ANNALES
POLONICI MATHEMATICI
LXIV.1 (1996)

Convergence results for unbounded solutions of first order non-linear differential-functional equations

by Henryk Leszczyński (Gdańsk)

Abstract. We consider the Cauchy problem in an unbounded region for equations of the type either $D_t z(t,x) = f(t,x,z(t,x),z_{(t,x)},D_x z(t,x))$ or $D_t z(t,x) = f(t,x,z(t,x), z, D_x z(t,x))$. We prove convergence of their difference analogues by means of recurrence inequalities in some wide classes of unbounded functions.

Introduction. Basic uniqueness results for first order differential equations were proved by Szarski [8], and then generalized by Kamont [3], Besala [1] and others. Let us mention [6] where the case of differential-functional equations was treated.

Uniqueness, existence and convergence results for parabolic equations require some assumptions on the class of solutions, namely one ought to assume that the solutions and their derivatives grow at most as $\exp\left(c\|x\|^2\right)$ (see [4]). The convergence of difference schemes was proved first locally, next in the unbounded case for differential problems [2], and finally for differential-functional systems using a special type of difference operators [7], and with general difference analogues consistent with the differential-functional problem [5].

We extend general methods of proving convergence by means of difference inequalities described in [8] and in the references mentioned there. Working in wide functional classes (see [6]), we prove recurrence estimates in a way similar to that used for parabolic equations. We deal simultaneously with two main types of functional dependence: first, with the variable $z_{(t,x)}$ as an extension of retardations and integrations over a rectangular bounded left-side neighbourhood of the point (t,x), and secondly, with z appearing as variable in a function of the Volterra type. These two quite general models of functional dependence coincide in classes of bounded solutions; however, if we investigate unbounded functions, then two slightly different sets of as-

1991 *Mathematics Subject Classification*: Primary 35A35; Secondary 35B35.
Key words and phrases: error estimates, recurrence inequalities, difference scheme.

sumptions imply uniqueness and convergence as shown in [6]. Finally, we remark that our results, formulated for one equation, can easily be proved for weakly coupled systems.

1. Basic notations and formulation of the first differential-functional problem. Let $E_0=[-\tau_0,0]\times\mathbb{R}^n$, $E=[0,a]\times\mathbb{R}^n$ and $D=[-\tau_0,0]\times[-\tau,\tau]$, where $\tau_0\in\mathbb{R}_+$, $a>0$ and $\tau=(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n_+$. If $z\in C(E_0\cup E,\mathbb{R})$ and $(t,x)\in E$, then $z_{(t,x)}:D\to\mathbb{R}$ is defined by $z_{(t,x)}(\overline{t},\overline{x})=z(t+\overline{t},x+\overline{x})$ for $(\overline{t},\overline{x})\in D$.

A function H is of class $\mathcal{H}$ iff $H\in C(E_0\cup E,(0,\infty))$, the functions $H_{|E_0}$ and $H_{|E}$ are continuously differentiable, and

(i) $H(\cdot,x)$ is non-decreasing for every $x\in\mathbb{R}^n$,

(ii) $x_iD_{x_i}H(t,x)\ge 0$ for $i=1,\ldots,n$ and $(t,x)\in E_0\cup E$, where $x=(x_1,\ldots,x_n)$.

If $H\in\mathcal{H}$, then a function ε is said to be of class $\mathcal{E}_H$ iff $\varepsilon\in C(E_0\cup E,(0,\infty))$, $\varepsilon(t,x)\le\varepsilon(\overline{t},\overline{x})$ for $\|x\|\ge\|\overline{x}\|$, $0\le t,\overline{t}\le a$, and the function $\widetilde{H}$ defined by $\widetilde{H}(t,x)=H(t,x)\varepsilon(t,x)$ for $(t,x)\in E_0\cup E$ belongs to $\mathcal{H}$.

A function z is of class $\mathcal{C}_H$ (resp. $\mathcal{C}_{H,\varepsilon}$) iff $z\in C(E_0\cup E,\mathbb{R})$ and $|z(t,x)|/H(t,x)$ (resp. $|z(t,x)|/\widetilde{H}(t,x)$, where $\widetilde{H}(t,x)=H(t,x)\varepsilon(t,x)$) is bounded.

The classes $\mathcal{C}_H$ and $\mathcal{C}_{H,\varepsilon}$ are equipped with the seminorms $\|\cdot\|_H(t)$ defined by

$$\|z\|_H(t)=\sup\{|z(\overline{t},\overline{x})|/H(\overline{t},\overline{x})\mid(\overline{t},\overline{x})\in E_0\cup E,\ \overline{t}\le t\}\tag{1.1}$$

for $(t,x)\in(0,a]$ and $z\in\mathcal{C}_H$. The space $C(D,\mathbb{R})$ is equipped with the maximum norm $\|\cdot\|_D$. Denote by $\Omega^{(0)}$ and $\Omega^{(1)}_H$ the sets

$$\Omega^{(0)}=E\times\mathbb{R}\times C(D,\mathbb{R})\times\mathbb{R}^n\quad\text{and}\quad\Omega^{(1)}_H=E\times\mathbb{R}\times\mathcal{C}_H\times\mathbb{R}^n.$$

If $\mathcal{L}\in C(E_0\cup E,\mathbb{R}_+)$ and $L_1,L_2\in\mathbb{R}_+$, then a function f is of class $\mathrm{Lip}(\Omega^{(0)};\mathcal{L},L_1,L_2)$ iff $f\in C(\Omega^{(0)},\mathbb{R})$ and

$$\begin{aligned}&|f(t,x,p,w,q)-f(t,x,\overline{p},\overline{w},\overline{q})|\\&\qquad\le L_1|p-\overline{p}|+\mathcal{L}(t,x)\|w-\overline{w}\|_D+L_2\|q-\overline{q}\|\end{aligned}\tag{1.2}$$

for all $(t,x,p,w,q),(t,x,\overline{p},\overline{w},\overline{q})\in\Omega^{(0)}$.

Now, let $f\in C(\Omega^{(0)},\mathbb{R})$. We consider the differential-functional equation

$$D_tz(t,x)=f(t,x,z(t,x),z_{(t,x)},D_xz(t,x)),\tag{1.3}$$

where $D_xz(t,x)=(D_{x_1}z(t,x),\ldots,D_{x_n}z(t,x))$, with the initial condition

$$z(t,x)=\phi(t,x),\qquad(t,x)\in E_0,\tag{1.4}$$

where $\phi\in C(E_0,\mathbb{R})$.

We assume throughout the paper that the Cauchy problem (1.3), (1.4) has a unique solution of class $\mathcal{C}_H$ defined on $E_0 \cup E$ (see [6]).

Let $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and $\kappa, \psi : [-\tau_0, a] \to (0, \infty)$ be continuously differentiable functions such that $\kappa'(t) \geq 0$ and $\psi'(t) \geq 0$ for $t \in [0, a]$; $\kappa'(t) = \psi'(t) = 0$ for $t \in [-\tau_0, 0]$; and $\Gamma'(t) \geq 0$ for $t \in \mathbb{R}_+$.

We define

$$H(t, x) = \Gamma(\psi(t)\sqrt{1 + \|x\|^2}) \tag{1.5}$$

for $(t, x) \in E_0 \cup E$, and

$$\mathcal{L}(t, x) = \overline{p}/\Gamma(\kappa(t)\sqrt{1 + \|x\|^2}) \tag{1.6}$$

for $(t, x) \in E$, where $\overline{p} \in \mathbb{R}_+$. Some conditions, assumed in [6], on the functions Γ, κ, ψ imply uniqueness for problem (1.3), (1.4) as well as for the Cauchy problem with another type of functional dependence. The properties of these functions assumed in the present paper are very close to those in [6].

Example. Let us list a few examples of functions Γ appearing in (1.5), (1.6):

(i) $\Gamma(t) = t^k$ for $t \in \mathbb{R}_+$ and $k > 0$.

(ii) $\Gamma(t) = \exp\big(t^m\big)$ for $t \in \mathbb{R}$ and $m \in \mathbb{N}$.

