ANNALES
POLONICI MATHEMATICI
LXIV.1 (1996)

Generalized symmetric spaces and minimal models

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Abstract. We prove that any compact simply connected manifold carrying a structure of Riemannian 3- or 4-symmetric space is formal in the sense of Sullivan. This result generalizes Sullivan's classical theorem on the formality of symmetric spaces, but the proof is of a different nature, since for generalized symmetric spaces techniques based on the Hodge theory do not work. We use the Thomas theory of minimal models of fibrations and the classification of 3- and 4-symmetric spaces.

1. Introduction. It is a classical result of Sullivan [15] that any Riemannian symmetric space is formal. Various investigations in symplectic geometry [13], Kählerian geometry [2], cohomology theory of transformation groups [1] and other geometric topics revealed deep relations between formality and geometric structures. Any Kählerian compact manifold is formal [2], there is a "formalizing tendency" of symplectic structures [13] etc. On the other hand, there is a broad class of Riemannian manifolds, which is a natural extension of that of symmetric spaces, namely, the generalized symmetric spaces [10]. Therefore, it is quite natural (in view of Sullivan's theorem [15]) to ask whether generalized symmetric spaces are formal.

In the present paper we show that there are new geometric structures implying formality of the underlying manifold, namely 3- and 4-symmetric spaces [4, 8]. These two classes of generalized symmetric spaces play an outstanding role in the whole theory because the geometry of manifolds carrying a 3- or 4-symmetric structure is very rich. 3-Symmetric spaces are nearly Kählerian, that is, $\nabla_X(J)X = 0$ for all vector fields $X \in \mathfrak{X}(M)$ and for the natural almost complex structure J determined by the 3-symmetric structure. The curvature tensor of a Riemannian 3-symmetric space is described in [4].

1991 *Mathematics Subject Classification*: 53C30, 55P62.
Key words and phrases: minimal model, Koszul complex, generalized symmetric space.

4-symmetric spaces were studied by J. A. Jiménez in [8]. They can be fibered in a natural way over symmetric spaces, there is an interesting duality theory, analogous to that of symmetric spaces, and some other deep results.

MAIN THEOREM. *Let M be any compact simply connected manifold carrying a structure of* 3- *or* 4-*symmetric Riemannian space. Then M is formal in the sense of Sullivan.*

The classical proof of formality of a symmetric space is based on the Hodge theory [21]. One takes the unique harmonic representative in each de Rham cohomology class and maps cohomology classes to their harmonic representatives. Since for symmetric Riemannian spaces, harmonic = $I_0(M,g)$-invariant ($I_0(M,g)$ is an isometry group), the linear homomorphism constructed above is multiplicative, and therefore, gives formality (see definitions below). For generalized symmetric spaces this proof does not work. Moreover, it is known that there are non-formal homogeneous spaces [5].

Our proof is of a different nature. It uses the technique of Koszul complexes, the Thomas theory of minimal models of fibrations and Jiménez's and Gray's classification of 3- and 4- symmetric spaces. This approach is of independent interest.

The paper is organized as follows. Section 2 is devoted to generalized symmetric spaces. Section 3 describes basic notions of rational homotopy theory with applications to compact homogeneous spaces. The Thomas theory of minimal models of Serre fibrations is presented in Section 4. Here we prove the basic algebraic result of the paper (Theorem 7) and apply it to bundles over homogeneous spaces (Theorem 8). In Section 5 we bring together all previous results to obtain the proof of the main theorem. The paper contains some new results on minimal models of homogeneous spaces. These results are stated in Theorems 3, 5, 7 and 8.

2. Preliminaries on generalized symmetric spaces. Throughout this paper we use the terminology and notations from [10]. Recall that a *generalized symmetric Riemannian space* is a Riemannian manifold (M,g) possessing at each point $x\in M$ an isometry $s_x : M\to M$ with the isolated fixed point x and satisfying the regularity condition

$$s_x s_y = s_{s_x(y)} s_x$$

for any $x,y\in M$. The family $\{s_x : x\in M\}$ is called a *regular Riemannian s-structure*.

DEFINITION. If there exists a positive integer k such that $s_x^k=\mathrm{id}$ for any $x\in M$, but $s_x^l\neq\mathrm{id}$ for $l<k$, then k is called the *order of the s-structure* $\{s_x : x\in M\}$ and denoted by $\mathrm{ord}\{s_x : x\in M\}$. The smallest

possible order $\operatorname{ord}\{s_x : x \in M\}$ of all regular s-structures which are admitted by a given generalized symmetric space (M, g) is called the *order of the generalized symmetric Riemannian space* (M, g) and denoted by $\operatorname{ord}(M, g)$. If $\operatorname{ord}(M, g) = 3$, we call (M, g) a 3-*symmetric space*, and if $\operatorname{ord}(M, g) = 4$, we call it a 4-*symmetric space*.

The homogeneous structure of k-symmetric spaces can be described as follows (see, e.g. [10]). Let G be the closure in the isometry group $I(M, g)$ of the subgroup generated by all $s_x s_y^{-1}$, $x, y \in M$. Then G acts transitively on M and we have

$$(*) \qquad M = G/H \quad \text{with} \quad (G^\sigma)_0 \subset H \subset G^\sigma,$$

where H is the isotropy subgroup of G at a fixed point in M and σ is the automorphism (of order k) of G induced by conjugation with s_0; G^σ, as usual, denotes the fixed point set of σ, and $(G^\sigma)_0$ its identity component. Let $\mathfrak{g}$ be the Lie algebra of G and σ_* the automorphism of $\mathfrak{g}$ induced by σ. Since H is compact, G/H is reductive, and $\mathfrak{g}$ admits an $\mathrm{Ad}(H)$- and σ_*-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathfrak{g}^{\sigma_*}$ is the Lie algebra corresponding to H, and $\mathfrak{m}$ can be identified with the tangent space of G/H at H, thus $\mathfrak{m}$ becomes equipped with an $\mathrm{Ad}(H)$- and σ_*-invariant scalar product. Conversely, given a connected Lie group G and an automorphism σ of order k of G and a subgroup H that satisfies $(*)$, assume that $\mathfrak{g}$ admits an $\mathrm{Ad}(H)$- and σ_*-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ admits an $\mathrm{Ad}(H)$- and σ_*-invariant scalar product. Then G/H can be made into a Riemannian k-symmetric space. It follows that the problem of classification of compact connected Riemannian k-symmetric spaces is equivalent to the problem of classifying automorphisms of order k of compact semisimple Lie algebras. The latter classification was done by V. Kac [7]; 3- and 4-symmetric spaces were classified by A. Gray [4] and J. A. Jiménez [8].

Since we use the classification of A. Gray and J. A. Jiménez in the proof, we reproduce it in *the compact case*.

Since any compact homogeneous space G/H of maximal rank is formal [5], we consider only the case of $\operatorname{rank} G > \operatorname{rank} H$.

