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Generalized telegraph equation and the Sova–Kurtz version of the Trotter–Kato theorem

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Abstract. The Sova–Kurtz approximation theorem for semigroups is applied to prove convergence of solutions of the telegraph equation with small parameter. Convergence of the solutions of the diffusion equation with varying boundary conditions is also considered.

Introduction. In the papers [1–2] convergence of the solutions to the telegraph equation with a small parameter and to the diffusion equation with varying boundary conditions were considered. They were viewed as examples of the Trotter–Kato theorem [3], [8], [15–16]. The aim of this article is to show that in dealing with the problem of convergence of those solutions, the Sova–Kurtz version of the approximation theorem for semigroups can be employed as well. The advantage of this method is that, on the one hand, various norms in the same Banach space may be considered, and, on the other hand, we avoid calculations of the resolvents of infinitesimal operators, which simplifies some proofs.

In order to illustrate the above idea, two propositions will be proved. While the proofs we present are new, the propositions are in essence equivalent to Proposition 1 of [1] and Corollary 1 of [2], respectively.

Let us note here that convergence of the solutions of the telegraph equation has aroused considerable attention. An abundant bibliography on this subject can be found in [1] and [7]. Our purpose, similarly to the paper [1], is to present a new approach rather than to cover all the previous results.

1. The Sova–Kurtz theorem. Let $(L, \|\cdot\|)$ and $(L_n, \|\cdot\|_n)$, $n \ge 1$, be Banach spaces and suppose that there are bounded linear transformations

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$P_n : L \to L_n$ such that for every $f \in L$, $\lim_{n\to\infty} \|P_n f\|_n$ exists and equals $\|f\|$. We say that a sequence $(f_n)_{n\ge 1}$ with $f_n \in L_n$, $n \ge 1$, is *convergent* iff there exists $f \in L$ such that

$$\lim_{n\to\infty} \|f_n - P_n f\|_n = 0. \tag{1.1}$$

In that case we write $\lim_{n\to\infty} f_n = f$.

Given a sequence of closed operators $A_n : L_n \supset D(A_n) \to L_n$, $n \ge 1$, define the (possibly multivalued) operator A acting in L by

$$\begin{aligned} D(A) = \{f \in L : {} & \text{there are } f_n \in D(A_n),\ n \ge 1, \text{ such that } \lim_{n\to\infty} f_n = f \\ & \text{and the sequence } (A_n f_n)_{n\ge 1} \text{ is convergent}\}, \\ Af = {} & \lim_{n\to\infty} A_n f_n. \end{aligned}$$

The operator A is usually called the *extended limit* of the operators A_n, and denoted by ex-lim A_n. The following theorem is due to M. Sova [14] and T. G. Kurtz [11], who introduced the concept of extended limit.

THEOREM 1. *Let $\{T_n(t) : t \ge 0\}$ be a sequence of semigroups acting in L_n, $n \ge 1$, with respective generators A_n. Assume also that there exists a constant M such that $\|T_n(t)\|_{\mathcal{L}(L_n, L_n)} \le M$ uniformly in $n \ge 1$, $t \ge 0$. Then the following are equivalent*:

(a) *there exists a strongly continuous semigroup $\{T(t) : t \ge 0\}$ acting in L such that*, *for every $f \in L$*,

$$\lim_{n\to\infty} T_n(t) P_n f = T(t) f, \tag{1.2}$$

(b) *the operator $A =$ ex-lim A_n has the following properties*:

(1.3) *$D(A)$ is dense in L*,

(1.4) *the range $\Re(\lambda - A)$ of $\lambda - A$ is dense in L*, *for some $\lambda > 0$*.

If (a) *or* (b) *holds then the operator $A =$ ex-lim A_n is single-valued and generates the semigroup $\{T(t) : t \ge 0\}$. Furthermore*, *the convergence in* (1.2) *is in fact uniform with respect to t in bounded intervals $\subset \mathbb{R}^+$.*

2. Generalized telegraph equation. Throughout this section E denotes a Banach space and $A : E \supset D(A) \to E$ the infinitesimal generator of a cosine operator function $\{C(t) : t \in \mathbb{R}\}$ ([13]) such that there exists a constant $M > 0$ that satisfies

$$\sup_{t\in\mathbb{R}} \|C(t)\|_{\mathcal{L}(E,E)} \le M.$$

Define $E_1 = \{x \in E : \text{the function } [0,1] \ni t \to C(t)x \text{ is strongly continuously differentiable}\}$ and set

$$\|x\|_{E_1} = \|x\|_E + \sup_{0\le u\le 1} \left\| \frac{dC(u)x}{du} \right\|_E .$$

Then E_1 is a Banach space ([9]).

