On the $C^0$-closing lemma

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Abstract. A proof of the $C^0$-closing lemma for noninvertible discrete dynamical systems and its extension to the noncompact case are presented.

1. Introduction. One of the most significant theorems in the theory of smooth dynamical systems is the $C^1$-closing lemma established by C. C. Pugh in [4], and its proof is fairly advanced. The $C^r$-closing lemma, $r > 1$, still remains an unsolved problem. A $C^0$-closing lemma for compact manifolds is stated without proof in [5]. The aim of this paper is to provide a proof of this lemma, as well as to extend it to the noncompact case.

We study the behaviour of the sequence $(f^n(x))_{n=0}^\infty$, where $f$ is a continuous function from a finite-dimensional manifold into itself. Using a method by H. Lehning [2] and the Tietze extension theorem, we prove Theorem 2 which is the standard $C^0$-closing lemma for a compact manifold, quoted (without proof) in [5]. Theorem 1 is a modification of the $C^0$-closing lemma, obtained without the assumption that the point $x_0$ is nonwandering. The main result of the present paper is Theorem 3, which is a generalization of the $C^0$-closing lemma to the case of a not necessarily compact manifold. Moreover, under suitable assumptions, in Theorem 4 we prove the $C^0$-closing lemma in the invertible case, using a lemma by Z. Nitecki and M. Shub [3].

Let $M$ denote a compact topological manifold with boundary, $N$ its dimension and $d$ a metric on $M$ compatible with the topology of $M$. $C(M)$ will denote the space of continuous functions from $M$ into itself and $d_\infty$ the metric of uniform convergence on $C(M)$: $d_\infty(f, g) = \max_{x \in M} d(f(x), g(x))$. If $B$ is a subset of $M$, we denote its interior by $\text{Int } B$, its closure by $\overline{B}$, its boundary by $\partial B$ and its diameter by $d(B)$. $B(x_0, r)$ will stand for the open ball with center $x_0$ and radius $r$, and $\overline{B}(x_0, r)$ for the closed ball with center $x_0$ and radius $r$. Fix once and for all a continuous function $f : M \to M$.

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2. Results

Definition 1. We call a point $x \in M$ eventually periodic if there exists $n$ such that $f^n(x)$ is periodic.

Theorem 1. Let $f : M \to M$ be a continuous function and $x_0 \in M$ a point. Then for every $\varepsilon > 0$ there exists a continuous function $g : M \to M$ and $\delta > 0$ such that $d_\infty(f, g) < \varepsilon$ and for every $x \in M$ satisfying $d(x, x_0) < \delta$, $x$ is an eventually periodic point of $g$.

Proof. Case I: $x_0$ is an eventually periodic point of $f$. There exists an open neighbourhood $U$ of $f(x_0)$ such that $U$ is homeomorphic to $I^N$, where $I$ is the closed unit interval, and $d(U) < \varepsilon$. We define $K = f^{-1}(U)$. Then $K$ is a neighbourhood of $x_0$ and $\text{Int} K \neq \emptyset$. We reduce $K$ if necessary so that it does not contain any point of the sequence $(f^n(x_0))$. This is possible as the set $(f^n(x_0))$ is finite. We choose a compact subset $k$ with $k \subset \text{Int} K$ and $x_0 \in \text{Int} k$. We now define the required function $g$:

- We put $g = f(x_0)$ on $k$ and $g = f$ on $\partial K$. This gives a continuous map from $k \cup \partial K$ to $U$, which is homeomorphic to $I^N$. Since $k \cup \partial K$ is a closed subset of $K$, by the Tietze extension theorem (see [1]), $g$ extends to a continuous map (still denoted by $g$) from $K$ to $U$. As $d(U) < \varepsilon$, $d(f(x), g(x)) < \varepsilon$ for each $x \in K$.

- We put $g = f$ outside $K$.

As $g$ agrees with $f$ on the boundary of $K$, it follows that $g$ is continuous on $M$. As $d(f(x), g(x)) < \varepsilon$ for each $x \in M$ and the function $x \to d(f(x), g(x))$ is continuous on the compact set $M$, we have $d_\infty(f, g) < \varepsilon$.

