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Convergence of iterates of Lasota–Mackey–Tyrcha operators

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Abstract. We provide sufficient and necessary conditions for asymptotic periodicity of iterates of strong Feller stochastic operators.

1. Let (X,d) be a locally compact, metric, Polish space and $\mathcal{B}$ denote the σ-algebra of Borel subsets in X. Given a σ-finite measure μ on $(X,\mathcal{B})$ we denote by $(L^1(\mu),\|\cdot\|)$ the Banach lattice of μ-integrable functions on X. Functions which are equal μ-almost everywhere are identified. A linear operator P on $L^1(\mu)$ is called *stochastic* (or *Markov* according to Lasota's terminology) if

$$Pf \ge 0 \quad \text{and} \quad \int_X Pf\,d\mu = 1$$

for all nonnegative and normalized (densities) $f \in L^1(\mu)$. The convex set of all densities is denoted by $\mathcal{D}_\mu$ (simply $\mathcal{D}$ if $X = \mathbb{R}_+$ and μ is the Lebesgue measure on $\mathbb{R}_+$). If there exists a Borel measurable function $k : X\times X \to \mathbb{R}_+$ such that

$$Pf(x) = \int_X k(x,y)f(y)\,d\mu(y)$$

then P is called a *kernel operator*.

We notice that each kernel stochastic operator may be extended to the Banach lattice $M(X)$ of all bounded signed Borel measures on $(X,\mathcal{B})$. Namely, if $\nu \in M(X)$ and A is Borel we define

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$$P\nu(A) = \int_X \int_X k(x,y)\mathbf{1}_A(x)\,d\mu(x)\,d\nu(y).$$

Obviously $P\nu \in L^1(\mu)$.

The paper is particularly devoted to stochastic kernel operators on $L^1(\mathbb{R}_+)$ with kernels

$$(*)\qquad k(x,y) = \begin{cases} -\dfrac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) & \text{if } 0 \le y \le \lambda(x) \\ 0 & \text{otherwise.} \end{cases}$$

They appear in mathematical modelling of the cell cycle. A systematic study of the asymptotic properties of iterates of $(*)$ is being continued by Lasota and his collaborators. The reader is referred to [6] for a comprehensive and updated review of the subject. Here we shall concentrate on the mathematical side rather than on biological applications. Our paper often refers to [1]. Following it we shall assume:

(H) $H : [0,\infty) \to [0,\infty)$ is nonincreasing and absolutely continuous, $H(0) = 1$ and $\lim_{x\to\infty} H(x) = 0$,

(Qλ) $Q : [0,\infty) \to [0,\infty)$ and $\lambda : [0,\infty) \to [0,\infty)$ are nondecreasing, absolutely continuous, $Q(0) = \lambda(0) = 0$ and $\lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty$.

The class of stochastic operators P with kernels $(*)$ satisfying (H) and (Qλ) is denoted by LMT (Lasota, Mackey, Tyrcha (cf. [7]) who contributed much to the discussed matters). It has recently been proved in [1] that if a LMT stochastic operator P additionally satisfies:

(α) $\int_0^\infty x^\alpha h(x)\,dx < \liminf_{x\to\infty} Q(\lambda(x))^\alpha - Q(x)^\alpha$ for some $0 < \alpha \le 1$, where $h(x) = -dH(x)/dx$ for almost all x, and

(c) there exists a nonnegative c such that $h(x) > 0$ for almost all $x \ge c$,

then there exists a unique $f_* \in \mathcal{D}$ such that

$$\lim_{n\to\infty} \|P^n f - f_*\| = 0 \quad \text{ for all } f \in \mathcal{D}$$

(P is *asymptotically stable*). In this paper we drop condition (c) and prove that the iterates of a LMT operator with (H), (Qλ) and (α) are strong operator topology convergent to a finite-dimensional projection (with a slight abuse of the terminology such operators are also called *stable* (cf. [9])).

We begin with considering a general case. Let us recall (cf. [10]) that a kernel stochastic operator P on $L^1(\mu)$ is called *strong Feller in the strict sense* if

(SFS) $X \ni y \to k(\cdot, y) \in \mathcal{D}_\mu$ is L^1-norm continuous.

Note that (SFS) implies the continuity of P^*h, where $h \in L^\infty(\mu)$ and P^* stands for the adjoint operator. This easily follows from $P^*h(y) = \int_X k(x,y)h(x)\,d\mu(x)$. It is also well known that if X is compact then (SFS) kernel stochastic operators are compact (see [10]). More details concerning asymptotic properties of iterates of compact (or quasi-compact) positive contractions on Banach lattices can be found in [3] and [4].