Table 1. Compact 3-symmetric spaces, $\operatorname{rank} G > \operatorname{rank} H$

	G/H
1	$\mathrm{Spin}(8)/(SU(3)/\mathbb{Z}_3)$
2	$\mathrm{Spin}(8)/G_2$
3	$G \times G \times G$

Table 2. Compact 4-symmetric spaces, $\operatorname{rank} G > \operatorname{rank} H$

	Classical type G/H
1	$SO(2n)/U(p) \times SO(q) \times SO(r)$ $2p+q+r=2n,$ both q and r odd, $n \geq 3,\ q \geq r \geq 1$
2	$SU(2n)/S(U(n) \times U(1))$
3	$SU(2p+q)/\operatorname{Sp}(p) \times SO(q)$
4	$\{U \times U \times U \times U\}/U$ where U is compact simple and simply connected, and U is diagonally embedded in $U \times U \times U \times U$
5	$\{U \times U\}/U^{\Theta}$ where U is compact simple and simply connected, and $U^{\Theta},$ the fixed point of $U,$ is diagonally embedded in $U \times U,$ where Θ is an involution

	Exceptional type G/H
6	$E_6/SU(2) \times SO(6)$
7	$E_6/SO(7) \times SO(3)$
8	$E_6/\operatorname{Sp}(3)T_1$

3. Koszul complexes and minimal models of homogeneous spaces. We assume the reader is familiar with rational homotopy theory (see e.g. [2], [6], [12], [17]).

We consider the category $\mathbb{R}\text{-DGA}_{(c)}$ of graded commutative differential algebras over the reals and suppose all the differentials to be of degree $+1$. We say that two graded differential algebras $(A, d_A), (B, d_B) \in \mathbb{R}\text{-DGA}_{(c)}$ are *c-equivalent* if there is a chain of algebras $(A_i, d_{A_i}) \in \mathbb{R}\text{-DGA}_{(c)},\ i = 1, \ldots, k,$ starting from $(A, d_A) = (A_1, d_{A_1})$ and ending with $(A_k, d_{A_k}) = (B, d_B)$ such that each pair $((A_i, d_{A_i}), (A_{i+1}, d_{A_{i+1}}))$ is related either by a morphism

$$(A_i, d_{A_i}) \to (A_{i+1}, d_{A_{i+1}})$$

or by a morphism

$$(A_{i+1}, d_{A_{i+1}}) \to (A_i, d_{A_i})$$

inducing an isomorphism in cohomology. A morphism inducing an isomorphism on the cohomology level is called a *quasi-isomorphism*. Any graded differential algebra (A, d_A) that we consider satisfies $H^0(A, d_A) = \mathbb{R}$ and $H^n(A, d_A)$ is a finite-dimensional vector space for each n. We denote the ideal of positive degree elements in A by A^+. If V is a vector space, then $\bigwedge V$ denotes the free graded commutative algebra generated by V. If $\{v_1, v_2, \ldots\}$ is a basis for V, then we write $V = \langle v_1, v_2, \ldots \rangle$ and $\bigwedge V = \bigwedge(v_1, v_2, \ldots)$. A graded differential algebra is *minimal* if (1) $A \cong \bigwedge V$ for some V and (2) there is a basis $\{v_1, v_2, \ldots\}$ such that, for each j, $dv_j \in$

$(\bigwedge(v_1,\ldots,v_{j-1}))^+(\bigwedge(v_1,\ldots,v_{j-1}))^+$. We say that $(\bigwedge V,d)\in\mathbb{R}\text{-DGA}_{(\mathrm{c})}$ is a *minimal model* of (A,d_A) if there is a quasi-isomorphism

$$\varrho:(\bigwedge V,d)\to(A,d_A).$$

We use the following

PROPOSITION 1 [12]. *Any two c-equivalent graded differential algebras have isomorphic minimal models.*

In this paper we consider only smooth manifolds and their "real minimal models". That is, for any smooth manifold M we call the graded differential algebra $\mathfrak{m}_{\mathcal{E}}$ (which is the minimal model of the de Rham algebra of M) the *minimal model* of M. We use the notation

$$\mathfrak{m}_M=\mathfrak{m}_{\mathcal{E}}.$$

By definition, we say that a minimal algebra $(\bigwedge V,d)$ is *formal* if it is c-equivalent to its cohomology algebra $H^*(\bigwedge V,d)$. A manifold M is called *formal* if $\mathfrak{m}_M$ is formal.

Remark. Of course, it is enough for our purposes to use the above notions, but [2, 6, 12, 17] contain a more subtle topological approach.

In what follows we consider *P-algebras* and their Koszul complexes. The appropriate notions are defined in [5]. Each time we deal with them, we change the notation for a free algebra. Namely, $\bigwedge P$ denotes the exterior algebra over a finite-dimensional graded vector space $P=\bigoplus_k P_k$, graded by odd degrees. If Q denotes an evenly graded vector space, then we use the notation $\bigvee Q$ for the symmetric algebra over Q.

DEFINITION. A *P-algebra* is a pair (S,σ), where:

(1) $P=\bigoplus_k P^k$ is a finite-dimensional positively graded vector space,
(2) $\sigma:P\to S$ is a linear mapping, homogeneous of degree 1, which satisfies

$$\sigma(x)\cdot z=z\cdot\sigma(x),\qquad x\in P,\ z\in S,$$

(3) S is a positively graded associative algebra with identity.

DEFINITION. With each P-algebra S there is associated the following graded differential algebra: In the tensor product $S\otimes\bigwedge P$ define a linear operator ∇_σ by setting

$$\nabla_\sigma(z\otimes 1)=0,\qquad z\in S,$$

$$\nabla_\sigma(z\otimes x_0\wedge\ldots\wedge x_p)=\sum_{i=0}^{p}(-1)^{i-q}z\sigma(x_i)\otimes x_0\wedge\ldots\wedge\widehat{x}_i\wedge\ldots\wedge x_p$$

(here and below $\widehat{\ }$ denotes the deleting of x_i).

Direct calculations imply that $S \otimes \bigwedge P$ becomes a graded differential algebra if one defines multiplication in $S \otimes \bigwedge P$ by the rule

$$(z \otimes \Phi)(w \otimes \Psi) = (-1)^{pq} zw \otimes \Phi \wedge \Psi, \quad z \in S,\ w \in S^q,\ \Phi \in \textstyle\bigwedge^p P,\ \Psi \in \bigwedge P.$$

This algebra is called the *Koszul complex*.

The grading in $S \otimes \bigwedge P$ is defined as usual: if $z \in S^q$ and $x_i \in P^{p_i}$, then

$$z \otimes x_1 \wedge \ldots \wedge x_m \in (S \otimes \textstyle\bigwedge P)^{q+p_1+\ldots+p_m}.$$

Under this grading, ∇_σ is a derivation of degree $+1$. There is a second grading, defined as follows:

$$S \otimes \textstyle\bigwedge P = \displaystyle\bigoplus_k (S \otimes \textstyle\bigwedge P)_k,$$

where $(S \otimes \bigwedge P)_k = S \otimes \bigwedge^k P$. One verifies that ∇_σ is homogeneous of degree -1 with respect to this grading, which is called the *lower grading*. The two gradings of $S \otimes \bigwedge P$ define the bigrading given by

$$S \otimes \textstyle\bigwedge P = \displaystyle\bigoplus_{k,r} (S \otimes \textstyle\bigwedge P)^r_k, \quad (S \otimes \bigwedge P)^r_k = (S \otimes \bigwedge^k P)^r.$$

The elements of $(S \otimes \bigwedge^k P)^r$ are called homogeneous of *lower degree* k and *bidegree* (r, k).