We choose $\delta > 0$ with $B(x_0, \delta) \subset \text{Int} k$. For $x$ such that $d(x, x_0) < \delta$ we have $g(x) = f(x_0)$ and hence $x$ is an eventually periodic point of $g$.

Case II: $x_0$ is not eventually periodic for $f$. Then the set $S = \{x_0, f(x_0), f^2(x_0), \ldots\}$ is infinite.

Using the compactness of $M$, we construct a finite cover of $M$ by open sets of diameters less than $\varepsilon$ and with closures homeomorphic to $I^N$. Let $p$ be the smallest positive integer such that there are an integer $n < p$ and an element $V$ of the cover containing $f^n(x_0)$ and $f^n(x_0)$ (such an integer exists because the set $S$ is infinite). This property implies in particular that the points $x_0, f(x_0), \ldots, f^{p-1}(x_0)$ are distinct.

Let $A$ be a closed subset of $M$, $B$ and $C$ two open subsets and $y \notin A$ be such that $f(y) \in B$ and $y \in C$. Using the continuity of $f$ at $y$, we can construct a compact subset $K$ homeomorphic to $I^N$ such that $K \cap A = \emptyset$, $K \subset C$, $y \in \text{Int} K$, $d(K) < \varepsilon$ and $f(K) \subset B$.

Using this property for $A = \{x_0, f(x_0), \ldots, f^{p-2}(x_0)\}$, $y = f^{p-1}(x_0)$, $B = V$ and $C = M$, we obtain a compact set $K_{p-1}$. We choose a compact set $k_{p-1} \subset \text{Int} K_{p-1}$ such that $f^{p-1}(x_0) \in \text{Int} k_{p-1}$.
We define the desired function \( g \) in the following way:

- We put \( g = f^n(x_0) \) on \( k_{p-1} \) and \( g = f \) on \( \partial K_{p-1} \); \( g \) takes its values in \( V \), which is homeomorphic to \( I^N \). Again, \( g \) extends to a continuous map from \( K_{p-1} \) to \( V \). As \( d(V) < \varepsilon \), \( d(f(x), g(x)) < \varepsilon \) for each \( x \in K_{p-1} \).
- We put \( g = f \) outside \( K_{p-1} \).

The function \( g \) is continuous and \( d_\infty(f, g) < \varepsilon \). We choose \( \delta > 0 \) such that \( B(x_0, \delta) \subset (g^{p^{-1}})^{-1}(k_{p-1}) \) (this is possible because \( g^{p^{-1}}(x_0) \in \text{Int} k_{p-1} \)). For \( x \in B(x_0, \delta) \) we have \( g^p(x) = f^n(x_0) = g^n(x_0) \). Hence the point \( g^p(x) \) is periodic of period \( p - n \). The result follows.

**Definition 2.** We call a point \( x \in M \) wandering if there is a neighbourhood \( U \) of \( x \) such that

\[
\bigcup_{n \geq 0} f^n(U) \cap U = \emptyset,
\]

and nonwandering otherwise.

**Theorem 2** (The C^0-closing lemma). Let \( f : M \to M \) be a continuous function and \( x_0 \in M \) a nonwandering point. Then for every \( \varepsilon > 0 \) there exists a continuous function \( g : M \to M \) such that \( d_\infty(f, g) < \varepsilon \) and \( x_0 \) is a periodic point of \( g \).

**Proof.** Clearly we can assume that \( x_0 \) is not periodic for \( f \).

Since \( x_0 \) is nonwandering, for every neighbourhood \( U \) of \( x_0 \) there exist \( x \in U \) and \( N \) such that \( f^N(x) \in U \).

Let \( \varepsilon > 0 \); there exists \( \delta_1 > 0 \) such that \( d(f(x), f(x_0)) < \varepsilon/8 \) if \( d(x, x_0) < \delta_1 \).

Let \( n \) be such that \( 1/n < \delta_1 \). There exists a neighbourhood \( U_0 \) of \( x_0 \) such that \( U_0 \subset B(x_0, 1/n) \), \( U_0 \) is homeomorphic to \( I^N \) and \( d(U_0) < \varepsilon/8 \). Let \( x_n \in U_0 \) be such that there exists \( N \) satisfying \( f^N(x_n) \in U_0 \) and \( f(x_n) \neq x_n \).