If X is not compact then (SFS) does not guarantee automatically any regularity of the trajectories $P^n f$. For instance, it may happen that for some $f \in \mathcal{D}_\mu$ the sequence $P^n f$ converges to a density, while for other f we have $\int_K P^n f\,d\mu \to 0$ for every compact $K \subset X$. Roughly speaking, starting from "good" states the process is rather concentrated, but starting from "bad" states it escapes to "infinity". Also all mixed situations may occur. The so-called Doeblin condition is never satisfied if the transition kernels $k(\cdot,\cdot)$ do not allow "long jumps" (i.e. if $d(y,z) \to \infty$ implies $\int_X |k(x,y) - k(x,z)|\,d\mu(x) \to 0$).

For noncompact X, in order to obtain asymptotic regularity of iterates of (SFS) stochastic operators, we must impose some extra assumptions. Following [5] we say that a stochastic operator P on $L^1(\mu)$ is *asymptotically periodic* if there exist densities $g_1, \ldots, g_r \in L^1(\mu)$ with disjoint supports, functionals $\Lambda_1, \ldots, \Lambda_r$ on $L^1(\mu)$ and a permutation α of $\{1, \ldots, r\}$ so that for all $f \in L^1(\mu)$ we have

$$\lim_{n\to\infty} \Big\| P^n f - \sum_{j=1}^{r} \Lambda_j(f) g_{\alpha^n(j)} \Big\| = 0.$$

P is said to be *constrictive* if there exists an L^1-norm compact set $\mathcal{F} \subseteq \mathcal{D}_\mu$ such that $\operatorname{dist}(P^n f, \mathcal{F}) \to 0$ for all $f \in \mathcal{D}_\mu$. It has been proved in [5] that each constrictive stochastic operator P on $L^1(\mu)$ is asymptotically periodic.

Given a (SFS) stochastic operator P on $L^1(\mu)$ we identify here an invariant sublattice on which P is asymptotically periodic. This sublattice appears to be trivial exactly when for each compact $K \subseteq X$ there exists $f \in \mathcal{D}_\mu$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N-1} \int_K P^j f\,d\mu = 0.$$

Condition (SFS) is usually easy to verify. We remark that many important kernels used in mathematical modelling of biological systems have this property. For instance, using [8], Theorem 7.4.8, we easily check that if $y_n \to y$ then

$$\begin{aligned}\int\limits_0^\infty \left|\frac{\partial}{\partial x}H(Q(\lambda(x))-Q(y))-\frac{\partial}{\partial x}H(Q(\lambda(x))-Q(y_n))\right|dx\\ =\int\limits_0^\infty |h(Q(\lambda(x))-Q(y))-h(Q(\lambda(x))-Q(y_n))|(Q\circ\lambda)'(x)\,dx\\ =\int\limits_0^\infty |h(t-Q(y))-h(t-Q(y_n))|\,dt\xrightarrow[n\to\infty]{}0,\end{aligned}$$

with our convention that $h(x)\equiv 0$ if $x\le 0$. Hence LMT operators satisfy (SFS).

2. The purpose of this section is to show asymptotic periodicity of (SFS) operators. The reader can view it as a generalization of [2].

We denote by $C_0(X)$ the Banach lattice of all continuous functions h on X such that for every $\varepsilon>0$ there exists a compact set $E_\varepsilon\subseteq X$ such that $|h(x)|\le\varepsilon$ for all $x\notin E_\varepsilon$ (endowed with the ordinary sup-norm $\|\cdot\|_{\sup}$). Given a stochastic operator P we denote by F the minimal (modulo sets of measure zero) measurable set which carries supports of all P-invariant densities (its existence follows from separability of the $L^1(\mu)$). Obviously $L^1(F)$ is P-invariant.

The next result, which will be the main ingredient of the proofs in Section 3, is also of some independent interest.

THEOREM 1. *Let P be a (SFS) stochastic operator on $L^1(\mu)$ such that P^* preserves $C_0(X)$. If*

(i) *there exists a compact set $K\subseteq X$ such that*

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\int\limits_K\sum_{j=0}^{N-1}P^jf\,d\mu>0\qquad\text{for all } f\in\mathcal{D}_\mu,$$

then F is nontrivial and P is asymptotically periodic on $L^1(F)$. In particular, there are only finitely many P-invariant ergodic densities.