Given $\varepsilon > 0$ let us also define an operator $A_\varepsilon : E_1 \times E \supset D(A_\varepsilon) \to E_1 \times E$ by

$$D(A_\varepsilon) = D(A) \times E_1, \qquad A_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{1}{\varepsilon} Ax \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{\varepsilon} y \end{pmatrix}. \tag{2.1}$$

Since $\frac{1}{\varepsilon}A$ is the infinitesimal generator of the cosine operator function $C_\varepsilon(t) = C\big(\frac{1}{\sqrt{\varepsilon}}t\big)$, by the main theorem of [9], the operator $\widehat{A}_\varepsilon : E_1 \times E \supset D(A_\varepsilon) \to E_1 \times E$ with

$$D(\widehat{A}_\varepsilon) = D(A_\varepsilon), \quad \widehat{A}_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{1}{\varepsilon} Ax \end{pmatrix},$$

generates a group of operators acting in $E_1 \times E$ and, consequently, by the Phillips perturbation theorem [12], the operator A_ε is the infinitesimal generator of a group

$$T_\varepsilon(t) = \begin{pmatrix} S_{00}(\varepsilon,t) & S_{01}(\varepsilon,t) \\ S_{10}(\varepsilon,t) & S_{11}(\varepsilon,t) \end{pmatrix},$$

acting in the same space.

Now we present a lemma due to J. Kisyński.

Lemma 1. *There exists $K > 0$ such that, for all $1 > \varepsilon > 0$ and $t \ge 0$,*

$$\begin{aligned} &\|S_{00}(\varepsilon,t)\|_{\mathcal{L}(E_1,E_1)} \le M, \\ &\|S_{11}(\varepsilon,t)\|_{\mathcal{L}(E,E)} \le M, \\ &\|S_{01}(\varepsilon,t)\|_{\mathcal{L}(E,E_1)} \le (\varepsilon + \sqrt{\varepsilon})M \le 2\sqrt{\varepsilon}M, \\ &\|S_{10}(\varepsilon,t)\|_{\mathcal{L}(E_1,E)} \le 2MK\frac{1}{\sqrt{\varepsilon}}. \end{aligned}$$

The proof is given in [1].

Lemma 2. *The operator A is the infinitesimal generator of a semigroup $\{T(t) : t \ge 0\}$ acting in E such that $\{T(t)_{|E_1} : t \ge 0\}$ is a strongly continuous semigroup in E_1.*

Proof. The fact that A is the generator of a semigroup $\{T(t) : t \ge 0\}$ acting in E is well known (see [5–6], [10], [13]). Furthermore, the formula

$$T(t)x = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} C(s)\, ds$$

holds for every $t > 0$ and $x \in E$ (see [5–6], [10]). It implies that $\{T(t)_{|E_1} : t \geq 0\}$ is a strongly continuous semigroup in E_1. Indeed, we have $C(s)T(t) = T(t)C(s)$ for every $s \in \mathbb{R}$ and $t \geq 0$, hence if $x \in E_1$ then also $T(t)x \in E_1$ and

$$\frac{dC(s)x}{ds}T(t)x = T(t)\frac{dC(s)x}{ds}.$$

Finally,

$$\|T(t)x - x\|_{E_1} = \|T(t)x - x\|_E + \sup_{0\leq s\leq 1}\left\|T(t)\frac{dC(s)x}{ds} - \frac{dC(s)x}{ds}\right\|_E, \tag{2.2}$$

the set $\{y \in E : y = dC(s)x/ds \text{ for some } 0 \leq s \leq 1\}$ is compact in E and $\sup_{t\geq 0}\|T(t)\|_{\mathcal{L}(E,E)} \leq M$. Thus, the right-hand side of (2.2) tends to 0 as $t \to 0$. ■

Set

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\varepsilon = \|x\|_{E_1} + \sqrt{\varepsilon}\|y\|_E. \tag{2.3}$$

Lemma 3. *The semigroups* $\{T_\varepsilon(t) : t \geq 0\}$, $1 > \varepsilon > 0$, *are equibounded when considered in the spaces* $(E_1\times E, \|\cdot\|_\varepsilon)$, *respectively. To be more specific, for every* $x \in E_1$, $y \in E$ *and* $1 > \varepsilon > 0$, *we have*

$$\left\|T_\varepsilon(t)\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\varepsilon \leq \max(M + 2KM, 3M)\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\varepsilon,$$

where K is the constant introduced in Lemma 1.