Since ∇_σ is the derivation of $S \otimes \bigwedge P$, the cohomology algebra $H^*(S \otimes \bigwedge P, \nabla_\sigma)$ inherits the gradings

$$H^r(S \otimes \textstyle\bigwedge P) = \displaystyle\bigoplus_k H^r_k(S \otimes \textstyle\bigwedge P), \quad H_k(S \otimes \bigwedge P) = \displaystyle\bigoplus_r H^r_k(S \otimes \textstyle\bigwedge P).$$

The following decomposition holds:

$$H^*(S \otimes \textstyle\bigwedge P) = H_0(S \otimes \bigwedge P) \oplus H_+(S \otimes \bigwedge P),$$

where

$$H_0(S \otimes \textstyle\bigwedge P) = \displaystyle\bigoplus_r H^r(S \otimes \textstyle\bigwedge^0 P), \quad H_+(S \otimes \bigwedge P) = \displaystyle\bigoplus_r H^r_{k>0}(S \otimes \textstyle\bigwedge P).$$

$H_0(S \otimes \bigwedge P)$ is a graded subalgebra and $H_+(S \otimes \bigwedge P)$ is a graded ideal.

Consider now $(S \otimes \bigwedge P, \nabla_\sigma) \in \mathbb{R}\text{-DGA}_{(c)}$ and define a linear map $\varrho : S \otimes \bigwedge P \to \bigwedge P$ by setting

$$\varrho(1 \otimes \Psi + z \otimes \Phi) = \Psi, \quad \Phi, \Psi \in \textstyle\bigwedge P,\ z \in S^+. \tag{1}$$

The direct computation shows that $\varrho \circ \nabla_\sigma = 0$ and therefore ϱ induces a morphism $\varrho^* : H^*(S \otimes \bigwedge P) \to \bigwedge P$ in the category $\mathbb{R}\text{-DGA}_{(c)}$.

Definition. The homomorphism ϱ^* is called the *Samelson projection* for (S, σ), the graded space $\widehat{P} = P \cap \operatorname{Im} \varrho^*$ is called the *Samelson subspace* of P, and the graded subspace $\widetilde{P}$ of P such that $P = \widetilde{P} \oplus \widehat{P}$ is called the *Samelson complement*.

Let now Q be an evenly graded finite-dimensional vector space with $Q^k = 0$ for $k \le 0$. Let $\bigvee Q$ be the corresponding symmetric algebra endowed with the induced grading

$$|y_1 \vee \ldots \vee y_q| = |y_1| + \ldots + |y_q|.$$

DEFINITION. A P-algebra (S, σ) with $S = \bigvee Q$ is called a *symmetric* P-algebra.

THEOREM 1 [5]. *Let* $(\bigvee Q, \sigma)$ *be a symmetric* P*-algebra such that* $H^*(\bigvee Q \otimes \bigwedge P)$ *has finite dimension. Then*

$$\dim P \ge \dim \widehat{P} + \dim Q,$$

where $\widehat{P}$ *is the Samelson subspace of* P.

COROLLARY [5]. *The following conditions are equivalent*:

$$\dim P = \dim \widehat{P} + \dim Q, \tag{2}$$

$$H_+(\bigvee Q \otimes \bigwedge \widetilde{P}) = 0. \tag{3}$$

THEOREM 2 [5]. *Let* $(\bigvee Q, \sigma)$ *be a symmetric* P*-algebra with Samelson subspace* $\widehat{P}$. *Then the graded differential algebra* $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ *is c-equivalent to* $(H^*(\bigvee Q \otimes \bigwedge P, \nabla_\sigma), 0)$ *if and only if* (3) *holds.*

DEFINITION. Let R be any ring. A sequence $a_1, a_2, \ldots$ in R is called *regular* if no a_i is a zero divisor in the factor ring $R/(a_1, \ldots, a_{i-1})$. Here and everywhere below $(a_1, \ldots, a_{i-1})$ stands for the ideal generated by $a_1, \ldots, a_{i-1}$.

THEOREM 3. *Let* $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ *be the Koszul complex associated with a symmetric* P*-algebra. Let* $y_1, \ldots, y_n$ *be a basis of* P *and let* $s = \dim Q$. *Suppose that* $H^*(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ *is finite-dimensional. Then the minimal model of* $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ *is formal if and only if the following conditions are satisfied*:

(i) $\dim P = n \ge \dim Q = s$,

(ii) $\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s)$ *constitute a regular sequence in* $\bigvee Q$ *and* $\nabla_\sigma(y_{s+1}), \ldots, \nabla_\sigma(y_n) \in (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$ (*after reordering if necessary*).

Proof. By Theorem 1, (i) is satisfied for any Koszul complex with finite-dimensional cohomology algebra. Thus, only two possibilities may occur:

(1) $\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s)$ form a regular sequence, $\nabla_\sigma(y_{s+1}), \ldots, \nabla_\sigma(y_n) \in (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$,

(2) $\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s)$ form a regular sequence, and at least one of $\nabla_\sigma(y_j)$ $(j > s)$, say $\nabla_\sigma(y_{s+1})$, is not in $(\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$, but the sequence $\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s), \nabla_\sigma(y_{s+1})$ is not regular.

If (1) is satisfied, then it is known that the minimal model of $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ is formal. Nevertheless, we give here a simple proof. We use the following lemma, proved in [20].

Lemma [20]. *Let $(A, d_A) \in \mathbb{R}\text{-DGA}_{(\mathrm{c})}$ be of finite type. Let ϑ be the ideal of A generated by the exterior generators, and let $\overline{A} = A/\vartheta$. If y is an exterior generator of A such that the image of $d_A y$ in $\overline{A}$ is non-zero, then $H^*(\widetilde{A}, \widetilde{d}) = H^*(A, d_A)$, where $\widetilde{A} = A/(y, d_A y)$ and $\widetilde{d}$ is the induced differential on $\widetilde{A}$.*

Now, to prove the formality of the minimal model of $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$ in case (1), it is enough to apply the previous lemma successively to each $\nabla_\sigma(y_j)$, $j = 1, \ldots, s$ (the regularity condition guarantees that this is possible). By this procedure one obtains

$$H^*(\bigvee Q \otimes \bigwedge P, \nabla_\sigma) = (\bigvee Q/(\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))) \otimes \bigwedge(y_{s+1}, \ldots, y_n),$$

and the natural projection

$$\varrho : (\bigvee Q \otimes \bigwedge P, \nabla_\sigma) \to (\bigvee Q/(\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))) \otimes \bigwedge(y_{s+1}, \ldots, y_n)$$

becomes a c-equivalence. Applying now Proposition 1, one obtains the proof in case (1).