(iii) $\Gamma(t) = e_l(t)$ for $t \in \mathbb{R}_+$ and $l \in \mathbb{N}$, where $e_0(t) = t$ and $e_{l+1}(t) = \exp(e_l(t))$ for $l = 0, 1, \ldots$ and $t \in \mathbb{R}$.

(iv) $\Gamma(t) = \Gamma^{\mathrm{T}}(t) = \exp(\int_0^t \widetilde{\Gamma}(r+1)\,dr)$ for $t \in \mathbb{R}$, where $\widetilde{\Gamma}(r) = (r - n)e_{n+1}(n + 1) + (n + 1 - r)e_n(n)$ for $r \in [n, n + 1]$ and $n = 0, 1, \ldots$ This example shows that Γ can grow faster than all e_l for $l = 0, 1, \ldots$ It is possible to construct still faster growing functions:

(v) $\Gamma(t) = \Gamma^{\mathrm{T}}_{,i}(t)$ for $t \in \mathbb{R}_+$ and $i = 0, 1, \ldots$, where $\Gamma^{\mathrm{T}}_{,0}(t) = \Gamma^{\mathrm{T}}(t)$ and $\Gamma^{\mathrm{T}}_{,k+1}(t) = \exp(\Gamma^{\mathrm{T}}_{,k}(t))$ for $k = 0, 1, \ldots$; and next:

(vi) $\Gamma(t) = \Gamma^{\mathrm{TT}}(t) = \exp(\int_0^t \widetilde{\Gamma}(r+1)\,dr)$ for $t \in \mathbb{R}_+$, where $\widetilde{\Gamma}(r) = (r - l)\Gamma^{\mathrm{T}}_{,l+1}(l + 1) + (l + 1 - r)\Gamma^{\mathrm{T}}_{,l}(l)$ for $r \in [l, l + 1]$ and $l = 0, 1, \ldots$, and so on.

(vii) If $\xi \in C(\mathbb{R}, \mathbb{R})$ and

$$\omega(t) = \exp(-\max\{0, (1 - t)^{-1}\})\Big(\int_1^\infty e^{-s} s^{-3/2}(s - 1)^{-1/2}\,ds\Big)^{-1}$$

for $t \in \mathbb{R}$, then

$$\Gamma(t) = C_0 \exp\Big(\int_{-\infty}^{\infty} \max_{s \in [-r-1, r+1]} |\xi(s)|\omega(\sqrt{1 + t^2} - r)\,dr\Big), \qquad t \in \mathbb{R}_+,$$

where $C_0 = 1 + \max\{|\xi(r)| \mid r \in [-1, 1]\}$, is differentiable and satisfies $\Gamma(|t|) \geq |\xi(t)|$ for $t \in \mathbb{R}$.

3. Formulation of the difference problem and consistency lemmas. Let $\overline{h} = (\overline{h}_0, \overline{h}') \in (0,\infty)^{1+n}$, where $\overline{h}' = (\overline{h}_1, \ldots, \overline{h}_n)$, and I_d be a non-empty subset of

$$\left\{ h = (h_0, h') \in \mathbb{R}_+^{1+n} \;\middle|\; \begin{array}{l} h' = (h_1, \ldots, h_n);\ h_i \in (0, \overline{h}_i],\ i = 1, \ldots, n; \\ h_0 N_0 = \tau_0 \text{ for some } N_0 \in \{0, 1, \ldots\} \end{array} \right\}.$$

Let $(t,x)^{(\eta)} = (t^{(\eta)}, x^{(\eta)})$, where $t^{(\eta)} = h_0\eta_0$ and $x^{(\eta)} = (h_1\eta_1, \ldots, h_n\eta_n)$ for $\eta = (\eta_0, \eta') \in \mathbb{Z}^{1+n}$ and $\eta = (\eta_1, \ldots, \eta_n)$. For $h = (h_0, h') \in I_d$ there is a natural constant N_* and $N_1, \ldots, N_n \in \mathbb{Z}_+$ such that $N_* h_0 \le a < (N_* + 1)h_0$ and $\tau_i \le N_i h_i < \tau_i + h_i$ for $i = 1, \ldots, n$.

Let $h = (h_0, h') \in I_d$. Then we define

$$\begin{aligned} E_{0,h} &= \{(t,x)^{(\eta)} \mid \eta = (\eta_0, \eta') \in \mathbb{Z}^{1+n},\ \eta_0 \in \{-N_0, \ldots, 0\}\}, \\ E_h &= \{(t,x)^{(\eta)} \mid \eta = (\eta_0, \eta') \in \mathbb{Z}^{1+n},\ \eta_0 \in \{0, \ldots, N_* - 1\}\}, \\ \widehat{E}_h &= \{(t,x)^{(\eta)} \mid \eta = (\eta_0, \eta') \in \mathbb{Z}^{1+n},\ \eta_0 \in \{-N_0, \ldots, N_*\}\}, \\ D_h &= \left\{ (t,x)^{(\eta)} \;\middle|\; \begin{array}{l} \eta = (\eta_0, \eta') \in \mathbb{Z}^{1+n},\ \eta_0 \in \{-N_0, \ldots, 0\}; \\ \eta' = (\eta_1, \ldots, \eta_n);\ |\eta_i| \le N_i, i = 1, \ldots, n \end{array} \right\}. \end{aligned}$$

Let z_h be a function defined on $\widehat{E}_h$. Set $z_h^{(\eta)} = z_h(t^{(\eta)}, x^{(\eta)})$ for $(t,x)^{(\eta)} \in \widehat{E}_h$. If $(t,x)^{(\eta)} \in E_h$ and z_h is a function defined on $\widehat{E}_h$, then the function $(z_h)_{(\eta)} : D_h \to \mathbb{R}$ is defined by

$$(z_h)_{(\eta)}((t,x)^{(\overline{\eta})}) = z_h((t,x)^{(\eta+\overline{\eta})}) \quad \text{for } (t,x)^{(\overline{\eta})} \in D_h.$$

Denote by $\mathcal{F}(\widehat{E}_h, \mathbb{R})$ the set of all functions from $\widehat{E}_h$ to $\mathbb{R}$. If $z \in C(E_0 \cup E, \mathbb{R})$, then $z_h \in \mathcal{F}(\widehat{E}_h, \mathbb{R})$ denotes the restriction of z to the mesh.

Let $\lambda \in \{1, 2, \ldots\}$ and

$$S_\lambda = \{ s = (s_1, \ldots, s_n) \in \mathbb{Z}^n \mid |s_i| \le \lambda,\ i = 1, \ldots, n \}.$$

We define the difference operators A, Δ_0 and $\Delta = (\Delta_1, \ldots, \Delta_n)$ by

$$\begin{aligned} A z_h^{(\eta)} &= \sum_{s \in S_\lambda} a_s^{(\eta)} z_h^{(\eta_0, \eta' + s)}, && (t,x)^{(\eta)} \in E_h, \\ \Delta_0 z_h^{(\eta)} &= h_0^{-1}(z_h^{(\eta_0+1, \eta')} - A z_h^{(\eta)}), && (t,x)^{(\eta)} \in E_h, \\ \Delta_l z_h^{(\eta)} &= h_l^{-1} \sum_{s \in S_\lambda} b_{s,l}^{(\eta)} z_h^{(\eta_0, \eta' + s)}, && (t,x)^{(\eta)} \in E_h,\ l = 1, \ldots, n, \end{aligned} \tag{2.1}$$

where $a_s^{(\eta)}$ and $b_{s,l}^{(\eta)}$ are real coefficients.

Assumption H_1. Suppose that the discrete operators A and Δ defined by (2.1) satisfy the following conditions

(i) $\sum_{s \in S_\lambda} a_s^{(\eta)} = 1$, $\sum_{s \in S_\lambda} a_s^{(\eta)}\, s_l = 0$ for $l = 1, \ldots, n$ and $(t,x)^{(\eta)} \in E_h$.

(ii) $\sum_{s\in S_\lambda} b_{s,l}^{(\eta)} = 0$, $\sum_{s\in S_\lambda} b_{s,l}^{(\eta)}\ s_j = \delta_{lj}$ for $j,l = 1,\dots,n$ and $(t,x)^{(\eta)} \in E_h$.