Proof. According to Lemma 1 we have, for $\binom{x}{y} \in E_1 \times E$,

$$\begin{aligned}
\left\|T_\varepsilon(t)\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\varepsilon &\leq \|S_{00}(\varepsilon,t)x\|_{E_1} + \|S_{01}(\varepsilon,t)y\|_{E_1} \\
&\quad + \sqrt{\varepsilon}\|S_{10}(\varepsilon,t)x\|_E + \sqrt{\varepsilon}\|S_{11}(\varepsilon,t)y\|_E \\
&\leq M\|x\|_{E_1} + (\sqrt{\varepsilon} + \varepsilon)M\|y\|_E + \sqrt{\varepsilon}\,\frac{2KM}{\sqrt{\varepsilon}}\|x\|_{E_1} + M\sqrt{\varepsilon}\|y\|_E \\
&\leq M(1 + 2K)\|x\|_{E_1} + 3M\sqrt{\varepsilon}\|y\|_E \\
&\leq \max(M + 2KM, 3M)\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\varepsilon. \ ■
\end{aligned}$$

Proposition 1. *Let* $(\varepsilon_n)_{n\geq 1}$ *be a sequence of positive numbers such that* $\lim_{n\to\infty}\varepsilon_n = 0$. *Set* $L_n = (E_1 \times E, \|\cdot\|_{\varepsilon_n})$, *and define operators* $P_n : E_1 \to L_n$ *by* $P_n x = \binom{x}{0}$. *Then the spaces* L_n *approximate the space* $L = E_1$ *in the sense of Section* 1 *and, for every* $x \in E_1$ *and* $t \geq 0$,

$$\lim_{n\to\infty} T_{\varepsilon_n}(t)x = T(t)x,$$

i.e.

$$\lim_{n\to\infty} \|T_{\varepsilon_n}(t)P_n x - P_n T(t)x\|_{\varepsilon_n} = \lim_{n\to\infty} \left\| T_{\varepsilon_n}(t)\begin{pmatrix} x \\ 0 \end{pmatrix} - \begin{pmatrix} T(t)x \\ 0 \end{pmatrix} \right\|_{\varepsilon_n} = 0.$$

P r o o f. It is obvious, by the very definition (2.3), that (1.1) is satisfied. According to Lemma 3 the semigroups $\{T_{\varepsilon_n}(t) : t \ge 0\}$, $n \ge 1$, $T_{\varepsilon_n}(t) : L_n \to L_n$, are equibounded. We now prove that both (1.3) and (1.4) hold.

Let A_0 denote the infinitesimal generator of the semigroup $\{T(t)_{|E_1} : t \ge 0\}$. By the Hille–Yosida theorem, $\overline{D(A_0)} = E_1$ and, for all $\lambda > 0$, $\Re(\lambda - A_0) = E_1$. Thus, it is enough to prove that $A_0 \subset \text{ex-lim}\, A_{\varepsilon_n}$, where the A_{ε_n} are defined by (2.1). Take $x \in D(A_0)$. Define

$$f_n = \begin{pmatrix} x \\ A_0 x \end{pmatrix} \in D(A_{\varepsilon_n}) \subset L_n.$$

We have $\|f_n - P_n x\|_{\varepsilon_n} = \sqrt{\varepsilon_n}\|A_0 x\|_E \to 0$ as $n \to \infty$, i.e. $\lim_{n\to\infty} f_n = x$. Moreover, since A_0 is a restriction of A, we have

$$A_{\varepsilon_n} f_n = \begin{pmatrix} A_0 x \\ \frac{1}{\varepsilon_n} Ax - \frac{1}{\varepsilon_n} A_0 x \end{pmatrix} = \begin{pmatrix} A_0 x \\ 0 \end{pmatrix}$$

and, consequently, $\lim_{n\to\infty} A_{\varepsilon_n} f_n = A_0 x$, whence $x \in D(\text{ex-lim}\, A_{\varepsilon_n})$, and $A_0 x = \text{ex-lim}\, A_{\varepsilon_n} x$, as desired.

