ANNALES
POLONICI MATHEMATICI
LXIV.2 (1996)

On the C^0-closing lemma

by Anna A. Kwiecińska (Kraków)

Abstract. A proof of the C^0-closing lemma for noninvertible discrete dynamical systems and its extension to the noncompact case are presented.

1. Introduction. One of the most significant theorems in the theory of smooth dynamical systems is the C^1-closing lemma established by C. C. Pugh in [4], and its proof is fairly advanced. The C^r-closing lemma, $r > 1$, still remains an unsolved problem. A C^0-closing lemma for compact manifolds is stated without proof in [5]. The aim of this paper is to provide a proof of this lemma, as well as to extend it to the noncompact case.

We study the behaviour of the sequence $(f^n(x))_{n=0}^{\infty}$, where f is a continuous function from a finite-dimensional manifold into itself. Using a method by H. Lehning [2] and the Tietze extension theorem, we prove Theorem 2 which is the standard C^0-closing lemma for a compact manifold, quoted (without proof) in [5]. Theorem 1 is a modification of the C^0-closing lemma, obtained without the assumption that the point x_0 is nonwandering. The main result of the present paper is Theorem 3, which is a generalization of the C^0-closing lemma to the case of a not necessarily compact manifold. Moreover, under suitable assumptions, in Theorem 4 we prove the C^0-closing lemma in the invertible case, using a lemma by Z. Nitecki and M. Shub [3].

Let M denote a compact topological manifold with boundary, N its dimension and d a metric on M compatible with the topology of M. $C(M)$ will denote the space of continuous functions from M into itself and d_∞ the metric of uniform convergence on $C(M)$: $d_\infty(f,g) = \max_{x\in M} d(f(x), g(x))$. If B is a subset of M, we denote its interior by $\operatorname{Int} B$, its closure by $\overline{B}$, its boundary by ∂B and its diameter by $d(B)$. $B(x_0,r)$ will stand for the open ball with center x_0 and radius r, and $\overline{B}(x_0,r)$ for the closed ball with center x_0 and radius r. Fix once and for all a continuous function $f : M \to M$.

1991 *Mathematics Subject Classification*: Primary 58F20.
Key words and phrases: closing lemma, nonwandering point, periodic point.

2. Results

DEFINITION 1. We call a point $x \in M$ *eventually periodic* if there exists n such that $f^n(x)$ is periodic.

THEOREM 1. *Let $f : M \to M$ be a continuous function and $x_0 \in M$ a point. Then for every $\varepsilon > 0$ there exists a continuous function $g : M \to M$ and $\delta > 0$ such that $d_\infty(f, g) < \varepsilon$ and for every $x \in M$ satisfying $d(x, x_0) < \delta$, x is an eventually periodic point of g.*

Proof. Case I: *x_0 is an eventually periodic point of f.* There exists an open neighbourhood U of $f(x_0)$ such that $\overline{U}$ is homeomorphic to I^N, where I is the closed unit interval, and $d(\overline{U}) < \varepsilon$. We define $K = f^{-1}(\overline{U})$. Then K is a neighbourhood of x_0 and $\operatorname{Int} K \neq \emptyset$. We reduce K if necessary so that it does not contain any point of the sequence $(f^n(x_0))$. This is possible as the set $(f^n(x_0))$ is finite. We choose a compact subset k with $k \subset \operatorname{Int} K$ and $x_0 \in \operatorname{Int} k$. We now define the required function g:

• We put $g = f(x_0)$ on k and $g = f$ on ∂K. This gives a continuous map from $k \cup \partial K$ to $\overline{U}$, which is homeomorphic to I^N. Since $k \cup \partial K$ is a closed subset of K, by the Tietze extension theorem (see [1]), g extends to a continuous map (still denoted by g) from K to $\overline{U}$. As $d(\overline{U}) < \varepsilon$, $d(f(x), g(x)) < \varepsilon$ for each $x \in K$.

• We put $g = f$ outside K.

As g agrees with f on the boundary of K, it follows that g is continuous on M. As $d(f(x), g(x)) < \varepsilon$ for each $x \in M$ and the function $x \to d(f(x), g(x))$ is continuous on the compact set M, we have $d_\infty(f, g) < \varepsilon$.