It remains to consider the second possibility. Since the sequence $\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s), \nabla_\sigma(y_{s+1}), \ldots$ is not regular, it follows that $\nabla_\sigma(y_{s+1}) \notin (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$, but $\nabla_\sigma(y_{s+1})$ is a zero divisor in the quotient algebra $\bigvee Q/(\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$, that is,

$$\begin{aligned} h \cdot \nabla_\sigma(y_{s+1}) = h_1 \cdot \nabla_\sigma(y_1) + \ldots + h_s \cdot \nabla_\sigma(y_s), \\ h \notin (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s)),\ h, h_i \in \bigvee Q,\ i = 1, \ldots, s. \end{aligned}$$

Put $v = h_1 y_1 + \ldots + h_s y_s - h y_{s+1}$. Obviously $\nabla_\sigma(v) = 0$ and v represents a cohomology class $[v] \in H^*(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$.

Observe that $y_{s+1} \notin \widehat{P}$ (recall that $\widehat{P}$ is the Samelson subspace). To see this, write down the Samelson projection ϱ, which in our case translates as

$$y_{s+1} = \varrho^*\Big(y_{s+1} + \sum_{j=1}^{s} g_j y_j + \sum_{l,m} g_{lm} y_l \wedge y_m + \ldots\Big).$$

The above equality implies $\nabla_\sigma(y_{s+1}) \in (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$, because y_{s+1} is the image of a cocycle under ϱ^*. Thus, one can choose the Samelson complement $y_{s+1} \in \widetilde{P}$. But then the cohomology class $[v]$ satisfies the condition

$$[v] \in H_1(\bigvee Q \otimes \bigwedge \widetilde{P}) \subset H_+(\bigvee Q \otimes \bigwedge \widetilde{P}).$$

Observe that $[v] \neq 0$. To prove this, suppose that

$$v = h_1 y_1 + \ldots + h_s y_s - h y_{s+1} = \nabla_\sigma(w). \tag{4}$$

Then, without loss of generality, w can be chosen in the form

$$(5)\quad w=\sum_{k,t} h_{k,t} y_k \wedge y_t = \sum_{k,t=1}^{s} h_{k,t} y_k \wedge y_t + \sum_{l=1}^{s} g_l y_l \wedge y_{s+1}, \qquad h_{k,t}, g_l \in \bigvee Q.$$

Then, applying ∇_σ to (5) one obtains from (4)

$$h_i = \sum_{k=1}^{s} (h_{k,i} - g_k \nabla_\sigma(y_{s+1})), \qquad h = \sum_{l=1}^{s} g_l \nabla_\sigma(y_l),$$

which implies $h \in (\nabla_\sigma(y_1), \ldots, \nabla_\sigma(y_s))$, a contradiction. Thus $H_+(\bigvee Q \otimes \bigwedge \widetilde{P}) \neq 0$ and applying the corollary of Theorem 1 completes the proof.

Recall the notion of the Cartan algebra of a homogeneous space. We consider compact Lie groups. To any compact homogeneous space G/H one can assign a graded differential algebra $(C, \nabla_\sigma) \in \mathbb{R}\text{-DGA}_{(\mathrm{c})}$ by the procedure described below. Let T, T' be maximal tori in G and H respectively $(T \supset T')$. Denote by $W(G)$ and $W(H)$ the Weyl groups associated with T and T' and consider the corresponding $W(G)$- and $W(H)$-actions on the Lie algebras $\mathfrak{T}$ and $\mathfrak{T}'$ of T and T'. These actions are extended in a natural way to actions on the polynomial algebras $\mathbb{R}[\mathfrak{T}]$ and $\mathbb{R}[\mathfrak{T}']$:

$$\sigma(f)(x) = f(\sigma^{-1}(x))$$

for any $\sigma \in W(G)$ (resp. $W(H)$), $f \in \mathbb{R}[\mathfrak{T}]$ (resp. $f \in \mathbb{R}[\mathfrak{T}']$) and $x \in \mathfrak{T}$ (resp. $x \in \mathfrak{T}'$). Let $\mathbb{R}[\mathfrak{T}]^{W(G)}$ and $\mathbb{R}[\mathfrak{T}']^{W(H)}$ be the subalgebras of $W(G)$- and $W(H)$-invariants. By the Chevalley theorem,

$$\mathbb{R}[\mathfrak{T}]^{W(G)} \cong \mathbb{R}[f_1, \ldots, f_n], \qquad n = \operatorname{rank} G,$$
$$\mathbb{R}[\mathfrak{T}']^{W(H)} \cong \mathbb{R}[u_1, \ldots, u_s], \qquad s = \operatorname{rank} H.$$

Consider the usual representation of the cohomology algebra $H^*(G, \mathbb{R})$ as the exterior algebra over the primitive elements

$$H^*(G, \mathbb{R}) \cong \bigwedge(y_1, \ldots, y_n).$$

Define

$$(6)\qquad \begin{aligned} &(C, \nabla_\sigma) = (\mathbb{R}[u_1, \ldots, u_s] \otimes \bigwedge(y_1, \ldots, y_n), \nabla_\sigma), \\ &\nabla_\sigma(u_i) = 0, \qquad i = 1, \ldots, s, \\ &\nabla_\sigma(y_j) = f_j|_{\mathfrak{T}'} = \widetilde{f}_j(u_1, \ldots, u_s), \qquad j = 1, \ldots, n. \end{aligned}$$

Definition. The algebra (C, ∇_σ) defined by (6) is called the *Cartan algebra* of the homogeneous space G/H.

Theorem 4 [5]. *The following isomorphism holds*:

$$\mathfrak{m}_{G/H} \cong \mathfrak{m}_{(C, \nabla_\sigma)}.$$

P r o o f. Let $\mathcal{E}$ be the de Rham algebra of G/H and $\mathcal{E}^{\text{inv}}$ denote the subalgebra of G-invariant forms. The following chain of c-equivalences is proved in [5]:

$$\mathcal{E} \overset{\text{c}}{\sim} \mathcal{E}^{\text{inv}} \overset{\text{c}}{\sim} (C, \nabla_\sigma).$$

Now, applying Proposition 1 completes the proof.

THEOREM 5. *Let $M = G/H$ be a homogeneous space of a compact Lie group G. Let (C, ∇_σ) be its Cartan algebra given by* (6). *Then M is formal if and only if the sequence $\widetilde{f}_1, \ldots, \widetilde{f}_n$ satisfies the following conditions* (*after an appropriate ordering*):

(i) $\widetilde{f}_1, \ldots, \widetilde{f}_s$ *constitute a regular sequence*,

(ii) $\widetilde{f}_{s+1}, \ldots, \widetilde{f}_n \in (\widetilde{f}_1, \ldots, \widetilde{f}_s)$.

P r o o f. Since (C, ∇_σ) is a particular case of $(\bigvee Q \otimes \bigwedge P, \nabla_\sigma)$, the result follows from Theorems 3 and 4.

4. Twisted tensor products of Koszul complexes and the Thomas theory of minimal models of rational fibrations. The proof of the main theorem requires some facts from rational homotopy theory of Serre fibrations. This theory was developed by Halperin, Grivel and Thomas [18], [19]. We use the Thomas approach.

