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On solutions of integral equations with analytic kernels and rotations

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Abstract. We deal with a class of integral equations on the unit circle in the complex plane with a regular part and with rotations of the form

$$x(t) + a(t)(Tx)(t) = b(t), \tag{$*$}$$

where $T = M_{n_1,k_1} \ldots M_{n_m,k_m}$ and M_{n_j,k_j} are of the form (3) below. We prove that under some assumptions on analytic continuation of the given functions, $(*)$ is a singular integral equation for m odd and is a Fredholm equation for m even. Further, we prove that T is an algebraic operator with characteristic polynomial $P_T(t) = t^3 - t$. By means of the Riemann boundary value problems, we give an algebraic method to obtain all solutions of equation $(*)$ in closed form.

1. Algebraic characterizations of integral operators with rotations. Let $\Gamma = \{t : |t| = 1\}$, $D^+ = \{t : |t| < 1\}$ and let $X = H^\mu(\Gamma)$ $(0 < \mu < 1)$. Consider the following operators in X:

$$(Sx)(t) = \frac{1}{\pi i} \int_\Gamma \frac{1}{s-t} x(s)\, ds, \tag{1}$$

$$(S_{n,k}x)(t) = \frac{1}{\pi i} \int_\Gamma \frac{s^{n-1-k}t^k}{s^n - t^n} x(s)\, ds, \tag{2}$$

$$(M_{n,k}x)(t) = \frac{1}{\pi i} \int_\Gamma \frac{s^{n-1-k}t^k \mathcal{M}_n(s,t)}{s^n - t^n} x(s)\, ds, \tag{3}$$

where $n, k \in \mathbb{N}$, $n > 1$, $0 \le k \le n-1$ and $\mathcal{M}_n(s,t)$ satisfies the Hölder condition with respect to $(s,t) \in \Gamma \times \Gamma$. Let $\varepsilon_{n,1} = e^{2\pi i/n}$, $\varepsilon_{n,j} = \varepsilon_{n,1}^j$, $j = 1, \ldots, n$. Consider the rotation operators $(W_n x)(t) = x(\varepsilon_{n,1} t)$. For $j \in$

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$\{1, \ldots, n\}$, write

$$P = \frac{1}{2}(I + S), \quad Q = \frac{1}{2}(I - S), \quad P_{n,j} = \sum_{k=1}^{n} \varepsilon_{n,j}^{n-1-k} W_n^{k+1}. \tag{4}$$

It is easy to check that $X = X^+ \oplus X^-$ and $X = \bigoplus_{j=1}^{n} X_{n,j}$, where $X^+ = PX$, $X^- = QX$ and $X_{n,j} = P_{n,j}X$ for $j \in \{1, \ldots, n\}$.

Lemma 1. *Let P, Q and $P_{n,j}$ be defined by* (4). *Then*

$$P_{n,j}X^+ \subset X^+, \quad P_{n,j}X^- \subset X^-$$

and

$$PX_{n,j} \subset X_{n,j}, \quad QX_{n,j} \subset X_{n,j} \quad (j \in \{1, \ldots, n\}).$$

Proof. This follows immediately from the equalities

$$SW_n = W_nS, \quad S_{n,k}S = SS_{n,k}, \quad S_{n,k}W_n = W_nS_{n,k},$$

for all $k \in \{0, \ldots, n-1\}$.

Lemma 2 ([2]–[3]). *$S_{n,k} = SP_{n,k} = P_{n,k}S$ for all $k \in \{0, \ldots, n-1\}$, where we set $P_{n,0} = P_{n,n}$.*

Lemma 3. *Suppose that $\mathcal{M}_n(s,t)$ is invariant with respect to the rotation operator W_n, i.e. $\mathcal{M}_n(\varepsilon_{n,1}s,t) = \mathcal{M}_n(s,\varepsilon_{n,1}t) = \mathcal{M}_n(s,t)$. Suppose, moreover, that $\mathcal{M}_n(t,t) = 1$ for all $t \in \Gamma$. Then*

$$M_{n,k} = S_{n,k} + N_nP_{n,k}, \quad N_nP_{n,k} = P_{n,k}N_n \quad \text{for all } k \in \{1, \ldots, n\}, \tag{5}$$

where

$$(N_nx)(t) = \frac{1}{\pi i}\int_{\Gamma} \mathcal{N}_n(s,t)x(s)\,ds, \quad \mathcal{N}_n(s,t) = \frac{\mathcal{M}_n(s,t) - 1}{s - t}.$$