We choose $\delta > 0$ with $B(x_0, \delta) \subset \operatorname{Int} k$. For x such that $d(x, x_0) < \delta$ we have $g(x) = f(x_0)$ and hence x is an eventually periodic point of g.

Case II: *x_0 is not eventually periodic for f.* Then the set $S = \{x_0, f(x_0), f^2(x_0), \ldots\}$ is infinite.

Using the compactness of M, we construct a finite cover of M by open sets of diameters less than ε and with closures homeomorphic to I^N. Let p be the smallest positive integer such that there are an integer $n < p$ and an element V of the cover containing $f^p(x_0)$ and $f^n(x_0)$ (such an integer exists because the set S is infinite). This property implies in particular that the points $x_0, f(x_0), \ldots, f^{p-1}(x_0)$ are distinct.

Let A be a closed subset of M, B and C two open subsets and $y \notin A$ be such that $f(y) \in B$ and $y \in C$. Using the continuity of f at y, we can construct a compact subset K homeomorphic to I^N such that $K \cap A = \emptyset$, $K \subset C$, $y \in \operatorname{Int} K$, $d(K) < \varepsilon$ and $f(K) \subset B$.

Using this property for $A = \{x_0, f(x_0), \ldots, f^{p-2}(x_0)\}$, $y = f^{p-1}(x_0)$, $B = V$ and $C = M$, we obtain a compact set K_{p-1}. We choose a compact set $k_{p-1} \subset \operatorname{Int} K_{p-1}$ such that $f^{p-1}(x_0) \in \operatorname{Int} k_{p-1}$.

We define the desired function g in the following way:

• We put $g = f^n(x_0)$ on k_{p-1} and $g = f$ on ∂K_{p-1}; g takes its values in $\overline{V}$, which is homeomorphic to I^N. Again, g extends to a continuous map from K_{p-1} to $\overline{V}$. As $d(\overline{V}) < \varepsilon$, $d(f(x), g(x)) < \varepsilon$ for each $x \in K_{p-1}$.

• We put $g = f$ outside K_{p-1}.

The function g is continuous and $d_\infty(f, g) < \varepsilon$. We choose $\delta > 0$ such that $B(x_0, \delta) \subset (g^{p-1})^{-1}(k_{p-1})$ (this is possible because $g^{p-1}(x_0) \in \operatorname{Int} k_{p-1}$). For $x \in B(x_0, \delta)$ we have $g^p(x) = f^n(x_0) = g^n(x_0)$. Hence the point $g^p(x)$ is periodic of period $p - n$. The result follows.

Definition 2. We call a point $x \in M$ *wandering* if there is a neighbourhoood U of x such that

$$\bigcup_{n>0} f^n(U) \cap U = \emptyset,$$

and *nonwandering* otherwise.

Theorem 2 (The C^0-closing lemma). *Let $f : M \to M$ be a continuous function and $x_0 \in M$ a nonwandering point. Then for every $\varepsilon > 0$ there exists a continuous function $g : M \to M$ such that $d_\infty(f, g) < \varepsilon$ and x_0 is a periodic point of g.*

Proof. Clearly we can assume that x_0 is not periodic for f.

Since x_0 is nonwandering, for every neighbourhood U of x_0 there exist $x \in U$ and N such that $f^N(x) \in U$.

Let $\varepsilon > 0$; there exists $\delta_1 > 0$ such that $d(f(x), f(x_0)) < \varepsilon/8$ if $d(x, x_0) < \delta_1$.

Let n be such that $1/n < \delta_1$. There exists a neighbourhood U_0 of x_0 such that $U_0 \subset B(x_0, 1/n)$, $\overline{U}_0$ is homeomorphic to I^N and $d(\overline{U}_0) < \varepsilon/8$. Let $x_n \in U_0$ be such that there exists N satisfying $f^N(x_n) \in U_0$ and $f(x_n) \neq x_n$. Such a point exists, because otherwise, as x_0 is nonwandering, we would find a sequence $\{x_n\}$ with $d(x_n, x_0) \to 0$ $(n \to \infty)$ and $f(x_n) = x_n$; since f is continuous, we would have $f(x_0) = x_0$, which is impossible, as x_0 is not periodic. Let n_0 be the smallest integer, $n_0 \geq 1$, such that $f^{n_0}(x_n) \in U_0$ and $x_n \neq f^{n_0}(x_n)$. Then the points $x_n, f(x_n), \ldots, f^{n_0-1}(x_n)$ are distinct. Indeed, if any two of them were equal, then x_n would be eventually periodic, which is impossible because $f^{n_0}(x_n) \in U_0$.