Such a point exists, because otherwise, as \( x_0 \) is nonwandering, we would find a sequence \( \{x_n\} \) with \( d(x_n, x_0) \to 0 \) \((n \to \infty)\) and \( f(x_n) = x_n \); since \( f \) is continuous, we would have \( f(x_0) = x_0 \), which is impossible, as \( x_0 \) is not periodic.

Let \( n_0 \) be the smallest integer, \( n_0 \geq 1 \), such that \( f^{n_0}(x_0) \in U_0 \) and \( x_n \neq f^{n_0}(x_n) \). Then the points \( x_n, f(x_n), \ldots, f^{n_0-1}(x_n) \) are distinct. Indeed, if any two of them were equal, then \( x_n \) would be eventually periodic, which is impossible because \( f^{n_0}(x_n) \in U_0 \).

We construct a cover of the manifold \( M \) by open sets of diameters less than \( \varepsilon/8 \) and with closures homeomorphic to \( I^N \) in the following way:

- We take \( U_0 \) as above; notice that except for \( x_n \) and \( f^{n_0}(x_n) \), \( U_0 \) does not contain any points of \( A = \{x_n, f(x_n), \ldots, f^{n_0}(x_n)\} \).
- We take \( U_1 \) which contains \( f(x_n) \) and does not contain any other points of \( A \).
• We continue in the same way till \( U_{n_0-1} \) which contains \( f^{n_0-1}(x_n) \) and does not contain any other points of \( A \).

• Other sets \( U_i \) of the cover do not contain any points of \( A \).

Since \( M \) is compact, \( \{U_i\} \) has a finite subcover and exactly as in the proof of Theorem 1, we construct a function \( g_n \) satisfying \( d(f,g_n) < \varepsilon/8 \). The important thing to notice here is that from the construction of \( \{U_i\} \) we have \( n = 0, p = n_0 \) and therefore \( g_n(x) = x_n \) on \( k_{n_0-1} \), where \( g_n^{n_0-1}(x_n) \in \text{Int} \ k_{n_0-1} \) (see the proof of Theorem 1), hence \( x_n \) is a periodic point of \( g_n \).

If there is a \( k \in \{0,1,\ldots,n_0-1\} \) such that \( g_n^k(x_n) = x_0 \), then \( x_0 \) is periodic.

So we assume that \( x_n \neq x_0, \ldots, g_n^{n_0-1}(x_n) \neq x_0 \). We construct a function \( h \) in the following way:

• We put \( h = x_0 \) on \( g_n^{n_0-1}(x_n) \) and \( h = g_n = x_n \) on \( \partial k_{n_0-1} \), and extend it to a continuous map from \( k_{n_0-1} \) to \( U_0 \). Therefore \( d(g_n(x), h(x)) < \varepsilon/8 \) for each \( x \in k_{n_0-1} \).

• We put \( h = g_n \) outside \( k_{n_0-1} \).

The function \( h \) is continuous, \( d_\infty(f,h) \leq d_\infty(f,g_n) + d_\infty(g_n,h) < \varepsilon/4 \) and \( h^{n_0}(x_n) = x_0 \).

For \( x \) such that \( d(x, x_0) < \delta_1 \) we have

\[
d(h(x), h(x_n)) \leq d(h(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), f(x_n)) + d(f(x_n), h(x_n)) < 3\varepsilon/4,
\]