Proof. The assumptions on $\mathcal{M}_n(s,t)$ imply the equalities

$$\begin{aligned}(M_{n,k}x)(t) &= (S_{n,k}x)(t) + \frac{1}{\pi i}\int_{\Gamma} \frac{s^{n-1-k}t^k}{s^n - t^n}(\mathcal{M}_n(s,t) - 1)x(s)\,ds \\ &= (S_{n,k}x)(t) + \sum_{j=1}^{n} \varepsilon_k^{n-1-j}\bigg(\frac{1}{\pi i}\int_{\Gamma} \frac{\mathcal{M}_n(s,t) - 1}{s - \varepsilon_j t}x(s)\,ds\bigg) \\ &= (S_{n,k}x)(t) + \frac{1}{\pi i}\int_{\Gamma} \mathcal{N}_n(s,t)(P_{n,k}x)(s)\,ds,\end{aligned}$$

which gives the first equality of the formula (5). The second equality of (5) immediately follows from the assumption that $\mathcal{M}_n(s,t)$ is invariant with respect to W_n.

Theorem 1. *Let $n > 1$ and $m \in \{0, \ldots, n-1\}$. Suppose that $\mathcal{M}_n(s,t)$ satisfy all assumptions of Lemma* 3 *and admit an analytic continuation in* *both variables into D^+. Write*

$$M = \sum_{k=0}^{m} M_{n,k},$$

where $M_{n,k}$ is of the form (3). *Then M is an algebraic operator and $M^3 = M$.*

P r o o f. From the assumptions on $\mathcal{M}_n(s,t)$, the function $\mathcal{N}_n(s,t) = (\mathcal{M}_n(s,t)-1)(s-t)^{-1}$ is continuous in $(s,t) \in \Gamma \times \Gamma$ and admits an analytic continuation in both variables into D^+. Hence, the Cauchy integral theorem gives

$$N_n^2 = 0, \quad SN_n = N_n, \quad N_nS = -N_n. \tag{6}$$

On the other hand, since $S^2 = I$, $S_{n,k}^2 = P_{n,k}$ and $P_{n,k}P_{n,j} = \delta_{kj}P_{n,j}$ we find

$$M_{n,k}^2 = P_{n,k}, \quad M_{n,k}^3 = M_{n,k}, \quad M_{n,k}M_{n,j} = \delta_{kj}M_{n,j}$$

and

$$M^3 = \sum_{k=0}^{m} M_{n,k}^3 = \sum_{k=0}^{m} M_{n,k} = M,$$

which gives the proof.

THEOREM 2. *Let $m \in \mathbb{N}^+$. Write*

$$T = M_{n_1,k_1}M_{n_2,k_2}\dots M_{n_m,k_m}, \tag{7}$$

where M_{n_j,k_j} ($j \in \{1,\dots,m\}$) are of the form (3)*. Suppose that $\mathcal{M}_{n_j}(t,t) = 1$, and $\mathcal{M}_{n_j}(s,t)$ ($j \in \{1,\dots,m\}$) admit an analytic continuation in both variables into D^+ and are invariant with respect to W_N, where $N = \prod_{j=1}^m n_j$. Then T is an algebraic operator and $T^3 = T$.*

P r o o f. Since every $\mathcal{M}_{n_j,k_j}(s,t)$ is invariant with respect to W_N, by Lemma 3, we conclude that

$$M_{n_\nu,k_\nu}M_{n_\mu,k_\mu} = M_{n_\mu,k_\mu}M_{n_\nu,k_\nu} \quad \text{for all } \mu,\nu \in \{1,\dots,m\}.$$

Hence, the proof immediately follows from Theorem 1.

R e m a r k 1. If $k,j \in \{0,\dots,n\}$ and $k \neq j$ then $M_{n,k}M_{n,j} = 0$ for $P_{n,k}P_{n,j} = 0$. Hence, if there are $i,j \in \{1,\dots,m\}$ and $i \neq j$ such that $n_i = n_j$ and $k_i \neq k_j$, then $T = 0$. Therefore, in the sequel, we only deal with the cases $n_i \neq n_j$ for $i \neq j$.

LEMMA 4. *Let $l \in \mathbb{N}^+$. Suppose that $\mathcal{M}_{n_j}(s,t)$ ($j \in \{1,\dots,m\}$) satisfy all assumptions of Theorem* 2*. Then*

$$\prod_{k=1}^{m}(S + N_{n_k}) = I + \sum_{k=1}^{2l}(-1)^k N_{n_k} \quad \textit{for } m = 2l \tag{8}$$

and

$$\prod_{k=1}^{m}(S+N_{n_k}) = S + \sum_{k=1}^{2l-1}(-1)^{k+1}N_{n_k} \quad \textit{for } m = 2l-1. \tag{9}$$

P r o o f. Since $SN_{n_2} = N_{n_2}$, $N_{n_1}S = -N_{n_1}$ and $N_{n_1}N_{n_2} = 0$ (see the proof of Theorem 2), for $l = 1$ we have

$$(S+N_{n_1})(S+N_{n_2}) = I + SN_{n_2} + N_{n_1}S + N_{n_1}N_{n_2} = I + N_{n_2} - N_{n_1}.$$

Hence, it is easy to check the formula (8) by induction on l.