We construct a cover of the manifold M by open sets of diameters less than $\varepsilon/8$ and with closures homeomorphic to I^N in the following way:

• We take U_0 as above; notice that except for x_n and $f^{n_0}(x_n)$, U_0 does not contain any points of $A = \{x_n, f(x_n), \ldots, f^{n_0}(x_n)\}$.

• We take U_1 which contains $f(x_n)$ and does not contain any other points of A.

• We continue in the same way till U_{n_0-1} which contains $f^{n_0-1}(x_n)$ and does not contain any other points of A.

• Other sets U_i of the cover do not contain any points of A.

Since M is compact, $\{U_i\}$ has a finite subcover and exactly as in the proof of Theorem 1, we construct a function g_n satisfying $d(f, g_n) < \varepsilon/8$. The important thing to notice here is that from the construction of $\{U_i\}$ we have $n = 0$, $p = n_0$ and therefore $g_n(x) = x_n$ on k_{n_0-1}, where $g_n^{n_0-1}(x_n) \in \operatorname{Int} k_{n_0-1}$ (see the proof of Theorem 1), hence x_n is a periodic point of g_n.

If there is a $k \in \{0, 1, \ldots, n_0 - 1\}$ such that $g_n^k(x_n) = x_0$, then x_0 is periodic.

So we assume that $x_n \neq x_0, \ldots, g_n^{n_0-1}(x_n) \neq x_0$. We construct a function h in the following way:

• We put $h = x_0$ on $g_n^{n_0-1}(x_n)$ and $h = g_n = x_n$ on ∂k_{n_0-1}, and extend it to a continuous map from k_{n_0-1} to U_0. Therefore $d(g_n(x), h(x)) < \varepsilon/8$ for each $x \in k_{n_0-1}$.

• We put $h = g_n$ outside k_{n_0-1}.

The function h is continuous, $d_\infty(f, h) \le d_\infty(f, g_n) + d_\infty(g_n, h) < \varepsilon/4$ and $h^{n_0}(x_n) = x_0$.

For x such that $d(x, x_0) < \delta_1$ we have

$$\begin{aligned} d(h(x), h(x_n)) &\le d(h(x), f(x)) + d(f(x), f(x_0)) \\ &\quad + d(f(x_0), f(x_n)) + d(f(x_n), h(x_n)) \\ &< 3\varepsilon/4, \end{aligned}$$

and hence $h(B(x_0, \delta_1)) \subset B(h(x_n), 3\varepsilon/4)$. There exists a cover $\{Z_i\}$ of $\overline{B}(h(x_n), 3\varepsilon/4)$ such that $\overline{B}(h(x_n), 3\varepsilon/4) \subset \bigcup_{i\in I} \operatorname{Int} Z_i$, where Z_i are homeomorphic to I^N and $d(Z_i) < \varepsilon/4$ for $i \in I$. There exists an i_0 such that $x_0 \in h^{-1}(\operatorname{Int} Z_{i_0})$. Since $h^{-1}(\operatorname{Int} Z_{i_0})$ is open, there exists $\delta_2 > 0$ such that $B(x_0, \delta_2) \subset h^{-1}(\operatorname{Int} Z_{i_0}) \cap B(x_0, \delta_1)$ and $B(x_0, \delta_2)$ does not meet the set $\{x_n, h(x_n), \ldots, h^{n_0-1}(x_n)\}$ (this last statement holds because $x_n \neq x_0$, $h(x_n) \neq x_0, \ldots, h^{n_0-1}(x_n) \neq x_0$, as $g_n^k(x_n) \neq x_0$ for $k = 0, 1, \ldots, n_0 - 1$, $h = g_n$ outside k_{n_0-1}, $x_n \notin k_{n_0-1} \subset K_{n_0-1}$ (see the construction in the proof of Theorem 1), therefore $h(x_n) = g_n(x_n)$ and hence $h^k(x_n) = g_n^k(x_n)$).