(iii) there is $c \in \mathbb{R}_+$ such that $|a_s^{(\eta)}|$, $|b_{s,l}^{(\eta)}| \le c$ for $s \in S_\lambda$, $(t,x)^{(\eta)} \in E_h$, $l = 1,\dots,n$ and $h \in I_d$.

(iv) there are constants $c_0, c_1 \in (0,\infty)$ such that $c_0 h_0 \le h_l \le c_1 h_0$ for $l = 1,\dots,n$.

This assumption is necessary to prove that the discrete operators Δ_l for $l = 0,1,\dots,n$ approximate the differential operators D_t and D_{x_l} for $l = 1,\dots,n$.

The operator $[\cdot]_h : \mathcal{F}(\widehat{E}_h,\mathbb{R}) \to C(E_0 \cup E,\mathbb{R})$ is defined by

$$[z_h]_h(\overline{t},\overline{x}) = \sum_{r=-\lambda}^{0} \sum_{s\in S_\lambda} p_{r,s}^{(\eta)}(\overline{t},\overline{x}) z_h^{(\eta_0+r,\eta'+s)} \tag{2.2}$$

for $z_h : \widehat{E}_h \to \mathbb{R}$, $t^{(\eta-\mathbf{1})} < \overline{t} \le t^{(\eta)}$ and $x^{(\eta-\mathbf{1})} < \overline{x} \le x^{(\eta)}$, where $(t,x)^{(\eta-\mathbf{1})}, (t,x)^{(\eta)} \in \widehat{E}_h$ and $\mathbf{1} = (1,\dots,1) \in \mathbb{Z}^{1+n}$. If $(t,x)^{(\eta_0+r,\eta'+s)} \notin \widehat{E}_h$, then $z_h^{(\eta_0+r,\eta'+s)}$ means the same as $z_h^{(-N_0,\eta'+s)}$.

Assumption H_2. Suppose that the functions $p_{r,s}^{(\eta)} : \mathbb{R}^{1+n} \to \mathbb{R}$ in formula (2.2) are bounded and satisfy

$$\sum_{r=-\lambda}^{0} \sum_{s\in S_\lambda} p_{r,s}^{(\eta)}(\overline{t},\overline{x}) = 1 \tag{2.3}$$

for $t^{(\eta-\mathbf{1})} < \overline{t} \le t^{(\eta)}$ and $x^{(\eta-\mathbf{1})} < \overline{x} \le x^{(\eta)}$, and

$$\widehat{p} = \sup \left\{ \sum_{r=-\lambda}^{0} \sum_{s\in S_\lambda} |p_{r,s}^{(\eta)}(\overline{t},\overline{x})| \;\middle|\; \begin{array}{c} t^{(\eta-\mathbf{1})} < \overline{t} \le t^{(\eta)}, \\ x^{(\eta-\mathbf{1})} < \overline{x} \le x^{(\eta)},\ h \in I_d, \\ (t,x)^{(\eta-\mathbf{1})}, (t,x)^{(\eta)} \in \widehat{E}_h. \end{array} \right\} < \infty \tag{2.4}$$

Now, we define difference schemes which correspond to the differential-functional problem (1.3), (1.4). Let $f \in \mathcal{C}_H(\Omega^{(0)})$. Then we consider the following difference-functional problem (cf. (1.3)):

$$\Delta_0 z_h^{(\eta)} = f((t,x)^{(\eta)}, z_h^{(\eta)}, ([z_h]_h)_{(\eta)}, \Delta z_h^{(\eta)}), \qquad (t,x) \in E_h, \tag{2.5}$$

with the initial condition

$$z_h^{(\eta)} = \overline{\phi}_h^{(\eta)}, \qquad (t,x) \in E_{0,h}. \tag{2.6}$$

In the literature the function f is often replaced by a function f_h which is defined by use of a finite Taylor expansion of f. This way one can obtain better difference approximations to the differential problem.

Let $H \in \mathcal{H}$ and $\varepsilon \in \mathcal{E}_H$. A function z_h is of class $\mathcal{F}_H^{(h)}$ (resp. of class $\mathcal{F}_{H,\varepsilon}^{(h)}$) iff there is a function $\overline{z} \in \mathcal{C}_H$ (resp. $\overline{z} \in \mathcal{C}_{H,\varepsilon}$) such that $\overline{z}_h = z_h$, where $h \in I_d$.

Now, we prove an auxiliary lemma on consistency.

LEMMA 2.1. *Suppose that* $H \in \mathcal{H}$ *and Assumptions* H_1 *and* H_2 *are satisfied. Let* $f \in \mathcal{C}_H(\Omega^{(0)})$ *with constants* $L_0, L_1, L_2 \in \mathbb{R}_+$ *and* $\mathcal{L} \in C(E, \mathbb{R}_+)$. *Let* $u \in C(E_0 \cup E, \mathbb{R})$ *be a solution of* (1.3), (1.4) *such that* $u, D_t u, D_{x_l} u \in \mathcal{C}_H$ *for* $l = 1, \ldots, n$, *and there is* $L_u \in \mathbb{R}_+$ *such that*

$$(2.7)\qquad |D_{x_l}u(t,x) - D_{x_l}u(\overline{t},\overline{x})| + |D_t u(t,x) - D_t u(\overline{t},\overline{x})| \le L_u\Big(|t-\overline{t}| + \sum_{j=1}^{n} |x_j - \overline{x}_j|\Big)\max\{H(t,x), H(\overline{t},\overline{x})\}$$

for $(t,x), (\overline{t},\overline{x}) \in E$ *and* $l = 1, \ldots, n$. *Then*

$$(2.8)\qquad |\Delta_0 u_h^{(\eta)} - f((t,x)^{(\eta)}, u_h^{(\eta)}, ([u_h]_h)_{(\eta)}, \Delta z_h^{(\eta)})| \le \mu_h^{(\eta)},$$

where

$$(2.9)\qquad \mu_h^{(\eta)} = h_0 L_u H_h^{(\eta_0+1,\eta')} + h_0^{-1}\widehat{a}\lambda^2\Big(\sum_{l=1}^{n} h_l\Big)^2 L_u H_h^{(\eta_0,|\eta'|+\lambda')} + \mathcal{L}((t,x)^{(\eta)})H_h^{(\eta_0,|\eta'|+\lambda'+1')}\widehat{p} \times \Big(h_0\lambda\|D_t u\|_H(a) + (\lambda+1)\sum_{l=1}^{n} h_l\|D_{x_l}u\|_H(a)\Big) + L_2\|(h_1^{-1}, \ldots, h_n^{-1})\|\widehat{b}\Big(\sum_{l=1}^{n} h_l\Big)^2 \lambda^2 L_u H_h^{(\eta_0,|\eta'|+\lambda')},$$

where $\lambda' = (\lambda, \ldots, \lambda) \in \mathbb{Z}^n$, $1' = (1, \ldots, 1) \in \mathbb{Z}^n$, *and*

$$(2.10)\qquad \widehat{a} = \sup_{\eta} \sum_{s \in S_\lambda \setminus \{0'\}} |a_s^{(\eta)}|, \qquad \widehat{b} = \sup_{l,\eta} \sum_{s \in S_\lambda \setminus \{0'\}} |b_{s,l}^{(\eta)}|.$$

Proof. First, using the mean value theorem we have

$$(2.11)\qquad |D_t u((t,x)^{(\eta)}) - \Delta_0 u_h^{(\eta)}| = \Big|D_t u((t,x)^{(\eta)}) - h_0^{-1}\Big\{u_h^{(\eta)} + h_0 D_t u((t,x)^{(\eta)} + \theta_0^{(\eta)}(h)) - \sum_{s \in S_\lambda} a_s^{(\eta)}\Big(u_h^{(\eta)} + \sum_{l=1}^{n} h_l s_l D_{x_l} u((t,x)^{(\eta)} + \theta_s^{(\eta)}(h))\Big)\Big\}\Big|$$

for $(t,x)^{(\eta)} \in E_h$, where

$$(2.12)\qquad \begin{aligned} &\theta_0^{(\eta)}(h) \in (0, h_0) \times \{0'\}, \\ &\theta_s^{(\eta)}(h) \in \{0\} \times [-\lambda h_1, \lambda h_1] \times \ldots \times [-\lambda h_n, \lambda h_n] \end{aligned}$$

for $s \in S_\lambda$.