Similarly, (9) is trivial for the case $l = 1$. Suppose that (9) is true for $l = s$. Then for $l = s+1$, we find

$$\begin{aligned}\prod_{k=1}^{2s+1}(S+N_{n_k}) &= \Big(S + \sum_{k=1}^{2s-1}(-1)^{k+1}N_{n_k}\Big)(S+N_{n_{2s}})(S+N_{n_{2s+1}})\\ &= \Big(S + \sum_{k=1}^{2s-1}(-1)^{k+1}N_{n_k}\Big)(I+N_{n_{2s+1}}-N_{n_{2s}})\\ &= S + \sum_{k=1}^{2s-1}(-1)^{k+1}N_{n_k} - SN_{n_{2s}} + SN_{n_{2s+1}}\\ &= S + \sum_{k=1}^{2s+1}(-1)^{k+1}N_{n_k},\end{aligned}$$

which proves the formula (9).

2. Integral equations with rotations. In the sequel, for every fixed $a \in X$, we write $(K_a x)(t) = a(t)x(t)$.

LEMMA 5. *Let $k, j \in \{1,\ldots,n\}$ be fixed. Then for every $a \in X$ there exists an element $b \in X$ such that $K_b X_j \subset X_k$ and*

$$P_{n,k}K_aP_{n,j} = K_bP_{n,j} \quad \textit{on } X.$$

(*Such a function $b(t)$ will be denoted by $a_{nkj}(t)$.*)

P r o o f. Note that $P_{n,k}$ defined by (4) is a polynomial in W_n with constant coefficients. On the other hand, we also have

$$P_{n,j}P_{n,k} = \delta_{jk}P_{n,k}, \quad W_n^l = \sum_{k=1}^{n}\varepsilon_{n,k}^l P_{n,k} \quad \text{for all } l \in \mathbb{N}.$$

Hence

$$P_{n,k}K_aP_{n,j} = \sum_{l=1}^{n}\varepsilon_{n,k}^{n-1-l}W_n^{l+1}K_aP_{n,j} = \sum_{l=1}^{n}\varepsilon_{n,k}^{n-1-l}a(\varepsilon_{n,l+1}t)W^{l+1}P_{n,j}$$

$$= \sum_{l=1}^{n} \varepsilon_{n,k}^{n-1-l} a(\varepsilon_{n,l+1}t) \sum_{\nu=1}^{n} \varepsilon_{n,\nu}^{l+1} P_{n,\nu} P_{n,j}$$

$$= \sum_{l=1}^{n} \varepsilon_{n,k}^{n-1-l} a(\varepsilon_{n,l+1}t) \varepsilon_{n,j}^{l+1} P_{n,j} = a_{nkj}(t) P_{n,j},$$

where

$$a_{nkj}(t) = \sum_{l=1}^{n} \varepsilon_{n,k}^{n-1-l} \varepsilon_{n,j}^{l+1} a(\varepsilon_{n,l+1}t). \tag{10}$$

It is easy to check that $P_{n,\nu} K_b P_{n,j} = 0$ for all $\nu \neq k$ and for $b(t) = a_{nkj}(t)$, which gives $K_b X_j \subset X_k$. The proof is complete.

COROLLARY 1. *Let $j \in \{1, \ldots, n\}$. Then for every $a \in X$ there is a function $b \in X$ that is invariant with respect to the rotation operator W_n and*

$$P_{n,j} K_a P_{n,j} = K_b P_{n,j} = P_{n,j} K_b \quad \text{on } X.$$

Proof. By Lemma 5, we have the equality

$$P_{n,j} K_a P_{n,j} = a_{njj}(t) P_{n,j},$$

where

$$a_{njj}(t) = \sum_{k=1}^{n} a(\varepsilon_{n,k+1}t). \tag{11}$$

We see from (11) that $a_{njj}(t)$ is a function invariant with respect to the rotation operator W_n. Hence $K_b P_{n,j} = P_{n,j} K_b$ for $b(t) = a_{njj}(t)$, which was to be proved.