R e m a r k 2. The above Proposition 1 and Proposition 1 of [1] are equivalent in the sense that both state that

$$\lim_{\varepsilon\to 0} S_{00}(\varepsilon, t)x = T(t)x \quad \text{for all } x \in E_1,$$
$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} S_{10}(\varepsilon, t)y = 0 \quad \text{for all } y \in E.$$

3. Diffusion equation. Let $a < b$ be fixed real numbers. Consider the space $L_\varepsilon = L = C_{[a,b]}$, $1 > \varepsilon > 0$, of all continuous functions $f : [a,b] \to \mathbb{R}$, equipped with the norm $\|f\| = \sup_{a\le x\le b} |f(x)|$. Given $\mu, \nu > 0$, define

$$D(A_{\mu,\nu}) = \{f \in C^2_{[a,b]} : f(a) - \mu f'(a) = 0, f(b) + \nu f'(b) = 0\},$$
$$A_{\mu,\nu} f = \frac{1}{2}\frac{d^2 f}{dx^2}.$$

Analogously, set $L' = \{f \in L : f(a) = f(b) = 0\}$ and

$$D(A_{00}) = \{f \in L' : f'' \in L'\}, \qquad A_{00} f = \frac{1}{2}\frac{d^2 f}{dx^2}.$$

Throughout this section we will not distinguish between norms in L and L' (both are the supremum norms).

LEMMA 4. *The operators $A_{\mu,\nu}$, $\mu,\nu > 0$, satisfy the positive maximum principle.*

(For the definition, see for example [4], p. 165.)

Proof. Fix $\mu,\nu > 0$ and $f \in D(A_{\mu,\nu})$, $f \not\equiv 0$. If the total maximum of f is attained at a point $x \neq a$, $x \neq b$, the conclusion is obvious. If $\max_{a\le x\le b} f(x) = f(a)$ then $f'(a) \le 0$,

$$0 \le f(a) = \mu f'(a) \le 0,$$

and, consequently, $f(a) = f'(a) = 0$. The function

$$f^*(x) = \begin{cases} f(x), & x \in [a,b], \\ f(2a-x), & x \in [2a-b,a], \end{cases}$$

is then of class C^2 in $[2a-b,b]$, and $\sup_{x\in[2a-b,b]} f^*(x) = f^*(a) = f(a)$. Of course

$$A_{\mu,\nu} f(a) = \tfrac{1}{2} f''(a) = \tfrac{1}{2}(f^*)''(a) \le 0,$$

as desired. If $\max_{a\le x\le b} f(x) = f(b)$ we proceed similarly. ∎

PROPOSITION 2. *For every $\mu,\nu > 0$, the operator $A_{\mu,\nu}$ is the infinitesimal generator of a positive contraction semigroup $\{S_{\mu,\nu}(t) : t \ge 0\}$ acting in L. The operator A_{00} generates a semigroup $\{S_{00}(t) : t \ge 0\}$ acting in L'. Furthermore,*

$$\lim_{\mu,\nu\to 0} S_{\mu,\nu}(t) f = S_{00}(t) f \quad \textit{for all } f \in L',\ t \ge 0.$$

Proof. To prove that $A_{\mu,\nu}$ is the infinitesimal generator of a strongly continuous semigroup note first that $D(A_{\mu,\nu})$ is dense in L. Indeed, the set of all twice continuously differentiable functions is dense in L, and, for every $\varepsilon > 0$ and every twice continuously differentiable function $f \in L$ with $f'' \in L$ such that $f \notin D(A_{\mu,\nu})$, there exists a function f_ε with $\|f_\varepsilon - f\| \le \varepsilon$ and $f_\varepsilon \in D(A_{\mu,\nu})$. Indeed, put