From (2.7), (2.11) and Assumption H_1 we obtain

$$|D_t u((t,x)^{(\eta)}) - \Delta_0 u_h^{(\eta)}| \le L_u h_0 H_h^{(\eta_0+1,\eta')} + h_0^{-1}\widehat{a}\lambda^2\Big(\sum_{l=1}^{n} h_l\Big)^2 L_u H_h^{(\eta_0,|\eta'|+\lambda')}, \tag{2.13}$$

where $\widehat{a}$ is defined by (2.10). In a similar way we obtain

$$|D_{x_l} u((t,x)^{(\eta)}) - \Delta_l u_h^{(\eta)}| = \Big|D_{x_l} u((t,x)^{(\eta)}) - h_l^{-1}\sum_{s\in S_\lambda} b_{s,l}^{(\eta)}\Big(u_h^{(\eta)} + \sum_{l=1}^{n} h_l s_l D_{x_l} u((t,x)^{(\eta)} + \theta_s^{(\eta)}(h))\Big)\Big| \tag{2.14}$$

for $(t,x)^{(\eta)} \in E_h$ and $l = 1,\dots,n$, and thus

$$|D_{x_l} u((t,x)^{(\eta)}) - \Delta_l u_h^{(\eta)}| \le h_l^{-1}\widehat{b}\Big(\sum_{l=1}^{n} h_l\Big)^2 \lambda^2 L_u H_h^{(\eta_0,|\eta'|+\lambda')} \tag{2.15}$$

for $(t,x)^{(\eta)} \in E_h$ and $l = 1,\dots,n$. If $(\overline{t},\overline{x}) \in E_0 \cup E$ and $t^{(\eta-\mathbf{1})} < \overline{t} \le t^{(\eta)}$, $x^{(\eta-\mathbf{1})} < \overline{x} \le x^{(\eta)}$, where $(t,x)^{(\eta-\mathbf{1})}, (t,x)^{(\eta)} \in \widehat{E}_h$, then

$$\begin{aligned} |[u_h]_h(\overline{t},\overline{x}) - u(\overline{t},\overline{x})| &= \Big|\sum_{r=-\lambda}^{0}\sum_{s\in S_\lambda} p_{r,s}^{(\eta)}(\overline{t},\overline{x})\Big(u(\overline{t},\overline{x}) + (t^{(\eta_0+r,\eta'+s)} - \overline{t})D_t u(\theta_{r,s}^{(\eta)}(\overline{t},\overline{x},h)) \\ &\quad + \sum_{l=1}^{n}(x^{(\eta_0+r,\eta'+s)} - \overline{x}_l)D_{x_l} u(\theta_{r,s}^{(\eta)}(\overline{t},\overline{x},h))\Big) - u(\overline{t},\overline{x})\Big| \\ &\le \widehat{p}\Big(\|D_t u\|_H(a)\,\lambda h_0 + \sum_{l=1}^{n} h_l(\lambda+1)\|D_{x_l} u\|_H(a)\Big) H_h^{(\eta_0,|\eta'|+\lambda')}, \end{aligned} \tag{2.16}$$

where $\widehat{p}$ is defined by (2.4) and $\theta_{r,s}^{(\eta)}(\overline{t},\overline{x},h)$ is an intermediate point between $(t,x)^{(\eta_0+r,\eta'+s)}$ and $(\overline{t},\overline{x})$.

Let $(t,x) \in E$. From (2.16) we have

$$\begin{aligned} \|([u_h]_h)_{(t,x)} - u_{(t,x)}\|_D &= \max_{(\overline{t},\overline{x})\in D}\|[u_h]_h(t+\overline{t},x+\overline{x}) - u(t+\overline{t},x+\overline{x})\| \\ &\le \widehat{p}\Big(\|D_t u\|_H(a)\lambda h_0 + \sum_{l=1}^{n} h_l(\lambda+1)\|D_{x_l} u\|_H(a)\Big) H_h^{(\widetilde{\eta}_0,|\widetilde{\eta}'|+\lambda')}, \end{aligned} \tag{2.17}$$

where $(\widetilde{\eta}_l - 1)h_l < x_l \le \widetilde{\eta}_l h_l$ for $l = 0,\dots,n$. Condition (2.8) follows from (2.13), (2.15) and (2.17). This finishes the proof.

3. Convergence theorem. In this section we will prove that natural assumptions imply the convergence of the difference scheme (2.5), (2.6).

THEOREM 3.1. *Suppose that*

1) $H \in \mathcal{H}$ *is given by* (1.5), *and* $\mathcal{L}$ *is defined by* (1.6), *where* $\kappa(t) > \psi(t) > 0$ *for* $t \in [-\tau_0, a]$, ψ, κ *are increasing on* $[0, a]$, *and* $f \in \mathcal{C}_H(\Omega^{(0)})$ *with* $L_1, L_2 \in \mathbb{R}_+$,

2) $u \in C(E_0 \cup E, \mathbb{R})$ *is a solution of* (1.3), (1.4) *such that* $u, D_t u, D_{x_l} u \in \mathcal{C}_H$ *for* $l = 1, \ldots, n$, *and there is* $L_u \in \mathbb{R}_+$ *such that condition* (2.7) *is satisfied*,

3) $v_h \in \mathcal{F}(\widehat{E}_h, \mathbb{R})$ *is a solution of* (2.5), (2.6),

4) *the following monotonicity condition is satisfied*:

$$a_s^{(\eta)} + h_0 \sum_{l=1}^{n} h_l^{-1} b_{s,l}^{(\eta)} D_{q_l} f(P^{(\eta)}) \geq 0 \tag{3.1}$$

for $s \in S_\lambda, (t,x)^{(\eta)} \in E_h$ *and* $P^{(\eta)} = ((t,x)^{(\eta)}, p, w, q) \in \Omega_H^{(0)}$,

5) *there is* $M_\phi \in \mathbb{R}_+$ *such that*

$$|\phi_h^{(\eta)} - \overline{\phi}_h^{(\eta)}| \leq h_0 M_\phi H_h^{(\eta)}, \qquad (t,x)^{(\eta)} \in E_{0,h}. \tag{3.2}$$

Then

$$|v_h^{(\eta)} - u_h^{(\eta)}| \leq h_0 \Psi(t^{(\eta)}) H_h^{(\eta)}, \qquad (t,x)^{(\eta)} \in \widehat{E}_h, \tag{3.3}$$

where $\Psi : [-\tau_0, a] \to \mathbb{R}$ *satisfies*

$$\Psi(t) \geq M_\phi, \qquad t \in [-\tau_0, 0], \tag{3.4}$$

$$\begin{aligned} \Psi'(t) - L_u - B_0(h) - L_1 \Psi(t) & \\ - B_1(h, \psi(t)) \Gamma(\psi(t) & \sqrt{1 + (r_1(t) + \|\tau\| + \|h'\|(\lambda+1))^2}) \\ & \times (\Gamma(\Psi(t)\widetilde{\theta}) \Gamma(\kappa(t)\widetilde{\theta}))^{-1} \geq 0 \end{aligned} \tag{3.5}$$

for $t \in [0, a]$ *and* $h \in I_d$, *where* $\widetilde{\theta} = \sqrt{1 + (r_0(t,h))^2}$, *and* B_0 *and* B_1 *are defined by*

$$B_0(h) = \lambda^2 L_u \Big(h_0^{-1} \sum_{l=1}^{n} h_l \Big)^2 (\widehat{a} + \widehat{b} L_2 h_0 \|(h_1^{-1}, \ldots, h_n^{-1})\|), \tag{3.6}$$

$$B_1(h, p) = \overline{p}\widehat{p} \Big(p + \lambda \|D_t u\|_H(a) + (\lambda+1) \sum_{l=1}^{n} h_l h_0^{-1} \|D_{x_l} u\|_H(a) \Big) \tag{3.7}$$

for $p \in \mathbb{R}_+$, *and*

$$
\begin{aligned}
r_0(t,h) &= \frac{1+\sqrt{1+(\lambda\|h'\|/h_0)^2}}{\psi'(t)/\psi(t)}\lambda\|h'\|/h_0, \\
r_1(t) &= \frac{\psi(t)(\|\tau\|+\|h'\|(\lambda+1))}{\kappa(t)-\psi(t)}.
\end{aligned} \tag{3.8}
$$