Consider now the equation with the operator T of the form (7):

$$x(t) + a(t)(Tx)(t) = b(t), \tag{12}$$

where $a, b \in X$ are given.

THEOREM 3. *Let $m \in \mathbb{N}^+$ and $n_1, \ldots, n_m$ be given distinct positive integers. Suppose that $\mathcal{M}_{n_j}(t,t) = 1$, and $\mathcal{M}_{n_j}(s,t)$ ($j \in \{1, \ldots, m\}$) admit an analytic continuation in both variables into D^+ and are invariant with respect to W_N, where $N = \prod_{j=1}^{m} n_j$. Then the equation* (12) *has solutions if and only if the equation*

$$y(t) + a^{(m)}(t)((S + N_{n_1}) \ldots (S + N_{n_m})y)(t) = b_{(m)}(t), \tag{13}$$

where $a^{(m)}(t)$ are constructed by Lemma 4 as follows:

$$a^{(0)}(t) = a(t), \quad a^{(1)}(t) = a^{(0)}_{n_1 k_1 k_1}(t), \ \ldots, \ a^{(m)}(t) = a^{(m-1)}_{n_m k_m k_m}(t) \tag{14}$$

and

$$b_{(m)}(t) = (P_{n_1,k_1} \ldots P_{n_m,k_m} b)(t),$$

has solutions in the space $X_{(m)} := P^{\otimes} X$, where $P^{\otimes} = P_{n_1,k_1} \ldots P_{n_m,k_m}$.

P r o o f. Suppose that (12) has a solution $x_0(t)$, i.e.

$$x_0(t) + a(t)(Tx_0)(t) = b(t). \tag{15}$$

Applying the operator $P^{\otimes}$ to both sides of (15), we obtain $y_0(t) = (P^{\otimes}x_0)(t)$ belonging to $X_{(m)}$ and satisfying (13).

Conversely, if $y_0(t)$ is a solution in $X_{(m)}$ of (13), then it is easy to check that $x(t) = b(t) - a(t)((S + N_{n_1}) \ldots (S + N_{n_m})y_0)(t)$ is a solution of (12), which gives the proof.

LEMMA 6. *Suppose that $\mathcal{M}_{n_j}(s,t)$ $(j \in \{1,\ldots,m\})$ satisfy all assumptions of Theorem 3. Then equation (13) has solutions in $X_{(m)}$ if and only if it has solutions in X. Moreover, if $y(t) \in X$ is a solution of (13), then $y_{(m)}(t) = (P^{\otimes}y)(t)$ is a solution of (13) in $X_{(m)}$.*

P r o o f. By Corollary 1, $a^{(m)}(t)$ is invariant with respect to all rotation operators W_{n_j} and by (5), we see that every operator $S + N_{n_j}$ commutes with any rotation operator W_{n_l}. Then $P^{\otimes}$ commutes with every $S + N_{n_j}$. Hence, if $y \in X$ is a solution of (13), then applying the operator $P^{\otimes}$ to both sides of (13), we see that again $y_{(m)}(t) = (P^{\otimes}y)(t)$ is also a solution of (13). The proof is complete.

R e m a r k 2. Lemma 6 shows that it is enough to solve the equation (13) in a given space $X = H^{\mu}(\Gamma)$.

LEMMA 7. *Let m be an odd positive integer. Suppose that $\mathcal{M}_{n_j}(s,t)$ $(j \in \{1,\ldots,m\})$ satisfy all assumptions of Theorem 3 and the function*

$$\mathcal{R}_m(s,t) = \frac{a^{(m)}(t)}{1 + a^{(m)}(t)} \sum_{k=1}^{m} (-1)^{k+1} \mathcal{N}_{n_k}(s,t), \tag{16}$$

where

$$\mathcal{N}_{n_k}(s,t) = \frac{\mathcal{M}_{n_k}(s,t) - 1}{s - t},$$

admits an analytic continuation in both variables in D^+. Then every solution $y(t) \in X$ of the equation (13) is given by the formula

$$y(t) = y^+(t) + y^-(t), \quad y^-(t) = \psi^-(t), \quad y^+(t) = \psi^+(t) - (R_m\psi^-)(t),$$

where

$$(R_m x)(t) = \frac{1}{\pi i} \int_{\Gamma} \mathcal{R}_m(s,t) x(s)\, ds \tag{17}$$

and $(\psi^+(t), \psi^-(t))$ is a solution of the following Riemann boundary value problem:

$$\psi^+(t) = \frac{a^{(m)}(t) - 1}{a^{(m)}(t) + 1} \psi^-(t) + \frac{b_{(m)}(t)}{1 + a^{(m)}(t)}. \tag{18}$$