DEFINITION. Let (A, d_A) and (B, d_B) be graded differential algebras, and let $f : (A, d_A) \to (B, d_B)$ be a homomorphism of graded differential algebras. Then f is said to be a *KS-extension* if there exists a subset $E \subset B$, $E = \{x_\alpha : \alpha \in A\}$ (A is an ordered set), such that

(i) if $j : \bigwedge(E) \to B$ is the homomorphism induced by the inclusion $E \to B$ and if $\varphi : A \otimes \bigwedge(E) \to B$ is the homomorphism induced by f and j, then φ is an isomorphism,

(ii) $d_B\varphi(1 \otimes x_\alpha) \in \varphi(A \otimes \bigwedge(E_\alpha))$, where $E_\alpha = \{x_\beta : \beta < \alpha\}$.

Since $d_B\varphi(a \otimes 1) = d_B f(a) = f(d_A(a)) = \varphi(d_A(a) \otimes 1)$ it follows that, if we use φ to identify B with $A \otimes \bigwedge(E)$, then d_B satisfies the conditions

$$d_B(a \otimes 1) = d_A(a) \otimes 1, \qquad d_B(x_\alpha) \in A \otimes \bigwedge(E_\alpha). \tag{7}$$

If $\varepsilon : A \to k$ is an augmentation, then one can define

$$d|_{\bigwedge(E)} = (\varepsilon \otimes \text{id})(d_B(x_\alpha))$$

and therefore $(\bigwedge(E), d)$ can be included in the sequence

$$(A, d_A) \to (A \otimes \bigwedge(E), d_B) \to (\bigwedge(E), d). \tag{8}$$

DEFINITION. If (A, d_A) is minimal and $(A \otimes \bigwedge(E), d_B)$ is a KS-extension, the sequence (8) is called the *minimal KS-extension*.

Let

$$F \to E \to B \tag{9}$$

be any Serre fibration. The following theorem was proved by Grivel, Halperin and Thomas (independently).

THEOREM 6 [18]. *Let* (9) *be a Serre fibration satisfying the following conditions*:

(i) F *is path-connected*,
(ii) $\pi_1(B)$ *acts nilpotently on* $H^j(F;k)$ *for all* $j \geq 1$,
(iii) *either* B *or* F *has finite* k*-type.*

Then there exists the following minimal KS-extension, *corresponding to* (9):

$$\mathfrak{m}_B \to \mathfrak{m}_B \otimes \mathfrak{m}_F \to \mathfrak{m}_F, \tag{10}$$

where $\mathfrak{m}_B \otimes \mathfrak{m}_F$ *is a model for* E, *but*, *owing to the possible twisting of the differential in* $\mathfrak{m}_B \otimes \mathfrak{m}_F$ (see (7)), *not necessarily minimal.*

Consider now two Koszul complexes $\mathcal{T}, \mathcal{S}$ associated with P-algebras:

$$\mathcal{T} = (\bigvee Q \otimes \bigwedge P, \nabla_1), \qquad \mathcal{S} = (\bigvee Q' \otimes \bigwedge P', \nabla_2),$$

and construct the KS-extension

$$\mathcal{T} \to \mathcal{T} \otimes_\tau \mathcal{S} \to \mathcal{S},$$

where the additional symbol τ expresses the "twisting" of the differential. Formulae (7) show that the derivation d_τ is determined by a linear map $\tau : P' \to Z(\mathcal{T})$ such that

$$d_\tau|_Q = d_\tau|_{Q'} = 0, \quad d_\tau|_P = \nabla_1|_P, \quad d_\tau|_{P'} = \nabla_2 + \tau|_{P'}. \tag{11}$$

It is convenient to use the notion of pureness, introduced in [13].

DEFINITION. Let $\bigwedge(V,d) = (\bigwedge V^{\mathrm{even}} \otimes \bigwedge V^{\mathrm{odd}}, d)$ be a minimal graded differential algebra. This algebra is called *pure* if

$$d(V^{\mathrm{odd}}) \subset \bigwedge V^{\mathrm{even}}.$$

Remark. This definition is essentially stronger than the definition of pureness in [18]. Observe that the statement about pureness in this article is stronger than in [18] and does not follow from the cited works.

THEOREM 7. *Let* $\mathcal{T} \otimes_\tau \mathcal{S}$ *be a twisted tensor product of Koszul complexes* $\mathcal{T}$ *and* $\mathcal{S}$, *determined by the KS-extension* (8) *corresponding to formulae* (11). *Suppose that*

$$\dim Q' = \dim P'$$

and $H^*(\mathcal{T}), H^*(\mathcal{S})$ *and* $H^*(\mathcal{T} \otimes_\tau \mathcal{S})$ *are finite-dimensional. Then the minimal model of* $\mathcal{T} \otimes_\tau \mathcal{S}$ *is formal if and only if the minimal model of* $\mathcal{T}$ *is formal.*

P r o o f. The first step is the following statement: *under the conditions of Theorem* 7, *the graded differential algebra* $\mathcal{T} \otimes_\tau \mathcal{S}$ *is pure*. To see this, write

$$(\mathcal{T} \otimes_\tau \mathcal{S}, d_\tau) = (\bigvee Q \otimes \bigwedge P \otimes \bigvee Q' \otimes \bigwedge P', d_\tau).$$

Let $P = \langle z_1, \ldots, z_n \rangle, P' = \langle y_1, \ldots, y_s \rangle, s = \dim Q'$. Introduce the notations

$$d_\tau z_i = g_i \in \bigvee Q, \quad d_\tau(y_j) = u_j + f_j, \quad u_j \in \bigvee Q \otimes \bigwedge P, \ f_j \in \bigvee Q'$$

(according to (11)). If $\mathcal{T} \otimes_\tau \mathcal{S}$ is not pure, then at least for one j,

$$d_\tau y_j = f_j + \sum_{i_1 < \ldots < i_t} h_j^{i_1 \ldots i_t} z_{i_1} \wedge \ldots \wedge z_{i_t}, \tag{12}$$

where $h_j^{i_1 \ldots i_t} \in \bigvee Q$ and $|z_{i_1} \wedge \ldots \wedge z_{i_t}|$ is even. Since $\dim Q' = \dim P' = s$, the sequence $f_1, \ldots, f_s$ is regular in $\bigvee Q'$. We claim that there exists an infinite sequence of linearly independent elements $q_\alpha f_j,\ \alpha = 1, 2, \ldots,\ q_\alpha \in \bigvee Q'$, which are not coboundaries. To prove this, consider the opposite condition

$$q f_j = d_\tau(w + u \otimes v + \overline{v}) = d_\tau(w) + d_\tau(u) \otimes v + u \otimes d_\tau(v) + d_\tau(\overline{v}),$$

where $w, u \in \bigvee Q \otimes \bigwedge P$ and $v, \overline{v} \in \bigvee Q' \otimes \bigwedge P'$. Since $\mathcal{T} \otimes_\tau \mathcal{S}$ is a free algebra, the above equality implies $d_\tau(w) = 0$. Let

$$v = \sum_{j_1 < \ldots < j_k} g_{j_1 \ldots j_k} y_{j_1} \wedge \ldots \wedge y_{j_k}, \qquad \overline{v} = \sum_{l_1 < \ldots < l_r} \overline{g}_{l_1 \ldots l_r} y_{l_1} \wedge \ldots \wedge y_{l_r}.$$