We define a function g in the following way:

• We put $g = h(x_n)$ on x_0 and $g = h$ on $\partial\overline{B}(x_0, \delta_2)$, and extend it to a continuous map from $\overline{B}(x_0, \delta_2)$ to Z_{i_0}. As $d(Z_{i_0}) < \varepsilon/4$, we have $d(h(x), g(x)) < \varepsilon/4$ for each $x \in \overline{B}(x_0, \delta_2)$.

• We put $g = h$ outside $\overline{B}(x_0, \delta_2)$.

The function g is continuous, x_0 is a periodic point of g and $d_\infty(g, f) \le d_\infty(g, h) + d_\infty(h, f) < \frac{1}{4}\varepsilon + \frac{3}{4}\varepsilon = \varepsilon$. The result follows.

In the above proof all the modifications of the function f have been made locally. Therefore the local compactness of M is sufficient. A theorem analogous to Theorem 2 can be proved in the following situation:

M is a finite-dimensional manifold with boundary (which is obviously locally compact), N its dimension, $C(M)$ the space of continuous functions from M into itself, $f|_Z$ the restriction of a function f to a subset $Z \subset M$, and τ_∞ the topology of uniform convergence on $C(M)$. We set $d_\infty(f,g) = \sup_{x\in M} d(f(x), g(x))$, which may not be a metric, nevertheless we have: $f_n \to f$ $(n \to \infty)$ in the topology τ_∞ if and only if $d_\infty(f_n, f) \to 0$ $(n \to \infty)$.

THEOREM 3 (A generalization of the C^0-closing lemma). *Let $f : M \to M$ be a continuous function, where M is a manifold with boundary, and $x_0 \in M$ a nonwandering point. Then for every $\varepsilon > 0$ there exists a continuous function $g : M \to M$ such that $d_\infty(f,g) < \varepsilon$ and x_0 is a periodic point of g.*

P r o o f. Let $\varepsilon > 0$ and V_0 be a neighbourhood of x_0 such that $\overline{V}_0$ is compact.

As in the proof of Theorem 2, we assume that x_0 is not periodic for f. We choose $\delta_1 > 0$ such that $d(f(x), f(x_0)) < \varepsilon/8$ if $d(x,x_0) < \delta_1$ and $B(x_0,\delta_1) \subset V_0$. Moreover, n fulfils the condition $1/n < \delta_1$, U_0 is a neighbourhood of x_0 such that $\overline{U}_0 \subset B(x_0, 1/n) \cap V_0$, $\overline{U}_0$ is homeomorphic to I^N and $d(\overline{U}_0) < \varepsilon/8$, x_n is a point for which there exists N such that $f^N(x_n) \in U_0$ and $f(x_n) \neq x_n$, and n_0 is the smallest integer ≥ 1 for which $x_n \neq f^{n_0}(x_n) \in U_0$.

We define the sets $V_0, V_1, \ldots, V_{n_0-1}$ in the following way:

• V_0 has already been defined above.

• V_1 is a neighbourhood of $f(x_0)$ with $\overline{V}_1$ compact and $f(\overline{V}_0) \subset V_1$. Such a neighbourhood exists, because for each $x \in f(\overline{V}_0)$ there exists a neighbourhood V_1^x such that $\overline{V_1^x}$ is compact. The set $f(\overline{V}_0)$ is compact, hence we can choose a finite cover $V_1^{x_1}, \ldots, V_1^{x_n}$. We define $V_1 = \bigcup_{k=1}^{n} V_1^{x_k}$.

• We continue in the same way till we get V_{n_0-1} which is a neighbourhood of $f^{n_0-1}(x_0)$ such that $\overline{V}_{n_0-1}$ is compact and $f^{n_0-1}(\overline{V}_0) \subset V_{n_0-1}$.

We define $Z = \bigcup_{n=0}^{n_0-1} \overline{V}_n$. The set Z is compact.