P r o o f. By Lemma 4, the equation (13) is equivalent to the equation

$$y(t)+a^{(m)}(t)\Big(\Big(S+\sum_{k=1}^{m}(-1)^{k+1}N_{n_k}\Big)y\Big)(t)=b_{(m)}(t). \tag{19}$$

Write $y^+(t)=(Py)(t)$ and $y^-(t)=(Qy)(t)$. Using the notations (16)–(17), we can rewrite (19) in the form

$$y^+(t)+\frac{1-a^{(m)}(t)}{1+a^{(m)}(t)}y^-(t)+(R_m y)(t)=\frac{b_{(m)}(t)}{1+a^{(m)}(t)}. \tag{20}$$

The assumption on $\mathcal{R}_m(s,t)$ implies $(R_m y^+)(t)=0$ and $(R_m y^-)(t)\in X^+$ for all $y\in X$. Hence (20) is a Riemann boundary value problem of the form (18), where $\psi^+(t)=y^+(t)+(R_m y^-)(t)$ and $\psi^-(t)=y^-(t)$. The proof is complete.

Note that all solutions of the Riemann boundary value problem can be found in closed form (see [1], [2], [4]). Consequently, from Lemma 7, we can formulate the following result.

COROLLARY 2. *All solutions of equation* (13) *can be given in closed form by means of the Riemann boundary value problem* (18).

LEMMA 8. *Let m be an even positive integer. Suppose that $\mathcal{M}_{n_j}(s,t)$ ($j\in\{1,\ldots,m\}$) satisfy all assumptions of Lemma* 7. *Then every solution $y(t)\in X$ of the equation* (13) *is given by the formula*

$$y^-(t)=\psi^-(t),\qquad y^+(t)=\psi^+(t)-(R_m\psi^-)(t), \tag{21}$$

where

$$\psi(z)=\frac{1}{\pi i}\int_{\Gamma}\frac{b_{(m)}(s)}{1+a^{(m)}(s)}\frac{ds}{s-z}. \tag{22}$$

P r o o f. By Lemma 4, (13) is equivalent to the equation

$$y(t)+a^{(m)}(t)\Big(\Big(I+\sum_{k=1}^{m}(-1)^{k}N_{n_k}\Big)y\Big)(t)=b_{(m)}(t). \tag{23}$$

Using notations (16)–(17), we can rewrite (23) in the form

$$y(t)+(R_m y)(t)=\frac{b_{(m)}(t)}{1+a^{(m)}(t)}. \tag{24}$$

The assumption on $\mathcal{R}_m(s,t)$ implies $(R_m y^+)(t)=0$ and $(R_m y^-)(t)\in X^+$. Hence, (24) is of the form

$$\psi^+(t)-\psi^-(t)=b_{(m)}(t)\{1+a^{(m)}(t)\}$$

which gives the formula (21)–(22) for any solution of (13).

Now we can formulate the main results of Theorem 3 and of Lemmas 6–7 about solutions of the equation (12) in the following form.

THEOREM 4. *Let* m *be an odd positive integer and let* T *be of the form* (7). *Suppose that* $\mathcal{M}_{n_j}(s,t)$ $(j \in \{1,\ldots,m\})$ *satisfy all assumptions of Theorem* 3. *Moreover*, *suppose that the function* $\mathcal{R}_m(s,t)$ *of the form* (16) *admits an analytic continuation in both variables into* D^+. *Then equation* (12) *has solutions if and only if the corresponding Riemann boundary value problem* (18) *has solutions. If that is the case*, *every solution of* (12) *is given by the formula*

$$x(t) = b(t) - a(t)((S + N_{n_1}) \ldots (S + N_{n_m})y)(t), \tag{25}$$

where $y(t) = \psi^+(t) - (R_m\psi^-)(t) + \psi^-(t)$ *and* $\psi^{\pm}(t)$ *is a solution of the Riemann boundary value problem* (18).

Similarly, Theorem 3 and Lemma 8 together imply the following.

THEOREM 5. *Let* m *be an even positive integer and let* T *be of the form* (7). *Suppose that* $\mathcal{M}_{n_j}(s,t)$ $(j \in \{1,\ldots,m\})$ *satisfy all assumptions of Theorem* 4. *Then equation* (12) *has solutions of the form* (25), *where* $y(t) = \psi^+(t) - (R_m\psi^-)(t) + \psi^-(t)$ *and* $\psi(z)$ *is of the form* (22).

Remark 3. Note that the equation (12) in the case of m even is not a singular integral equation. In that case, (12) is a Fredholm integral equation (see Lemma 4).

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