$$\begin{aligned} f_\varepsilon(x) &= f(x) + \alpha e^{-\beta(x-a)/(b-x)} + \gamma e^{-\delta(b-x)/(x-a)} \qquad \text{for } a < x < b, \\ \alpha &= \min\left(\frac{\varepsilon}{2}, \frac{1}{2}|\mu f'(a) - f(a)|\right) \operatorname{sign}(\mu f'(a) - f(a)), \\ \gamma &= -\min\left(\frac{\varepsilon}{2}, \frac{1}{2}|\nu f'(b) + f(b)|\right) \operatorname{sign}(\nu f'(b) + f(b)), \\ \beta &= \frac{b-a}{\mu}\left[\frac{\mu f'(a) - f(a)}{\alpha} - 1\right], \qquad \delta = \frac{b-a}{\nu}\left[\frac{\nu f'(b) + f(b)}{-\gamma} - 1\right] \end{aligned}$$

(if either $\mu f'(a) - f(a) = 0$ or $\nu f'(b) + f(b) = 0$, then put $\alpha = \beta = 0$ and

$\gamma = \delta = 0$, respectively) and note that

$$\beta \geq \left(2\frac{|\mu f'(a) - f(a)|}{|\mu f'(a) - f(a)|} - 1\right)\frac{b-a}{\mu} = \frac{b-a}{\nu} > 0,$$
$$\delta \geq \left(2\frac{|\nu f'(b) + f(b)|}{|\nu f'(b) + f(b)|} - 1\right)\frac{b-a}{\nu} = \frac{b-a}{\nu} > 0.$$

Furthermore,

$$f_\varepsilon(a) = f(a) + \alpha, \quad f'_\varepsilon(a) = f'(a) - \frac{\alpha\beta}{b-a},$$
$$f_\varepsilon(b) = f(b) + \gamma, \quad f'_\varepsilon(b) = f'(b) + \frac{\gamma\delta}{b-a}.$$

Thus $f_\varepsilon \in D(A_{\mu,\nu})$ and $\|f - f_\varepsilon\| \leq |\alpha| + |\gamma| \leq \varepsilon$.

The operators $A_{\mu,\nu}$ satisfy the positive maximum principle and are closed. We have to show that for all $\lambda > 0$, $g \in L$ and $\mu, \nu > 0$ there exists $f \in D(A_{\mu,\nu})$ such that

$$\lambda f(x) - \tfrac{1}{2} f''(x) = g(x) \quad \text{for } x \in [a,b]$$

(i.e. $\Re(\lambda - A_{\mu,\nu}) = L$). Note that the general solution to the above equation is

$$f(x) = f_{g,\lambda}(x) + C_1 e^{-\sqrt{2\lambda}x} + C_2 e^{\sqrt{2\lambda}x}, \tag{3.1}$$

where $f_{g,\lambda}(x) = \frac{1}{\sqrt{2\lambda}} \int_a^b e^{-\sqrt{2\lambda}|x-y|} g(y)\,dy$. The boundary conditions $f(a) - \mu f'(a) = 0$ and $f(b) + \nu f'(b) = 0$ lead to the system of equations for C_1, C_2:

$$C_1 e^{-\sqrt{2\lambda}a} + C_2 e^{\sqrt{2\lambda}a} + f_{g,\lambda}(a) = \mu\sqrt{2\lambda}(-C_1 e^{-\sqrt{2\lambda}a} + C_2 e^{\sqrt{2\lambda}a} + f_{g,\lambda}(a)),$$
$$C_1 e^{-\sqrt{2\lambda}b} + C_2 e^{\sqrt{2\lambda}b} + f_{g,\lambda}(b) = -\nu\sqrt{2\lambda}(-C_1 e^{-\sqrt{2\lambda}b} + C_2 e^{\sqrt{2\lambda}b} - f_{g,\lambda}(b))$$

(in deriving it the relations $f'_{g,\lambda}(a) = \sqrt{2\lambda} f_{g,\lambda}(a)$ and $f'_{g,\lambda}(b) = -\sqrt{2\lambda} f_{g,\lambda}(b)$ were employed). Thus

$$C_1 e^{-\sqrt{2\lambda}a}(1 + \mu\sqrt{2\lambda}) + C_2 e^{\sqrt{2\lambda}a}(1 - \mu\sqrt{2\lambda}) = (\mu\sqrt{2\lambda} - 1) f_{g,\lambda}(a),$$
$$C_1 e^{-\sqrt{2\lambda}b}(1 - \nu\sqrt{2\lambda}) + C_2 e^{\sqrt{2\lambda}b}(1 + \nu\sqrt{2\lambda}) = (\nu\sqrt{2\lambda} - 1) f_{g,\lambda}(b),$$

i.e.