We define a cover $\{U_i\}$ of Z by open sets of diameters less than $\varepsilon/8$ with closures homeomorphic to I^N as in the proof of Theorem 2:

• U_0 has been defined above. Except for x_n and $f^{n_0}(x_n)$ it does not contain any points of $A = \{x_n, f(x_n), \ldots, f^{n_0}(x_n)\}$.

• U_i, for $1 \leq i \leq n_0 - 1$, does not contain any points of A except for $f^i(x_n)$.

• Other sets of the cover $\{U_i\}$ do not contain any points of A.

The set Z is compact, therefore we can choose a finite subcover. Modifying a little the construction from the proof of Theorem 1, we can obtain a function g_n defined on Z such that $d_\infty(f|_Z, g_n) < \varepsilon/8$ and $g_n^{n_0}(x_n) =$

x_n. We construct a compact set K_{n_0-1} homeomorphic to I^N taking $A = \{x_n, f(x_n), \ldots, f^{n_0-2}(x_n)\}$, $y = f^{n_0-1}(x_n)$, $B = U_0$ and $C = \bigcup_{n=0}^{n_0-1} V_n$. Then we define g_n as in the proof of Theorem 1.

As in the proof of Theorem 2 we define a function h on Z such that $h^{n_0}(x_n) = x_0$, and a function $g|_Z$ such that $g|_Z$ is continuous, x_0 is a periodic point of $g|_Z$ and $d_\infty(f|_Z, g|_Z) < \varepsilon$ (this last construction is possible, because we have assumed that $B(x_0, \delta_1) \subset V_0 \subset Z$).

Finally, we define

$$g = \begin{cases} g|_Z & \text{on } Z, \\ f & \text{outside } Z. \end{cases}$$

As all the constructions modify f only in $\operatorname{Int} Z$, we have $g = f$ on ∂Z, hence g is continuous. Moreover, $d_\infty(f, g) = d_\infty(f|_Z, g|_Z) < \varepsilon$. The result follows.

Let M be a C^∞-smooth, compact manifold of dimension ≥ 2 with distance d coming from a Riemannian metric. Then the C^0-closing lemma in the invertible case can be easily obtained by using a lemma proved by Z. Nitecki and M. Shub [3].

$Z(M)$ will denote the space of homeomorphisms from M into itself and d_1 the metric of uniform convergence on $Z(M)$: $d_1(f, g) = \max_{x \in M}(d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)))$. We define a wandering point as in Definition 2 with $\bigcup_{n>0} f^n(U) \cap U = \emptyset$ replaced by $\bigcup_{n \in \mathbb{Z},\, n \neq 0} f^n(U) \cap U = \emptyset$. As previously, a point which is not wandering is called nonwandering.

THEOREM 4. *Let $f : M \to M$ be a homeomorphism, where M is a C^∞ compact manifold of dimension ≥ 2 with distance d coming from a Riemannian metric, and $x_0 \in M$ a nonwandering point. Then for every $\varepsilon > 0$ there exists a homeomorphism $g : M \to M$ such that $d_1(f, g) < \varepsilon$ and x_0 is a periodic point of g.*

Proof. Clearly we can assume that x_0 is not periodic for f.

Let ε be a small positive constant. As f is uniformly continuous, there exists η with $0 < \eta < \varepsilon$ such that $d(f(x_1), f(x_2)) < \varepsilon$ if $d(x_1, x_2) < \eta$. We choose δ, with $0 < \delta < \eta/2$, such that $d(f^{-1}(x_1), f^{-1}(x_2)) < \eta/2$ if $d(x_1, x_2) < \delta$. This is possible, as f^{-1} is also uniformly continuous. Let $z_0 \in B(x_0, \delta)$ be such that there exists $m \in \mathbb{Z}$, $m \neq 0$, satisfying $f^m(z_0) = z_m \in B(x_0, \delta)$, $z_0 \neq x_0$ and $z_m \neq x_0$. Without loss of generality we can assume that $m > 0$ (otherwise we take z_m instead of z_0) and we take the smallest m satisfying the above conditions.