and hence \( h(B(x_0, \delta_1)) \subset B(h(x_n), 3\varepsilon/4) \). There exists a cover \( \{Z_i\} \) of \( B(h(x_n), 3\varepsilon/4) \) such that \( B(h(x_n), 3\varepsilon/4) \subset \bigcup_{i \in I} \text{Int} Z_i \), where \( Z_i \) are homeomorphic to \( I^N \) and \( d(Z_i) < \varepsilon/4 \) for \( i \in I \). There exists an \( i_0 \) such that \( x_0 \in h^{-1}(\text{Int} Z_{i_0}) \). Since \( h^{-1}(\text{Int} Z_{i_0}) \) is open, there exists \( \delta_2 > 0 \) such that \( B(x_0, \delta_2) \subset h^{-1}(\text{Int} Z_{i_0}) \cap B(x_0, \delta_1) \) and \( B(x_0, \delta_2) \) does not meet the set \( \{x_n, h(x_n), \ldots, h^{n_0-1}(x_n)\} \) (this last statement holds because \( x_n \neq x_0 \), \( h(x_n) \neq x_0, \ldots, h^{n_0-1}(x_n) \neq x_0 \), as \( g_n^k(x_n) \neq x_0 \) for \( k = 0,1,\ldots,n_0-1 \), \( h = g_n \) outside \( k_{n_0-1} \), \( x_n \notin k_{n_0-1} \subset K_{n_0-1} \) (see the construction in the proof of Theorem 1), therefore \( h(x_n) = g_n(x_n) \) and hence \( h^k(x_n) = g_n^k(x_n) \)).

We define a function \( g \) in the following way:

• We put \( g = h(x_n) \) on \( x_0 \) and \( g = h \) on \( \partial B(x_0, \delta_2) \), and extend it to a continuous map from \( B(x_0, \delta_2) \) to \( Z_{i_0} \). As \( d(Z_{i_0}) < \varepsilon/4 \), we have \( d(h(x), g(x)) < \varepsilon/4 \) for each \( x \in B(x_0, \delta_2) \).

• We put \( g = h \) outside \( B(x_0, \delta_2) \).

The function \( g \) is continuous, \( x_0 \) is a periodic point of \( g \) and \( d_\infty(g,f) \leq d_\infty(g,h) + d_\infty(h,f) < \frac{3}{4} \varepsilon + \frac{1}{4} \varepsilon = \varepsilon \). The result follows.
In the above proof all the modifications of the function $f$ have been made locally. Therefore the local compactness of $M$ is sufficient. A theorem analogous to Theorem 2 can be proved in the following situation:

$M$ is a finite-dimensional manifold with boundary (which is obviously locally compact), $N$ its dimension, $C(M)$ the space of continuous functions from $M$ into itself, $f|_Z$ the restriction of a function $f$ to a subset $Z \subset M$, and $τ_∞$ the topology of uniform convergence on $C(M)$. We set $d_∞(f, g) = \sup_{x \in M} d(f(x), g(x))$, which may not be a metric, nevertheless we have: $f_n \to f$ $(n \to \infty)$ in the topology $τ_∞$ if and only if $d_∞(f_n, f) \to 0$ $(n \to \infty)$.

**Theorem 3** (A generalization of the $C^0$-closing lemma). Let $f : M \to M$ be a continuous function, where $M$ is a manifold with boundary, and $x_0 \in M$ a nonwinding point. Then for every $ε > 0$ there exists a continuous function $g : M \to M$ such that $d_∞(f, g) < ε$ and $x_0$ is a periodic point of $g$.

**Proof.** Let $ε > 0$ and $V_0$ be a neighbourhood of $x_0$ such that $M$ is compact.

As in the proof of Theorem 2, we assume that $x_0$ is not periodic for $f$. We choose $δ_1 > 0$ such that $d(f(x), f(x_0)) < ε/8$ if $d(x, x_0) < δ_1$ and $B(x_0, δ_1) \subset V_0$. Moreover, $n$ fulfills the condition $1/n < δ_1$, $U_0$ is a neighbourhood of $x_0$ such that $V_1 \subset B(x_0, 1/n) \cap V_0$, $U_0$ is homeomorphic to $I^N$ and $d(U_0) < ε/8$, $x_n$ is a point for which there exists $N$ such that $f^n(x_n) \in U_0$ and $f(x_n) \neq x_n$, and $n_0$ is the smallest integer $≥ 1$ for which $x_n \neq f^{n_0}(x_n) \in U_0$.