Then

$$\begin{aligned} q f_j = d_\tau(u) \otimes & \sum_{j_1 < \ldots < j_k} g_{j_1 \ldots j_k} y_{j_1} \wedge \ldots \wedge y_{j_k} \\ & \pm u \otimes \sum_m \sum_{j_1 < \ldots < j_k} d_\tau(y_m) g_{j_1 \ldots j_k} \otimes y_{j_1} \wedge \ldots \wedge \widehat{y}_{j_m} \wedge \ldots \wedge y_{j_k} \\ & \pm \sum_p \sum_{l_1 < \ldots < l_t} \overline{g}_{l_1 \ldots l_{t_r}} \otimes d_\tau(y_{l_p}) \otimes y_{l_1} \wedge \ldots \wedge \widehat{y}_{l_p} \wedge \ldots \wedge y_{l_{t_r}} \end{aligned}$$

(the sign of summands does not influence the argument).

In any case the above equality can be valid only if $k = r = 1$ (otherwise the right-hand side is either zero, or contains elements of non-zero y'-degree, both cases contradicting the left-hand side). If $k = 1$ the same freeness argument implies

$$q f_j = d_\tau\Big(\sum_q g_j^q y_q\Big). \tag{13}$$

Consider all y_j for which $\tau(y_j) \in Z_+(S)$. Without loss of generality one can assume that all expressions (12) either differ by variables $z_{i_1} \wedge \ldots \wedge z_{i_t}$, or $h_j^{i_1 \ldots i_t}$ are linearly independent, since otherwise one could obtain, for example,

$$d_\tau(y_p) = f_p + \sum h_p^{i_1 \ldots i_t} z_{i_1} \wedge \ldots \wedge z_{i_t}, \qquad d_\tau(y_q) = f_q + \sum h_q^{i_1 \ldots i_t} z_{i_1} \wedge \ldots \wedge z_{i_t}$$

with $h_q^{i_1\ldots i_t}=\mu h_p^{i_1\ldots i_t}$ and $d_\tau(y_q-\mu y_p)=f_q-\mu f_p$, and taking the appropriate variable change, one would obtain $\tau(y_q)\in Z_0(S)$, lowering the number of variables whose image is contained in $Z_+(S)$. Thus, (13) can be rewritten as

$$qf_j=d_\tau\Big(\sum_q g_j^q y_q\Big)=\sum_q f_q g_j^q+\sum_q\Big(\sum h_q^{i_1\ldots i_t}z_{i_1}\wedge\ldots\wedge z_{i_t}\Big)g_j^q.$$

Using the freeness condition, one obtains

$$\sum_l h_l^{i_1\ldots i_t}\otimes g_j^l=0,\qquad h_l^{i_1\ldots i_t}\in\bigvee Q,\ g_j^l\in\bigvee Q',$$

where by assumption $h_l^{i_1\ldots i_t}$ are linearly independent. Thus necessarily $g_j^l=0$. Therefore, finally,

$$qf_j\in(f_1,\ldots,\widehat{f_j},\ldots,f_s)\tag{14}$$

(because g_j^j certainly belongs to the set $\{g_j^l\}$). Since $f_1,\ldots,f_s$ is a regular sequence, $q\in(f_1,\ldots,\widehat{f_j},\ldots,f_s)$. Since $\dim Q'=s$, one can find an infinite sequence of polynomials q_α, $\alpha=1,2,\ldots$, with $q_\alpha\notin(f_1,\ldots,\widehat{f_j},\ldots,f_s)$. Then (14) implies the existence of an infinite sequence of linearly independent cohomology classes $[q_\alpha f_j]$, $\alpha=1,2,\ldots$, in $H^*(\mathcal{T}\otimes_\tau\mathcal{S},d_\tau)$, which is a contradiction. The first statement is proved.

Suppose now that the minimal model of $\mathcal{T}$ is formal. Then

$$\mathcal{T}\overset{\mathrm{c}}{\sim}(\bigvee Q\otimes\bigwedge\widehat{P},\overline{\nabla}_1)\otimes(\bigwedge\widetilde{P},0),\tag{15}$$

where, as in the previous section, $\widehat{P}$ is the Samelson subspace and $\widetilde{P}$ is the Samelson complement. To prove the above quasi-isomorphism, apply Theorem 3 and the well-known derivation change:

$$\begin{aligned}&\overline{\nabla}_1(z_1)=\nabla_1(z_1),\ \ldots,\quad\overline{\nabla}_1(z_t)=\nabla_1(z_t),\quad t=\dim Q,\\&\overline{\nabla}_1(z_{t+1})=\nabla_1(z_{t+1})-\nabla_1(z_1)z_1-\ldots-\nabla_1(z_t)z_t.\end{aligned}$$

In (15), $\overline{\nabla}_1(z_1),\ldots,\overline{\nabla}_1(z_t)$ constitute a regular sequence in $\bigvee Q$. Finally, $\mathcal{T}$ is quasi-isomorphic to the algebra

$$(\bigvee Q\otimes\bigwedge\widehat{P})\otimes_\tau(\bigvee Q'\otimes\bigwedge P')\otimes(\bigwedge\widetilde{P},0)=\mathcal{T}_1\otimes(\bigwedge\widetilde{P},0)$$

and the twisted tensor product $\mathcal{T}_1$ is pure. Therefore, $d_\tau(z_1),\ldots,d_\tau(z_t)$, $d_\tau(y_1),\ldots,d_\tau(y_s)$ is a sequence of polynomials in $\bigvee Q\otimes\bigvee Q'$ and $\mathcal{T}_1$ is again a Koszul complex. Since the number of the polynomials $(s+t)$ equals the dimension of $Q\oplus Q'$, the minimal model of $\mathcal{T}_1$ (and $\mathcal{T}\otimes_\tau\mathcal{S}$) is formal, by the corollary to Theorem 1 and Theorem 2.