Moreover, we have

$$
\begin{aligned}
&\Psi'(t) - L_1 - L_u + (\Psi(t) - h_0 L_u)\psi'(t)\Gamma'(\psi(t))/\Gamma(\psi(t)) \\
&\quad - B_0(h)\Gamma(\psi(t)\sqrt{1+(\|h'\|\lambda/h_0 + r_0(t,h))^2})/\Gamma(t) \\
&\quad - B_1(h,\Psi(t))\Gamma\Big(\psi(t)\sqrt{1+(r_0(t,h)+\|\tau\|+\|\overline{h}'\|(1+\lambda))^2}\Big) \\
&\quad \times (\Gamma(\kappa(t))\Gamma(\psi(t)))^{-1} \ge 0
\end{aligned} \tag{3.9}
$$

for $t \in [0,a]$ with $\Psi(t)$ so large that $\Psi(t) - \overline{h}_0 L_u \ge 0$.

Proof. Let $w_h^{(\eta)} = u_h^{(\eta)} - v_h^{(\eta)}$ for $(t,x)^{(\eta)} \in \widehat{E}_h$. From (2.5) we obtain the recurrence equality

$$
\begin{aligned}
w_h^{(\eta_0+1,\eta')} &= Aw_h^{(\eta)} \\
&\quad + h_0(f((t,x)^{(\eta)}, u_h^{(\eta)}, ([u_h]_h)_{(\eta)}, \Delta u_h^{(\eta)}) \\
&\quad - f((t,x)^{(\eta)}, u_h^{(\eta)}, ([u_h]_h)_{(\eta)}, \Delta v_h^{(\eta)})) \\
&\quad + h_0(f((t,x)^{(\eta)}, u_h^{(\eta)}, ([u_h]_h)_{(\eta)}, \Delta v_h^{(\eta)}) \\
&\quad - f((t,x)^{(\eta)}, v_h^{(\eta)}, ([v_h]_h)_{(\eta)}, \Delta v_h^{(\eta)})) \\
&\quad + h_0(\Delta_0 u_h^{(\eta)} - f((t,x)^{(\eta)}, u_h^{(\eta)}, ([u_h]_h)_{(\eta)}, \Delta u_h^{(\eta)}))
\end{aligned} \tag{3.10}
$$

for $(t,x)^{(\eta)} \in E_h$. Using the mean value theorem, the Lipschitz condition for f, and Lemma 2.1, we obtain the recurrence estimate

$$
\begin{aligned}
|w_h^{(\eta_0+1,\eta')}| &\le \sum_{s\in S_\lambda} |w_h^{(\eta_0,\eta'+s)}| \Big| a_s^{(\eta)} + h_0 \sum_{l=1}^n h_l^{-1} b_{s,l}^{(\eta)} D_{q_l} f(P^{(\eta)}) \Big| \\
&\quad + h_0 L_1 |w_h^{(\eta)}| + h_0 \mathcal{L}((t,x)^{(\eta)}) \, \|([w_h]_h)_{(\eta)}\|_D + h_0 \mu_h^{(\eta)}
\end{aligned} \tag{3.11}
$$

for $(t,x)^{(\eta)} \in E_h$. Now, using (3.1) and Assumption H_1, we easily obtain from (3.11) the following inequality which is much easier to analyse:

$$
\begin{aligned}
|w_h^{(\eta_0+1,\eta')}| &\le \max_{s\in S_\lambda} |w_h^{(\eta_0,\eta'+s)}| + h_0 L_1 |w_h^{(\eta)}| \\
&\quad + h_0 \mathcal{L}((t,x)^{(\eta)}) \|([w_h]_h)_{(\eta)}\|_D + h_0 \mu_h^{(\eta)}
\end{aligned} \tag{3.12}
$$

for $(t,x)^{(\eta)} \in E_h$. If we prove that the function $\mathcal{W}_h^{(\eta)} = h_0 \Psi(t^{(\eta)}) H_h^{(\eta)}$, $(t,x)^{(\eta)} \in \widehat{E}_h$, satisfies a comparison inequality with respect to (3.12), then

(3.3) will be established. Thus, in order to finish the proof of our theorem it is enough to prove the following

LEMMA 3.1. *If the assumptions of Theorem* 3.1 *are satisfied*, *then*

$$\mathcal{W}_h^{(\eta_0+1,\eta')} \geq \mathcal{W}_h^{(\eta_0,|\eta'|+\lambda')} + h_0 L_1 \mathcal{W}_h^{(\eta)} + h_0 \mathcal{L}((t,x)^{(\eta)}) \|([\mathcal{W}_h]_h)_{(\eta)}\|_D + h_0 \mu_h^{(\eta)} \tag{3.13}$$

for $(t,x)^{(\eta)} \in E_h$, *and*

$$\mathcal{W}_h^{(\eta)} \geq h_0 M_\phi H_h^{(\eta)}, \quad (t,x)^{(\eta)} \in E_{0,h}. \tag{3.14}$$

Proof. Condition (3.14) follows immediately from (3.4) and (3.2). Condition (3.13) is a consequence of

$$\begin{aligned}(\Psi(t+h_0) - h_0 L_u)\Gamma(\psi(t+h_0)\sqrt{1+r^2}) &\geq (\Psi(t) + h_0 B_0(h))\Gamma(\psi(t)\sqrt{1+(r+\|h'\|\lambda)^2}) \\ &\quad + h_0 B_1(h,\Psi(t))\Gamma(\psi(t)\sqrt{1+(r+\|\tau\|+\|h'\|(\lambda+1))^2}) \\ &\quad \times (\Gamma(\psi(t)\sqrt{1+r^2}))^{-1}\end{aligned} \tag{3.15}$$

for $t \in [0, a-h_0]$ and $r = \|x\| \in \mathbb{R}_+$, where $B_0(h)$ and $B_1(h,\Psi(t))$ are given by (3.6) and (3.7). This implication follows from (3.9).

If r is greater than $r_0(t,h)$ given by (3.8), then

$$\psi(t+h_0)\sqrt{1+r^2} \geq \psi(t)\sqrt{1+(r+\|h'\|\lambda)^2}, \tag{3.16}$$

and (3.15) follows from

$$\begin{aligned}\Xi(\theta) := \Psi(t+\theta) - \theta L_u - \Psi(t) - \theta B_0(h) - \theta L_1 \Psi(t) \\ - \theta B_1(h,\Psi(t))\Gamma(\psi(t)\sqrt{1+(r_1(t)+\|\tau\|+\|h'\|(\lambda+1))^2}) \\ \times (\Gamma(\psi(t)\widetilde{\theta})\Gamma(\kappa(t)\widetilde{\theta}))^{-1} \geq 0,\end{aligned} \tag{3.17}$$

where $\theta \in [0,h_0]$, $t \in [0, a-h_0]$ and $\widetilde{\theta}$ is the same as in (3.5). Now, (3.17) holds true because $\Xi(0) = 0$ and $\Xi'(\theta) \geq 0$ for $\theta \in [0,h_0]$ as we have (3.5).