If $m \geq 2$, we set $z_i = f^i(z_0)$ for $i \in \mathbb{Z}$ and we consider the finite collection $\{(p_i, q_i) \in M \times M : i = 0, \ldots, m-1\}$, where $p_0 = x_0$, $p_1 = z_1$, $p_2 = z_2$, $\ldots$, $p_{m-2} = z_{m-2}$, $p_{m-1} = z_{m-1} = f^{-1}(z_m)$ and $q_0 = z_0$, $q_1 = z_1$, $q_2 = z_2, \ldots, q_{m-2} = z_{m-2}$, $q_{m-1} = f^{-1}(x_0)$. The points p_i and q_i $(i = 1, \ldots, k)$ satisfy:

(a) $p_i \neq p_j$, $q_i \neq q_j$ for $0 \leq i < j \leq m-1$,
(b) $d(p_i, q_i) < \eta/2$ for $i = 0, \ldots, m-1$.

According to the lemma of Nitecki and Shub ([3], Lemma 13) there exists a diffeomorphism $h : M \to M$ with the following properties:

(a) $d_1(h, \mathrm{id}) < \eta$,
(b) $h(p_i) = q_i$ for $i = 0, \ldots, m-1$.

We put $g = f \circ h$. Obviously g is a homeomorphism, x_0 is a periodic point of g and as $d_1(h, \mathrm{id}) < \eta < \varepsilon$,

$$d_1(f, g) = \max_{x \in M}(d(f(x), (f \circ h)(x)), d(f^{-1}(x), (h^{-1} \circ f^{-1})(x)))$$
$$= \max_{x \in M}(d(f(x), (f \circ h)(x)), d(x, h^{-1}(x))) < \varepsilon.$$

If $m = 1$, we choose ϱ, with $0 < \varrho < \eta$, such that $d(f^{-1}(x_1), f^{-1}(x_2)) < \eta/2$ if $d(x_1, x_2) < \varrho$. We choose θ, with $0 < \theta < \varrho/2$, satisfying $d(f(x_1), f(x_2)) < \varrho/2$ if $d(x_1, x_2) < 2\theta$. Let $y_0 \in B(x_0, \theta)$ be such that there exists $n > 0$ satisfying $f^n(y_0) = y_n \in B(x_0, \theta)$, $y_0 \neq x_0$ and $y_n \neq x_0$; we take the smallest n satisfying these conditions. If $n \geq 2$ our problem reduces to the previous case and if $n = 1$ then we consider the points $p_0 = x_0$, $p_1 = y_1 = f^{-1}(y_2)$, $q_0 = y_0$, $q_1 = f^{-1}(x_0)$. The points p_0, p_1, q_0, q_1 satisfy:

(a) $p_0 \neq p_1$, $q_0 \neq q_1$,
(b) $d(p_0, q_0) = d(x_0, y_0) < \theta < \varrho/2 < \eta/2$ and $d(p_1, q_1) = d(y_1, f^{-1}(x_0)) = d(f^{-1}(y_2), f^{-1}(x_0)) < \eta/2$ as $d(x_0, y_2) \leq d(x_0, y_1) + d(y_1, y_2) = d(x_0, y_1) + d(f(y_0), f(y_1)) < \theta + \varrho/2 < \varrho$.

Again the lemma of Nitecki and Shub gives a diffeomorphism $h : M \to M$ with the following properties:

(a) $d_1(h, \mathrm{id}) < \eta$,
(b) $h(x_0) = y_0$, $h(y_1) = f^{-1}(x_0)$.

We put $g = f \circ h$. Again g is a homeomorphism, x_0 is a periodic point of g and as before $d_1(f, g) < \varepsilon$. The result follows.

References

[1] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[2] H. Lehning, *Dynamics of typical continuous functions*, preprint, 1993.

[3] Z. Nitecki and M. Shub, *Filtrations, decompositions and explosions*, Amer. J. Math. 97 (1975), 1029–1047.

[4] C. C. Pugh, *Improved closing lemma*, ibid. 89 (1967), 1010–1021.

[5] W. Szlenk, *Introduction to the Theory of Smooth Dynamical Systems*, Polish Sci. Publ., Warszawa, 1984.

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland

Reçu par la Rédaction le 10.4.1995
Révisé le 12.12.1995