Let the minimal model $\mathcal{T}\otimes_\tau\mathcal{S}$ be formal. Consider the representation

$$(\mathcal{T}\otimes_\tau\mathcal{S},d_\tau)=(\bigvee Q\otimes\bigvee Q'\otimes\bigwedge P\otimes\bigwedge P',d_\tau),$$

where $d_\tau(z_1),\ldots,d_\tau(z_n),d_\tau(y_1),\ldots,d_\tau(y_s)$ is a sequence of polynomials in $\bigvee Q\otimes\bigvee Q'$ of the form

$$\begin{aligned} d_\tau(z_i)&=f_i(q_1,\ldots,q_t), && i=1,\ldots,n,\\ d_\tau(y_j)&=\overline{\tau}_j(q_1,\ldots,q_t)+g_j(u_1,\ldots,u_s), && j=1,\ldots,s, \end{aligned}$$

where $Q=\langle q_1,\ldots,q_t\rangle$ and $Q'=\langle u_1,\ldots,u_s\rangle$. By Theorem 3 there is a regular sequence of polynomials f_i in $\bigvee Q$, say $f_1,\ldots,f_t$, and we claim that *the sequence*

$$f_1,\ldots,f_t,\overline{\tau}_1+g_1,\ldots,\overline{\tau}_s+g_s \tag{16}$$

is regular in $\bigvee Q\otimes\bigvee Q'$. To prove the above claim, consider the ideal $I=(f_1,\ldots,f_t,\overline{\tau}_1+g_1,\ldots,\overline{\tau}_s+g_s)\subset\bigvee Q\otimes\bigvee Q'$ generated by the sequence (16) and denote by $V(I)$ the affine algebraic variety in $Q\oplus Q'$ generated by I. Observe that since $f_1,\ldots,f_t$ form a regular sequence in $\bigvee Q$, we have

$$V(I)=\{(\underbrace{0,\ldots,0}_{t},u_1,\ldots,u_s)\in Q\oplus Q'\},$$

because $\dim Q=t$ and $f_i(q_1,\ldots,q_t,u_1,\ldots,u_s)=f_i(q_1,\ldots,q_t)$ (the latter equality expresses the fact that the f_i on the left-hand side are considered as polynomials in $\bigvee(Q\oplus Q')$). Since $I_1=(f_1,\ldots,f_t)\subset\bigvee Q$ is a complete intersection ($h(I_1)=\mu(I_1)=d(I_1)$, its height equals the number of generators and the depth because of regularity), the algebraic variety $V(I_1)$ is an ideal-theoretic complete intersection (see [11], p. 135) and therefore $\dim V(I_1)=0$. Since all the polynomials are homogeneous, $V(I_1)=0$. Then

$$(\overline{\tau}_i+g_i)(\underbrace{0,\ldots,0}_{t},u_1,\ldots,u_s)=g_i(u_1,\ldots,u_s)=0,$$

because the $\overline{\tau}_i$ are homogeneous. Since $g_1,\ldots,g_s$ is a regular sequence in $\bigvee Q'$ and $\dim Q'=s$, the above argument applied to $g_1,\ldots,g_s$ gives $u_1=\ldots=u_s=0$ and $V(I)=0$. Therefore, I is a complete intersection.

By [11], p. 135, (16) is regular in $\bigvee Q\otimes\bigvee Q'$. Since $t+s=\dim Q+\dim Q'$ and the minimal model of $\mathcal{T}\otimes_\tau\mathcal{S}$ is formal, Theorem 3 implies

$$f_{t+1},\ldots,f_n\in(f_1,\ldots,f_t,\overline{\tau}_1+g_1,\ldots,\overline{\tau}_s+g_s).$$

Comparing $(q_1,\ldots,q_t)$-degrees and $(u_1,\ldots,u_s)$-degrees, one obtains necessarily

$$f_{t+1},\ldots,f_n\in(f_1,\ldots,f_t)$$

in $\bigvee Q$ and by Theorem 3 the minimal model of $\mathcal{T}$ is formal (it is necessary, however, to use the regularity of $g_1,\ldots,g_s$). Theorem 7 is proved.

Now we apply the above theorem to total spaces of some bundles.

Let $P\xrightarrow{G}M$ be a principal bundle and $L\subset G$ be a closed subgroup of maximal rank. Suppose that G is compact. Let $E=P\times_G G/L$ be the total space of the associated bundle

$$E \xrightarrow{G/L} M. \tag{17}$$

THEOREM 8. *Let $P \xrightarrow{G} M$ and L be as above. Suppose that $M = H/K$ is a homogeneous space of a compact connected Lie group H. Then the total space of the associated bundle $E = P \times_G G/L$ is formal if and only if M is formal.*

Proof. The associated bundle (17) becomes the Serre fibration

$$G/L \to E \to M \tag{18}$$

satisfying the conditions of Theorem 6 (Grivel, Halperin, Thomas). By the conditions of Theorem 8, $\operatorname{rank} G = \operatorname{rank} L$ and therefore $m_{G/L}$ satisfies the conditions of Theorem 7 (for $\mathcal{S}$). Taking the minimal KS-extension corresponding to the fibration (18), one can notice that this KS-extension satisfies the conditions of Theorem 7, since the minimal model of a Koszul complex is again a Koszul complex (see [12], Section 8). Now, an application of Theorem 7 completes the proof.

5. Proof of the main theorem. The proof is based on the classification of 3- and 4-symmetric spaces and Theorem 8. Let G be a compact Lie group, H be its closed subgroup,

$$G \xrightarrow{H} G/H$$

be an H-principal bundle. Let $H \supset K$, where K is a closed Lie subgroup in G, and consider the associated bundle with fiber H/K, which leads to the Serre fibration

$$H/K \to G/K \to G/H. \tag{19}$$

We now consider 3- and 4-symmetric spaces separately.

3-*symmetric case*. The case $G \times G \times G$ is evident, since any compact Lie group is formal. The case $\mathrm{Spin}(8)/G_2$ can be treated as follows. The homogeneous space $\mathrm{Spin}(8)/G_2$ is a $\mathbb{Z}_2$-covering of $SO(8)/G_2$ and therefore has the same minimal model. But then one constructs the fibration (19), taking $H = SO(7)$, $G = SO(8)$ and $K = G_2$. Then

$$SO(7)/G_2 \to SO(8)/G_2 \to SO(8)/SO(7),$$

which means that one obtains a bundle over S^7 with fiber S^7. Since $m_{S^7} = (\bigwedge(z), 0), |z| = 7$, the Thomas theorem (see Theorem 6) implies formality of $SO(8)/G_2$ (there is no twisting in the tensor product).

Consider now the case $\mathrm{Spin}(8)/(SU(3)/\mathbb{Z}_3)$. To treat it, we need the explicit expression for its Cartan algebra, given in Section 2. An explicit expression for $W(G)$- and $W(H)$-invariants in the cases under consideration can be found in [3]. In fact, it is enough for us to write appropriate degrees (see [3]). Calculating formulae (6) for the Cartan algebra (C,∇_σ), one obtains in this particular case

$$\begin{aligned}&(C,\nabla_\sigma)=(\mathbb{R}[u_1,u_2]\otimes\bigwedge(y_1,y_2,y_3,y_4),\nabla_\sigma),\\&\nabla_\sigma(u_i)=0,\quad i=1,2,\\&\nabla_\sigma(y_j)=\widetilde{f}_j(u_1,u_2),\quad j=1,\dots,4,\end{aligned}$$

and $|u_1|=4$, $|u_2|=6$, $|\widetilde{f}_1|=4$, $|\widetilde{f}_2|=8$, $|\widetilde{f}_3|=12$, $|\widetilde{f}_4|=8$. Therefore $\widetilde{f}_1=\alpha\cdot u_1$, $\alpha\in\mathbb{R}$, and $\widetilde{f}_2=\mu\cdot u_1^2$, $\mu\in\mathbb{R}$. Now we claim that $\alpha\neq 0$. To prove this, one needs explicit expressions for f_i and u_j, which can be written (after introducing appropriate coordinates in the Cartan subalgebra of $SO(8)$) as

$$f_1=x_1^2+\dots+x_4^2,\quad u_1=x_1^2+x_2^2+x_3^2,\quad x_1+x_2+x_3=0$$

(we need no other expressions).