For $r \leq r_0(x_0,h)$ and $t \in [0, a-h_0]$, formula (3.15) is a consequence of the inequality $\Xi_1(\theta;t,r) \geq 0$ for $\theta \in [0,h_0]$, where

$$\begin{aligned}\Xi_1(\theta;t,r) \\ := (\Psi(t+\theta) - \theta L_u)\Gamma(\psi(t+\theta)\sqrt{1+r^2}) - \theta L_1 \Gamma(\psi(t)\sqrt{1+r^2}) \\ - (\Psi(t) + \theta B_0(h))\Gamma(\psi(t)\sqrt{1+(r+\theta\lambda\|h'\|/h_0)^2}) \\ - \theta B_1(h,\Psi(t))\Gamma\Big(\psi(t)\sqrt{1+(r_0(t,h)+\|\tau\|+\|\overline{h}'\|(\lambda+1))^2}\Big)/\Gamma(\kappa(t))\end{aligned} \tag{3.18}$$

for $\theta \in [0, h_0]$ and $t \in [0, a - h_0]$. From (3.18) we find a lower estimate of $\Xi_1'(\theta; t, h)$:

$$\begin{aligned}(3.19)\quad &\Xi_1'(\theta; t, h)\\ &\geq (\Psi'(t+\theta) - L_u)\Gamma(\psi(t+\theta)\sqrt{1+r^2})\\ &\quad + (\Psi(t+\theta) - \theta L_u)\Gamma'(\psi(t+\theta)\sqrt{1+r^2})\psi'(t+\theta)\sqrt{1+r^2}\\ &\quad - L_1\Gamma(\psi(t)\sqrt{1+r^2}) - B_0(h)\Gamma(\psi(t)\sqrt{1+(r+\|h'\|\lambda)^2})\\ &\quad - (\Psi(t) + \theta B_0(h))\Gamma'(\psi(t)\sqrt{1+(r+\lambda\|H'\|)^2})\psi'(t)\\ &\quad - B_1(h, \Psi(t))\Gamma\Big(\psi(t)\sqrt{1+(r_0(t,h) + \|\tau\| + \overline{h}'\|(\lambda+1))^2}\Big)/\Gamma(\kappa(t)).\end{aligned}$$

From (3.19) and (3.9) we obtain $\Xi'(\theta; t, h) \geq 0$, and (3.18) implies $\Xi(0; t, h) = 0$. Therefore, $\Xi(h_0; t, h) \geq 0$ for $t \in [0, a - h_0]$. This completes the proof.

Remark. Condition 1) of Theorem 3.1 can be much weaker, namely $\mathcal{L}$ might be defined by

$$\mathcal{L}(t, x) = \Gamma(\psi(t)\sqrt{1+\|x\|^2})/\Gamma(\kappa(t)\sqrt{1+\|x\|^2}), \qquad (t, x) \in E.$$

In this case the function Ψ satisfies stronger conditions than (3.4), (3.5) and (3.9).

If $\Gamma(t) \geq \text{const}\, e_2(t)$, then $\mathcal{L}$ might be defined by

$$\mathcal{L}(t, x) = (\Gamma(\psi(t)\sqrt{1+\|x\|^2}))^\nu/\Gamma(\kappa(t)\sqrt{1+\|x\|^2}), \qquad (t, x) \in E,$$

where $\nu \geq 0$, and the proof works for a sufficiently large function Ψ.

4. Convergence result for another functional dependence. If $L, L_1, L_2 \in \mathbb{R}_+$, then f is of class $\mathrm{Lip}(\Omega_H^{(1)}; L, L_1, L_2)$ iff $f \in C(\Omega_H^{(1)}, \mathbb{R})$ and

$$\begin{aligned}(4.1)\quad &|f(t, x, p, w, q) - f(t, x, \overline{p}, \overline{w}, \overline{q})|\\ &\qquad\leq L_1|p - \overline{p}| + LH(t, x)\|w - \overline{w}\|_H(t) + L_2\|q - \overline{q}\|\end{aligned}$$

for all $(t, x, p, w, q), (t, x, \overline{p}, \overline{w}, \overline{q}) \in \Omega_H^{(1)}$. For $f \in C(\Omega_H^{(1)}, \mathbb{R})$ we consider the Cauchy problem for the equation

$$(4.2)\qquad D_t z(t, x) = f(t, x, z(t, x), z, D_x z(t, x)).$$

We assume that the Cauchy problem (4.2), (1.4) has a unique solution of class $\mathcal{C}_H$ defined on $E_0 \cup E$.

Let $f \in \mathcal{C}_H(\Omega_H^{(1)})$. Then the difference analogue of (4.2) reads

$$(4.3)\qquad \Delta_0 z_h^{(\eta)} = f((t, x)^{(\eta)}, z_h^{(\eta)}, [z_h]_h, \Delta z_h^{(\eta)}), \qquad (t, x)^{(\eta)} \in E_h.$$

Problem (4.3) is also considered with initial condition (2.6).

LEMMA 4.1. *Suppose that Assumptions H_1 and H_2 are satisfied and $H \in \mathcal{H}$, $\varepsilon \in \mathcal{E}_H$. Let $f \in \mathcal{C}_H(\Omega_H^{(1)})$ with constants $L, L_0, L_1, L_2 \in \mathbb{R}_+$. Let $u \in C(E_0 \cup E, \mathbb{R})$ be a solution of* (4.2), (1.4) *such that $u, D_t u, D_{x_l} u \in \mathcal{C}_{H,\varepsilon}$ for $l = 1, \ldots, n$, and there is $L_u \in \mathbb{R}_+$ such that*

$$
(4.4)\qquad |D_{x_l}(t,x) - D_{x_l}u(\overline{t},\overline{x})| + |D_t u(t,x) - D_t u(\overline{t},\overline{x})| \\
\le L_u\Big(|t-\overline{t}| + \sum_{j=1}^{n} |x_j - \overline{x}_j|\Big) \max\{H(t,x)\varepsilon(t,x),\ H(\overline{t},\overline{x})\varepsilon(\overline{t},\overline{x})\}
$$

for $(t,x), (\overline{t},\overline{x}) \in E$ and $l = 1, \ldots, n$. Assume also that for $h \in I_d$ and $s \in S_{\lambda+1}$ there is $R(h) \in \mathbb{R}_+$ such that $\limsup\{R(h) \mid h \in I_d\} < \infty$, and

$$
(4.5)\qquad \sum_{l=1}^{n} x_l D_{x_l}(\widetilde{H}(t+h_0, x + (x^{(s)}))/H(t,x)) \le 0
$$

for $\|x\| \ge R(h)$ and $(t+h_0, x) \in E$, where

$$
(4.6)\qquad \widetilde{H}(t,x) = H(t,x)\varepsilon(t,x), \qquad (t,x) \in E.
$$

Then

$$
(4.7)\qquad |\Delta_0 u_h^{(\eta)} - f((t,x)^{(\eta)}, u_h^{(\eta)}, [u_h]_h, \Delta u_h^{(\eta)})| \le \mu_h^{(\eta)},
$$

where

$$
(4.8)\qquad \mu_h^{(\eta)} = h_0 L_u \widetilde{H}_h^{(\eta_0+1,\eta')} + h_0^{-1}\widehat{a}\lambda^2\Big(\sum_{l=1}^{n} h_l\Big)^2 L_u \widetilde{H}_h^{(\eta_0,|\eta'|+\lambda')} \\
+ L\widetilde{H}_h^{(\eta)}\widehat{p}\Big(h_0\lambda\|D_t u\|_H(a) + (\lambda+1)\sum_{l=1}^{n} h_l \|D_{x_l}u\|_H(a)\Big) \\
\times \sup_{\overline{\eta}} \sup_{x^{(\overline{\eta}-1)} < \overline{x} \le x^{(\overline{\eta})}} \frac{\widetilde{H}_h^{(\overline{\eta}_0,|\overline{\eta}'|+\lambda')}}{H(\overline{t},\overline{x})} \\
+ L_2\|(h_1^{-1}, \ldots, h_n^{-1})\|\widehat{b}\Big(\sum_{l=1}^{n} h_l\Big)^2 \lambda^2 L_u \widetilde{H}_h^{(\eta_0,|\eta'|+\lambda')},
$$

where $(t,x)^{(\eta)} \in E_h$, and $\widetilde{H}$ is defined by (4.6).

