \( (-1)^n \neq 1 \). This means that \( n \) is odd. Consequently, \( a^{n+1} \cdot |a|^{-n-1} = -a \cdot |a|^{-1} \notin R \) and \( (a^{n+1} \cdot |a|^{-n-1})^n = (-1)^{n+1} = 1 \), which contradicts (8).

In this way we have proved that \( F = K \). Thus, by (19), \( A \) is a hyperplane of \( X \) (i.e. codim \( A = 1 \)) and, according to Corollary 1 and (20),

\[(21) \quad \text{for } K = C, \quad W = C \setminus \{0\}, \]
\[(22) \quad \text{for } K = R, \quad W = (0, \infty) \text{ or } W = R \setminus \{0\}, \]

whence (8) yields condition (14).

Define a linear functional \( g : X \to K \) by

\[(23) \quad g(ax_0 + y) = a \quad \text{for } a \in K, \ y \in A.\]

It is easy to check that, on account of (9) and (18),

\[(24) \quad g(x) = f(x)^n - 1 \quad \text{for } x \in S_f,\]

which, in view of (8), (18), and (22), means that, in the case where \( f(X) \subset R \), both conclusions of (i) are valid. In the case where \( n = 1 \) and \( f(X) \not\subset R \), (21), (18), and (24) imply that \( f(x) = g(x) + 1, \ x \in X \). Therefore, on account of Lemma 13, it remains to show that \( g \) is Christensen measurable.

If \( n = 1 \) and \( f(X) \not\subset R \), this is obvious, because \( f \) is Christensen measurable. On the other hand, if \( f(X) \subset R \), then \( g(x) = f(x)^n - 1 \) for \( x \in g^{-1}((-1, \infty)) \). Furthermore, for each set \( U \subset R, \ g^{-1}(U) = g^{-1}(U^+) \cup (-g^{-1}(-U^-)) \cup g^{-1}(U_0) \), where \( U^+ = U \cap (0, \infty), \ U^- = U \cap (-\infty, 0) \) and
$U_0 = U \cap \{0\}$. This implies that $g$ is Christensen measurable, which ends the proof.

Remark. It is easy to check that each function $f : X \to K$ satisfying (14) and conditions (i), (ii) of the Theorem is a solution of equation (2).

Finally, since in the case where $X$ is locally compact, $C_0$ coincides with the set of all the Haar measure zero subsets of $X$ (see [9], p. 256), from the Theorem we get the following

Corollary 2. Let $k \in \mathbb{N}$ and let $f : K^k \to K$ be a Lebesgue measurable solution of equation (2). Then either $f$ is continuous or the set $S_f$ is of Lebesgue measure zero.

Acknowledgements. I wish to thank Professor Karol Baron for calling my attention to the problem.

References


Institute of Mathematics
Pedagogical University of Rzeszów
Rejtana 16A
35-310 Rzeszów, Poland

*Reçu par la Rédaction le 17.12.1990
Révisé le 23.10.1995*