Now, observe that (C,∇_σ) is not minimal. To obtain its minimal model, one can notice that (C,∇_σ) is free and apply the Sullivan algorithm ([12], Section 8). This algorithm eliminates non-decomposable generators. In our case we eliminate u_1 and y_1. As a result, $\widetilde{\nabla}_\sigma(y_2)=0$ ($\widetilde{\nabla}_\sigma$ denotes the differential in the minimal model of (C,∇_σ)). Thus, finally, the minimal model of (C,∇_σ) contains only one even-degree generator. Therefore it is evidently formal since the condition of Theorem 5 is satisfied automatically.

4-*symmetric case*. Use the Jiménez classification (Table 2) and Theorem 8. The proof is based on fibering 4-symmetric spaces over formal bases with fiber of maximal rank. The method is the same, but calculations differ in each particular case, therefore we have to reproduce all of them. Let us consider the 4-symmetric spaces according to their numbers in Table 2.

Case 1. Since $SO(2p)/U(p)$ is of maximal rank, it suffices to apply Theorem 8 provided that we prove that

$$SO(2n)/SO(2p)\times SO(q)\times SO(r)$$

is formal. Recall that q and r are odd (see Table 2). Therefore, $q=2s+1$, $r=2t+1$ and since $2n=2p+2s+2t+1+1=2(p+s+t+1)$,

$$(**)\quad \operatorname{rank}SO(2n)-\operatorname{rank}(SO(2p)\times SO(q)\times SO(r))=n-p-s-t=1.$$

Recall the explicit expressions of invariant polynomials for $SO(2n)$ (see [3]). If $x_1,\dots,x_n$ are coordinates in a Cartan subalgebra of the Lie algebra of $SO(2n)$, then

$$f_i=x_1^{2i}+\dots+x_n^{2i},\quad i=1,\dots,n-1,\quad f_n=x_1\dots x_n.$$

Taking into consideration the explicit form of the embedding

$$SO(2p)\times SO(q)\times SO(r)\to SO(2n)$$

one easily obtains $\widetilde{f}_n = 0$. Then apply Theorem 5 and the equality $(**)$ to complete the proof.

Remark. The proof could also be derived from [5] (p. 480, Example 8) using the equality $j_{\Theta=0}(Pf) = 0$ on page 481. In fact, we constructed a new example of a homogeneous space with the group $SO(2n)$ which is formal.

Case 2. Consider the evident fibration

$$SU(n+1)/SU(n)\times T_1 \to SU(2n)/SU(n)\times T_1 \to SU(2n)/SU(n+1).$$

Its base is formal by [5] (p. 475), the fiber is of maximal rank and Theorem 8 is applicable.

Case 3. This case can be treated in two steps. Consider the following two fibrations:

$$\begin{gathered}\mathrm{Sp}(p)/U(p) \to SU(2p+q)/U(p)\times SO(q) \to SU(2p+q)/\mathrm{Sp}(p)\times SO(q),\\ SO(2p+q)/U(p)\times SO(q) \to SU(2p+q)/U(p)\times SO(q)\\ \to SU(2p+q)/SO(2p+q).\end{gathered}$$

Both have fibres of maximal rank. Since $SU(2p+q)/SO(2p+q)$ is formal as a symmetric space, the second fibration yields the formality of the space $SU(2p+q)/U(p)\times SO(q)$ (Theorem 8). Since the total space of the first fibration is formal [5] (p. 483), the same theorem yields the formality of the base.

Cases 4 and 5. These are evident, since one obtains a homeomorphism either to a product of Lie groups, or of symmetric spaces.

Case 6. Consider the fibration

$$SO(9)/SO(6)\times SO(3) \to E_6/SO(6)\times SO(3) \to E_6/SO(9).$$

Since $SO(9)/SO(6)\times SO(3)$ is of maximal rank, it is enough to show that $E_6/SO(9)$ is formal. To prove that, use Theorem 5 again. The degrees of invariants of $W(E_6)$ and $W(SO(9))$ are given in [16] (the exceptional type) and [3] (the classical type). Using the explicit expressions one obtains the Cartan algebra of $E_6/SO(9)$:

$$\begin{aligned}&(C,\nabla_\sigma) = (\mathbb{R}[u_1,u_2,u_3,u_4]\otimes \bigwedge(y_1,\ldots,y_6),\nabla_\sigma),\\ &\nabla_\sigma(u_i) = 0, \quad i=1,\ldots,4,\\ &\nabla_\sigma(y_j) = \widetilde{f}_j(u_1,u_2,u_3,u_4),\end{aligned}$$

and the degrees are the following:

$$\begin{gathered}|u_1| = 4, \quad |u_2| = 18, \quad |u_3| = 12, \quad |u_4| = 16,\\ |\widetilde{f}_1| = 4, \quad |\widetilde{f}_2| = 10, \quad |\widetilde{f}_3| = 12, \quad |\widetilde{f}_4| = 16, \quad |\widetilde{f}_5| = 18, \quad |\widetilde{f}_6| = 24,\end{gathered}$$

which implies $\widetilde{f}_1 = \alpha\cdot u_1$, $\widetilde{f}_2 = 0$, $\widetilde{f}_3 = v\cdot u_3$, $\widetilde{f}_5 = 0$ and Theorem 5 applies ($\widetilde{f}_1, \widetilde{f}_3, \widetilde{f}_4, \widetilde{f}_6$ constitute necessarily a regular sequence).

Case 7. The homogeneous space $E_6/SO(7) \times SO(3)$ can be considered as the base of the fibration

$$S^6 \to E_6/SO(6) \times SO(3) \to E_6/SO(7) \times SO(3)$$

and Theorem 8 applies.

Case 8. The appropriate fibration is

$$\mathrm{Sp}(4)/\mathrm{Sp}(3) \times T_1 \to E_6/\mathrm{Sp}(3) \times T_1 \to E_6/\mathrm{pSp}(4),$$

where pSp(4) denotes the projective symplectic group. The base of the fibration is the symmetric space $E_6/\,\mathrm{pSp}(4)$. To finish the proof it is enough to notice that by [4] and [8] any 3- or 4-symmetric space is the product of spaces appearing in Tables 1 and 2 and that the case $\operatorname{rank} G = \operatorname{rank} H$ follows from [5].

The proof is complete.

Remark. All generalized symmetric spaces of dimension ≤ 5 classified by O. Kowalski [9] are formal, but the reason for this is of different nature: by the Neisendorfer theorem [14] any manifold of dimension ≤ 6 is formal.

Acknowledgments. The third author was supported by Polish KBN research grant, 2P301/009/05.

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Reçu par la Rédaction le 11.4.1994
Révisé le 30.4.1995