Remark. Condition (4.5) is satisfied if, for example, H is defined by (1.5), and

$$
(4.9)\qquad \varepsilon(t,x) = \Gamma(\xi(t)\sqrt{1+\|x\|^2})/\Gamma(\psi(t)\sqrt{1+\|x\|^2}),
$$

where $\xi \in C^1([-\tau_0, a], \mathbb{R}_+)$ is increasing, and $0 < \xi(t) < \psi(t)$ for $t \in [-\tau_0, a]$. We should only assume that $\overline{h}_0 < (\psi(t) - \xi(t))/\xi'(a)$ for $t \in [-\tau_0, a]$.

Proof of Lemma 4.1. Estimates (2.13)–(2.16) remain true if we replace H by $\widetilde{H}$ defined by (4.6). Thus, in order to obtain (4.7) with μ_h defined by (4.8), it is sufficient to estimate $LH_h^{(\eta)}\|[u_h]_h - u\|_H(a)$ by a suitably chosen expression basing on formula (2.16) with H replaced by $\widetilde{H}$. The double supremum appearing in (4.8) is finite because (4.5) guarantees that $\widetilde{H}_h^{(\overline{\eta}_0,|\overline{\eta}'|+\lambda')}/H(\overline{t},\overline{x})$ is bounded by

$$M_h = \sup\left\{ \widetilde{H}(t+h_0, x+(x^{(s)}))/H(t,x) \;\middle|\; \begin{array}{l} \|x\| \le R(h), \\ (t+h_0,x) \in E, \\ s \in S_{\lambda+1} \end{array} \right\}, \tag{4.10}$$

where $h \in I_d$. The rest works as in the proof of Lemma 2.1.

Theorem 4.1. *Suppose that*

1) *Assumptions* H_1 *and* H_2 *are satisfied*,

2) $H \in \mathcal{H}$ *and* $\varepsilon \in \mathcal{E}_H$ *are given by* (1.5) *and* (4.9), *respectively*, *where* $\xi \in C([-\tau_0,a],\mathbb{R}_+)$ *has positive derivative on* $[0,a]$; $0 < \xi(t) < \psi(t)$ *for* $t \in [-\tau_0,a]$; $\widetilde{H} \in \mathcal{H}$ *is defined by* (4.6), *and* $f \in \mathcal{C}_H(\Omega_H^{(1)})$ *with constants* L, L_1, L_2,

3) $u \in C(E_0 \cup E, \mathbb{R})$ *is a solution of* (4.2), (1.4) *such that* $u, D_t u, D_{x_l} u \in \mathcal{C}_{H,\varepsilon}$ *for* $l = 1,\ldots,n$, *and there exists* $L_u \in \mathbb{R}_+$ *such that condition* (4.4) *is satisfied*, *and the constants* M_h *defined by* (4.10) *for* $h \in I_d$ *are such that* $\limsup\{M_h \mid h \in I_d\} \le M$,

4) $v_h \in \mathcal{F}_{H,\varepsilon}^{(h)}$ *is a solution of* (4.3), (2.6),

5) *the monotonicity condition* (3.1) *is satisfied for* $s \in S_\lambda$, $(t,x)^{(\eta)} \in E_h$ *and* $P^{(\eta)} = ((t,x)^{(\eta)}, p, z, q) \in \Omega_H^{(1)}$,

6) *there is* $M_\phi \in \mathbb{R}$ *such that*

$$|\phi_h^{(\eta)} - \overline{\phi}_h^{(\eta)}| \le h_0 M_\phi \widetilde{H}_h^{(\eta)}, \qquad (t,x)^{(\eta)} \in E_h, \tag{4.11}$$

7) $\overline{h}_0$ *is so small that* $\psi(t) - \xi(t) - \overline{h}_0\xi'(t) > 0$ *for* $t \in [0,a]$.

Then

$$|v_h^{(\eta)} - u_h^{(\eta)}| \le h_0 \Psi(t^{(\eta)}) \widetilde{H}_h^{(\eta)}, \qquad (t,x)^{(\eta)} \in \widehat{E}_h, \tag{4.12}$$

where $\Psi : [-\tau_0,a] \to \mathbb{R}_+$ *satisfies inequality* (3.4) *and*

$$\begin{aligned} &\Psi'(\theta) - L_u - L_1 - \widetilde{B}_0(h) \\ &\qquad - \widetilde{B}(h,\Psi(\theta))\Gamma\big(\xi(\theta)\sqrt{1+(r_1(t))^2}\big)/\Gamma(\psi(0)) \ge 0 \end{aligned} \tag{4.13}$$

for $\theta \in [0,a]$, *where* $\widetilde{B}_0(h)$, $r_1(t)$ *and* $\widetilde{B}(h,p)$ *for* $p \in \mathbb{R}_+$ *are defined by*

$$\widetilde{B}_0(h)=\lambda^2 L_u\Big(\sum_{l=1}^n h_l/h_0\Big)^2(\widehat{a}+L_2\widehat{b}\|(h_0h_1^{-1},\ldots,h_0h_n^{-1})\|),$$

$$\widetilde{B}(h,p)=L\widehat{p}\Big(\lambda\|D_tu\|_H(a)+(\lambda+1)\sum_{l=1}^n h_lh_0^{-1}\|D_{x_l}u\|_H(a)+p\Big), \tag{4.14}$$

$$r_1(t)=(\lambda+1)\|\overline{h}'\|(1+\psi(t)(\psi(t)-\xi(t)-\overline{h}_0\xi'(t))^{-1}),$$

where $\theta,t\in[0,a]$, $h\in I_d$, *and*

$$\begin{aligned}(\Psi'(\theta)-L_u)&\Gamma(\xi(\theta))+\Gamma'(\xi(\theta))\xi'(\theta)(\Psi(\theta)-h_0L_u)\\&-\Gamma(\xi(\theta)\sqrt{1+(r_0(t))^2})\\&\times(L_1\Psi(\theta)+\widetilde{B}(h,\Psi(t))\Gamma(\xi(\theta)\sqrt{1+(r_1(\theta))^2})/\Gamma(\psi(0)))\\&-(\Psi(\theta)+h_0\widetilde{B}_0(h))\\&\times\Gamma'(\xi(\theta)\sqrt{1+(r_0(\theta)+\lambda\|h'\|)^2})\xi(\theta)\,\lambda\|h'\|/h_0\\&-\widetilde{B}_0(h)\Gamma(\xi(\theta)\sqrt{1+(r_0(\theta)+\lambda\|h'\|)^2})\ge 0\end{aligned} \tag{4.15}$$

for $\theta\in[0,a]$, $h\in I_d$ *with* $\overline{h}_0$ *so small that* $\Psi(\theta)-\overline{h}_0L_u\ge 0$ *on* $[-\tau_0,a]$.

Proof. Let $w_h^{(\eta)}=u_h^{(\eta)}-v_h^{(\eta)}$ for $(t,x)^{(\eta)}\in\widehat{E}_h$. From (4.11) it follows that (4.12) is satisfied for $(t,x)^{(\eta)}\in E_{0,h}$. Subtracting the recurrence expressions of $v_h^{(\eta_0+1,\eta')}$ from $u_h^{(\eta_0+1,\eta')}$ leads to the recurrence error estimates similar to (3.12), as in the proof of Theorem 3.1 (compare (3.10) and (3.11)):

$$\begin{aligned}|w_h^{(\eta+1,\eta')}|\le \max_{s\in S_\lambda}&|w_h^{(\eta_0,\eta'+s)}|+h_0L_1|w_h^{(\eta)}|\\&+h_0L\widetilde{H}_h^{(\eta)}\|[w_h]_h\|_H(a)+h_0\mu_h^{(\eta)}\end{aligned} \tag{4.16}$$

for $(t,x)^{(\eta)}\in E_h$, where μ_h is defined by (4.8). In order to establish (4.12) on $\widehat{E}_h$ it is sufficient to prove the following