We define the sets $V_0, V_1, \ldots, V_{n_0-1}$ in the following way:

- $V_0$ has already been defined above.
- $V_1$ is a neighbourhood of $f(x_0)$ with $V_1$ compact and $f(V_0) \subset V_1$. Such a neighbourhood exists, because for each $x \in f(V_0)$ there exists a compact $V_1$ such that $V_1$ is compact. The set $f(V_0)$ is compact, hence we can choose a finite cover $V_1, \ldots, V_{n_0-1}$. We define $V_1 = \bigcup_{k=1}^{n_0-1} V_{1, k}$.
- We continue in the same way till we get $V_{n_0-1}$ which is a neighbourhood of $f^{n_0-1}(x_0)$ such that $V_{n_0-1}$ is compact and $f^{n_0-1}(V_0) \subset V_{n_0-1}$.

We define $Z = \bigcup_{n=0}^{n_0-1} V_n$. The set $Z$ is compact.

We define a cover $\{U_i\}$ of $Z$ by open sets of diameters less than $ε/8$ with closures homeomorphic to $I^N$ as in the proof of Theorem 2:

- $U_0$ has been defined above. Except for $x_n$ and $f^{n_0}(x_n)$ it does not contain any points of $A = \{x_n, f(x_n), \ldots, f^{n_0}(x_n)\}$.
- $U_i$, for $1 \leq i \leq n_0-1$, does not contain any points of $A$ except for $f^i(x_n)$.
- Other sets of the cover $\{U_i\}$ do not contain any points of $A$.

The set $Z$ is compact, therefore we can choose a finite subcover. Modifying a little the construction from the proof of Theorem 1, we can obtain a function $g_n$ defined on $Z$ such that $d_∞(f|_Z, g_n) < ε/8$ and $g_n^{n_0}(x_n) =
x_n. We construct a compact set K_{n-1} homeomorphic to I^N taking A = {x_n, f(x_n), \ldots, f^{n-2}(x_n)}, y = f^{n-1}(x_n), B = U_0 and C = \bigcup_{n=0}^{n-1} V_n. Then we define g_0 as in the proof of Theorem 1.

As in the proof of Theorem 2 we define a function h on Z such that h^n(x_n) = x_0, and a function g|Z such that g|Z is continuous, x_0 is a periodic point of g|Z and d_\infty(f|Z, g|Z) < \varepsilon (this last construction is possible, because we have assumed that B(x_0, \delta_1) \subset V_0 \subset Z).

Finally, we define

\[ g = \begin{cases} g|Z & \text{on } Z, \\ f & \text{outside } Z. \end{cases} \]

As all the constructions modify f only in Int Z, we have g = f on \partial Z, hence g is continuous. Moreover, d_\infty(f, g) = d_\infty(f|Z, g|Z) < \varepsilon. The result follows.

Let M be a C^\infty-smooth, compact manifold of dimension \geq 2 with distance d coming from a Riemannian metric. Then the C^0-closing lemma in the invertible case can be easily obtained by using a lemma proved by Z. Nitecki and M. Shub [3].

Z(M) will denote the space of homeomorphisms from M into itself and d_1 the metric of uniform convergence on Z(M): d_1(f, g) = \max_{x \in M}(d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))). We define a wandering point as in Definition 2 with \bigcup_{n=0}^{\infty} f^n(U) \cap U = \emptyset replaced by \bigcup_{n \in \mathbb{Z}, n \neq 0} f^n(U) \cap U = \emptyset. As previously, a point which is not wandering is called nonwandering.

**Theorem 4.** Let f : M \to M be a homeomorphism, where M is a C^\infty compact manifold of dimension \geq 2 with distance d coming from a Riemannian metric, and x_0 \in M a nonwandering point. Then for every \varepsilon > 0 there exists a homeomorphism g : M \to M such that d_1(f, g) < \varepsilon and x_0 is a periodic point of g.

**Proof.** Clearly we can assume that x_0 is not periodic for f.

Let \varepsilon be a small positive constant. As f is uniformly continuous, there exists \eta with 0 < \eta < \varepsilon such that d(f(x_1), f(x_2)) < \varepsilon if d(x_1, x_2) < \eta. We choose \delta, with 0 < \delta < \eta/2, such that d(f^{-1}(x_1), f^{-1}(x_2)) < \eta/2 if d(x_1, x_2) < \delta. This is possible, as f^{-1} is also uniformly continuous. Let z_0 \in B(x_0, \delta) be such that there exists m \in \mathbb{Z}, m \neq 0, satisfying f^m(z_0) = z_m \in B(x_0, \delta), z_0 \neq x_0 and z_m \neq x_0. Without loss of generality we can assume that m > 0 (otherwise we take z_m instead of z_0) and we take the smallest m satisfying the above conditions.