$$\begin{aligned} C_1 - H(\mu\sqrt{2\lambda}) e^{2\sqrt{2\lambda}a} C_2 &= H(\mu\sqrt{2\lambda}) f_{g,\lambda}(a) e^{\sqrt{2\lambda}a}, \\ -H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}b} C_1 + C_2 &= H(\nu\sqrt{2\lambda}) f_{g,\lambda}(b) e^{-\sqrt{2\lambda}b}, \end{aligned} \tag{3.2}$$

where $H(z) = (z-1)/(z+1)$. The system has a unique solution since its determinant

$$W = 1 - H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}(b-a)}$$

is not 0. Thus the first part of the proposition is proved.

Since it is well known that A_{00} is the generator of a strongly continuous semigroup in L' (for a non-standard proof based on the approximation theorem one may consult [2]), it remains to prove, as in Proposition 1, that $\text{ex-lim}_{\mu,\nu\to 0} A_{\mu,\nu} \supset A_{00}$. Let $f \in D(A_{00})$ be given. Define

$$f_{\mu,\nu}(x) = f(x) + \mu f'(a)e^{-(x-a)^2/(b-x)} - \nu f'(b)e^{-(x-b)^2/(x-a)}, \qquad a < x < b.$$

Since

$$\lim_{x\to a^+} e^{-(x-b)^2/(x-a)} = \lim_{x\to b^-} e^{-(x-a)^2/(b-x)} = 0$$

and

$$\lim_{x\to a^+} \frac{e^{-(x-b)^2/(x-a)}}{x-a} = \lim_{x\to b^-} \frac{e^{-(x-a)^2/(b-x)}}{x-b} = 0,$$

it follows that $f_{\mu,\nu} \in L$, $f_{\mu,\nu}(a) = f(a) + \mu f'(a) = \mu f'(a)$, $f_{\mu,\nu}(b) = f(b) - \nu f'(b) = -\nu f'(b)$ and

$$f'_{\mu,\nu}(a) = \left(f(x) - \mu f'(a)\frac{(x-a)(2b-a-x)}{(b-x)^2}e^{-(x-a)^2/(b-x)}\right)_{|x=a} = f'(a),$$

$$f'_{\mu,\nu}(b) = \left(f(x) + \nu f'(b)\frac{(x-b)(x-2a+b)}{(x-a)^2}e^{-(x-b)^2/(x-a)}\right)_{|x=b} = f'(b).$$

It is easily proven that the second derivative at $x = a$ and $x = b$ also exists, and $f_{\mu,\nu}$ is of class C^2. Thus $f_{\mu,\nu} \in D(A_{\mu,\nu})$, and $\lim_{\mu,\nu\to 0} f_{\mu,\nu} = f$. Finally,

$$\begin{aligned}\lim_{\mu,\nu\to 0} A_{\mu,\nu}f_{\mu,\nu}(x) &= f''(x) + \lim_{\mu\to 0} \mu f'(a)\bigg(\frac{(x-a)^2(2b-a-x)^2}{(b-x)^4} \\ &\quad - \frac{2(b-x)^2 + 2(2b-a-x)(x-a)}{(b-x)^3}\bigg)e^{-(x-a)^2/(b-x)} \\ &\quad - \lim_{\nu\to 0} \nu f'(b)\bigg(\frac{2(x-a)^2 - 2(x-2a+b)(x-b)}{(x-a)^3} \\ &\quad - \frac{(x-b)^2(x-2a+b)^2}{(x-a)^4}\bigg)e^{-(x-b)^2/(x-a)} \\ &= A_{00}f(x)\end{aligned}$$

(uniformly with respect to $x \in [a,b]$). This proves that $f \in \text{ex-lim}\, A_{\mu,\nu}$, as desired. ∎

Remark 3. Let us note that in the paper [2], where a result similar to Proposition 2 was proved, tiresome estimations concerning the resolvent given by (3.1) with C_1 and C_2 being the solution of (3.2) were necessary. On the other hand, however, there was no need for Lemma 4.

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