Lemma 4.2. *If the assumptions of Theorem* 4.1 *are satisfied, then the function* $\widetilde{\mathcal{W}}_h:\widehat{E}_h\to\mathbb{R}_+$ *given by* $\widetilde{\mathcal{W}}_h^{(\eta)}=h_0\Psi(t^{(\eta)})\widetilde{H}_h^{(\eta)}$ *for* $(t,x)^{(\eta)}\in\widehat{E}_h$ *satisfies the inequality*

$$\begin{aligned}|\widetilde{\mathcal{W}}_h^{(\eta_0+1,\eta')}|\ge\widetilde{\mathcal{W}}_h^{(\eta_0,|\eta'|+\lambda')}&+h_0L_1|\widetilde{\mathcal{W}}_h^{(\eta)}|\\&+h_0L\widetilde{H}_h^{(\eta)}\|[\widetilde{\mathcal{W}}_h]_h\|_H(a)+h_0\mu_h^{(\eta)}\end{aligned} \tag{4.17}$$

for $(t,x)^{(\eta)}\in E_h$, *and*

$$\widetilde{\mathcal{W}}_h^{(\eta)}\ge h_0M_\phi\widetilde{H}_h^{(\eta)},\qquad (t,x)^{(\eta)}\in E_{0,h}. \tag{4.18}$$

P r o o f. Inequality (4.18) is obvious. Formula (4.17) follows from

$$(4.19)\quad \begin{aligned}&(\Psi(t+h_0)-h_0L_u)\Gamma(\xi(t+h_0)\sqrt{1+r^2})\\ &\geq h_0L_1\Psi(t)\Gamma(\xi(t)\sqrt{1+r^2})+h_0\widetilde{B}(h,\Psi(t))\\ &\quad\times\sup\left\{\frac{\Gamma(\xi(t^{(\overline{\eta})})\sqrt{1+\|x^{(|\overline{\eta}|)}+\lambda h'\|^2})}{\Gamma(\psi(\overline{t})\sqrt{1+\|\overline{x}\|^2})}\,\middle|\,\begin{array}{l}t^{(\overline{\eta}-\mathbf{1})}<\overline{t}\leq t^{(\overline{\eta})},\\ x^{(\overline{\eta}-\mathbf{1})}<\overline{x}\leq x^{(\overline{\eta})},\\ (t,x)^{(\overline{\eta}-\mathbf{1})},(t,x)^{(\overline{\eta})}\in\widehat{E}_h,\\ -h_0N_0\leq h_0\overline{\eta}_0\leq t,\end{array}\right\}\\ &\quad+(\Psi(t)+h_0\widetilde{B}_0(h))\Gamma(\xi(t)\sqrt{1+(r+\lambda\|h'\|)^2})\end{aligned}$$

for $t\in[0,a-h_0]$, $r\in\mathbb{R}_+$ and $h=(h_0,h')\in I_d$.

First, observe that

$$(4.20)\quad \begin{aligned}&\Gamma(\xi(t^{(\overline{\eta})})\sqrt{1+\|x^{(|\overline{\eta}|)}+\lambda h'\|^2})/\Gamma(\psi(\overline{t})\sqrt{1+\|\overline{x}\|^2})\\ &\qquad\leq\Gamma(\xi(t^{(\eta)})\sqrt{1+(r_1(t))^2})/\Gamma(\psi(\overline{t})\sqrt{1+\|\overline{x}\|^2})\end{aligned}$$

for $t^{(\overline{\eta}-\mathbf{1})}<\overline{t}\leq t^{(\overline{\eta})}$, $x^{(\overline{\eta}-\mathbf{1})}<\overline{x}\leq x^{(\overline{\eta})}$, $(t,x)^{(\overline{\eta}-\mathbf{1})},(t,x)^{(\overline{\eta})}\in\widehat{E}_h$, $-N_0\leq\overline{\eta}_0\leq\eta_0$. We claim that

$$(4.21)\quad \begin{aligned}\widetilde{\Xi}(\theta;t):=\Psi(t+\theta)-\Psi(t)-\theta(L_u+L_1+\widetilde{B}_0(h)\\ +\widetilde{B}(h,\Psi(t))\Gamma(\xi(t)\sqrt{1+(r_1(t))^2})/\Gamma(\psi(0)))\geq 0\end{aligned}$$

for $t\in[0,a-h_0]$ and $\theta\in[0,h_0]$, because $\widetilde{\Xi}(0;t)=0$ and $\widetilde{\Xi}'(\theta;t)\geq 0$ on the considered interval as we have (4.13). Thus, if $r\geq r_0(t)$, then (4.19) results from (4.20) and (4.21).

If $r\leq r_0(t)$, then we define

$$(4.22)\quad \begin{aligned}\widetilde{\Xi}_1(\theta;t,r)&=(\Psi(t+\theta)-\theta L_u)\Gamma(\xi(t+\theta)\sqrt{1+r^2})\\ &\quad-\theta\Gamma(\xi(t)\sqrt{1+r^2})(L_1\Psi(t)\\ &\quad+\widetilde{B}(h,\Psi(T))\Gamma(\xi(t)\sqrt{1+(r_1(t))^2})/\Gamma(\psi(0)))\\ &\quad-(\Psi(t)+\theta\widetilde{B}_0(h))\Gamma(\xi(t)\sqrt{1+(r+\theta\lambda\|h'\|/h_0)^2})\end{aligned}$$

for $t\in[0,a-h_0]$ and $\theta\in[0,h_0]$. Next, from (4.15) we have

$$(4.23)\quad \begin{aligned}&\widetilde{\Xi}_1'(\theta;t,r)\\ &=(\Psi'(t)-L_u)\Gamma(\xi(t+\theta)\sqrt{1+r^2})\\ &\quad+\Gamma'(\xi(t+\theta)\sqrt{1+r^2})\xi'(t+\theta)\sqrt{1+r^2}(\Psi(t+\theta)-\theta L_u)\\ &\quad-\Gamma(\xi(t)\sqrt{1+r^2})(L_1\Psi(t)+\widetilde{B}(h,\Psi(t))\Gamma(\xi(t)\sqrt{1+(r_1(t))^2})/\Gamma(\psi(0)))\\ &\quad-(\Psi(t)+\theta\widetilde{B}_0(h))\Gamma'(\xi(t)\sqrt{1+(r+\theta\lambda\|h'\|/h_0)^2})\psi(t)\lambda\|h'\|/h_0\\ &\quad-\widetilde{B}_0(h)\Gamma(\xi(t)\sqrt{1+(r+\theta\lambda\|h'\|/h_0)^2})\geq 0\end{aligned}$$

for $t \in [0, a-h_0]$ and $\theta \in [0, h_0]$. From (4.21) it is easy to obtain $\widetilde{\Xi}_1(0; t, r) = 0$. From this and (4.23) we have $\widetilde{\Xi}_1(h_0; t, r) \geq 0$. This finishes the proof of our lemma.

References

[1] P. Besala, *On solutions of first order partial differential equations defined in an unbounded zone*, Bull. Acad. Polon. Sci. 12 (1964), 95–99.

[2] —, *Finite difference approximation to the Cauchy problem for non-linear parabolic differential equations*, Ann. Polon. Math. 46 (1985), 19–26.

[3] Z. Kamont, *On the Cauchy problem for system of first order partial differential equations*, Serdica 5 (1979), 327–339.

[4] M. Krzyżański, *Partial Differential Equations of Second Order*, PWN, Warszawa, 1971.

[5] H. Leszczyński, *General finite difference approximation to the Cauchy problem for non-linear parabolic differential-functional equations*, Ann. Polon. Math. 53 (1991), 15–28.

[6] —, *Uniqueness results for unbounded solutions of first order non-linear differential-functional equations*, Acta Math. Hungar. 64 (1994), 75–92.

[7] M. Malec et A. Schiaffino, *Méthode aux différences finies pour une équation non-linéaire différentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy*, Boll. Un. Mat. Ital. (7) 1-B (1987), 99–109.

[8] K. Prządka, *Difference methods for non-linear partial differential functional equations of the first order*, Math. Nachr. 138 (1988), 105–123.

[9] J. Szarski, *Differential Inequalities*, PWN, Warszawa, 1967.

Institute of Mathematics
University of Gdańsk
57 Wita Stwosza St.
80-952 Gdańsk, Poland
E-mail: hleszcz@ksinet.univ.gda.pl

Reçu par la Rédaction le 28.8.1992
Révisé le 9.2.1996