If m \geq 2, we set z_i = f^i(z_0) for i \in \mathbb{Z} and we consider the finite collection \{(p_i, q_i) \in M \times M : i = 0, \ldots, m - 1\}, where p_0 = x_0, p_1 = z_1, p_2 = z_2, \ldots, p_{m-2} = z_{m-2}, p_{m-1} = f^{-1}(z_0) and q_0 = x_0, q_1 = z_1, q_2 = z_2, \ldots, q_{m-2} = z_{m-2}, q_{m-1} = f^{-1}(x_0). The points p_i and q_i (i = 1, \ldots, k) satisfy:
(a) \( p_i \neq p_j, q_i \neq q_j \) for \( 0 \leq i < j \leq m - 1 \),
(b) \( d(p_i, q_i) < \eta/2 \) for \( i = 0, \ldots, m - 1 \).

According to the lemma of Nitecki and Shub ([3], Lemma 13) there exists a
diffeomorphism \( h : M \to M \) with the following properties:

(a) \( d_1(h, \text{id}) < \eta \),
(b) \( h(p_i) = q_i \) for \( i = 0, \ldots, m - 1 \).

We put \( g = f \circ h \). Obviously \( g \) is a homeomorphism, \( x_0 \) is a periodic point
of \( g \) and as \( d_1(h, \text{id}) < \eta < \varepsilon \),
\[
d_1(f, g) = \max_{x \in M} (d(f(x), (f \circ h)(x)), d((f^{-1}(x), (h^{-1} \circ f^{-1})(x)))
\]
\[
= \max_{x \in M} (d(f(x), (f \circ h)(x)), d(x, h^{-1}(x))) < \varepsilon.
\]

If \( m = 1 \), we choose \( g \), with \( 0 < \varrho < \eta \), such that \( d(f^{-1}(x_1), f^{-1}(x_2)) < \eta/2 \)
if \( d(x_1, x_2) < \varrho \). We choose \( \theta \), with \( 0 < \theta < \varrho/2 \), satisfying \( d(f(x_1), f(x_2)) < \varrho/2 \)
if \( d(x_1, x_2) < 2\theta \). Let \( y_0 \in B(x_0, \theta) \) be such that there exists \( n > 0 \)
satisfying \( f^n(y_0) = y_n \in B(x_0, \theta), y_n \neq x_0 \) and \( y_n \neq x_0 \); we take the
smallest \( n \) satisfying these conditions. If \( n \geq 2 \) our problem reduces to
the previous case and if \( n = 1 \) then we consider the points \( p_0 = x_0, p_1 = y_1 = f^{-1}(y_2), q_0 = y_0, q_1 = f^{-1}(x_0) \). The points \( p_0, p_1, q_0, q_1 \) satisfy:

(a) \( p_0 \neq p_1, q_0 \neq q_1 \),
(b) \( d(p_0, y_0) = d(x_0, y_0) < \theta < \varrho/2 < \eta/2 \) and \( d(p_1, q_1) = d(y_1, f^{-1}(x_0)) \)
\[
= d(f^{-1}(y_2), f^{-1}(x_0)) < \eta/2 \text{ as } d(x_0, y_1) \leq d(x_0, y_2) + d(y_1, y_2) = d(x_0, y_1) +
\]
\[
d(f(y_0), f(y_1)) < \theta + \varrho/2 < \varrho.
\]

Again the lemma of Nitecki and Shub gives a diffeomorphism \( h : M \to M \)
with the following properties:

(a) \( d_1(h, \text{id}) < \eta \),
(b) \( h(x_0) = y_0, h(y_1) = f^{-1}(x_0) \).

We put \( g = f \circ h \). Again \( g \) is a homeomorphism, \( x_0 \) is a periodic point of \( g \)
and as before \( d_1(f, g) < \varepsilon \). The result follows.

References


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