Proof. The set of all subprobabilistic positive measures on X is a compact convex set with respect to the vague topology (we say that a variation norm bounded sequence of measures ν_n is *vaguely convergent* to ν if $\lim_{n\to\infty}\int_X h\,d\nu_n=\int_X h\,d\nu$ for all $h\in C_0(X)$). Given $f\in\mathcal{D}_\mu$ we may choose a sequence $n_k\nearrow\infty$ so that the measures with densities

$$\frac{1}{n_k}\sum_{j=0}^{n_k-1}P^jf=A_{n_k}f$$

are vaguely convergent. By (i) the limit ν is nonzero and $PA_{n_j}f$ tends to

$P\nu$ vaguely. Since

$$\|A_{n_k} f - P A_{n_k} f\| = \left\| \frac{P^{n_k} f - f}{n_k} \right\| \xrightarrow[k\to\infty]{} 0$$

we conclude that $\nu = P\nu \in L^1(\mu)$ is a fixed point of P. Normalizing ν if necessary we obtain a P–invariant density.

Now we show that the linear subspace (sublattice) $\mathrm{Fix}(P)$ of all P-invariant functions is finite-dimensional. Assume we are given pairwise orthogonal P-invariant densities $f_1, \ldots, f_k$. By (i) we have $\int_K f_j \, d\mu > 0$. Consider the following family of (restricted to K) continuous functions:

$$g_j = (P^* \mathbf{1}_{F_j})|_K, \quad \text{where} \quad F_j = \mathrm{supp}(f_j).$$

Clearly

$$g_j(x) = 1 \quad \text{for all } x \in \overline{F}_j \cap K,$$

and

$$g_j(x) = 0 \quad \text{if } x \in \bigcup_{l \neq j} \overline{F}_l \cap K.$$

As a result, $\|g_j - g_l\|_{\sup} = 1$ for $j \neq l$. The condition (SFS) combined with the Arzelà theorem easily gives $\|\cdot\|_{\sup}$-compactness of $\overline{P^* B_1}|_K$, where B_1 stands for the unit ball of $L^\infty(\mu)$. Hence, k is bounded and there are only finitely many ergodic P-invariant densities $f_1, \ldots, f_r$.

For fixed $1 \le j \le r$ we show that P is asymptotically periodic on $L^1(F_j)$. First we notice that each trajectory

$$\gamma(f) = \{P^n f\}_{n \ge 0}, \quad \text{where} \quad f \in L^1(F_j),$$

is L^1-norm relatively compact. We may confine discussion to $0 \le f \le f_j$. Clearly $\gamma(f)$ is weakly compact, which follows from invariance and weak compactness of the order interval $[0, f_j] = \{f \in L^1(F_j) : 0 \le f \le f_j\}$ (see [11], II.5.10). Let $P^{n_l} f$ be an arbitrary sequence. We choose a subsequence $P^{n_{l_m}} f$ which is weakly convergent to $\tilde{f}$. Suppose $P^{n_{l_m}} f$ is not norm relatively compact. Choosing a further subsequence if necessary we may assume that

$$\|P^{n_{l_{m+1}}+1} f - P^{n_{l_m}+1} f\| > \varepsilon$$

for some $\varepsilon > 0$ and all m. By Prokhorov's theorem the sequence of densities $P^n f$ is tight. Hence there exists a compact set $K_\varepsilon \subseteq X$ such that for all n,

$$\int_{X \setminus K_\varepsilon} P^n f \, d\mu \le \varepsilon/4.$$

Now we find $h_m \in L^\infty(F_j)$ with $|h_m| \le 1$ so that

$$\int_X P(P^{n_{l_{m+1}}} f - P^{n_{l_m}} f) h_m \, d\mu > \varepsilon.$$

Then

$$\int\limits_{K_\varepsilon} (P^{n_{l_{m+1}}} f - P^{n_{l_m}} f) P^* h_m \, d\mu \ge \varepsilon/2.$$

As before $\{P^* h_m|_{K_\varepsilon}\}_{m=1}^\infty$ is relatively compact for the uniform convergence on K_ε. Choosing again a subsequence we may assume that $P^* h_m \to h$ uniformly on K_ε. This leads to a contradiction as

$$\begin{aligned} \varepsilon/2 &\le \overline{\lim_{m\to\infty}} \int\limits_{K_\varepsilon} (P^{n_{l_{m+1}}} f - P^{n_{l_m}} f) P^* h_m \, d\mu \\ &= \lim_{m\to\infty} \int\limits_{K_\varepsilon} (P^{n_{l_{m+1}}} f - P^{n_{l_m}} f) h \, d\mu = 0. \end{aligned}$$

We denote by Ω_j the subspace of all L^1-norm recurrent $f \in L^1(F_j)$. It is well known that Ω_j consists of all limit vectors in $L^1(F_j)$ (see [3], [4] for all details). Given a sequence $\underline{n} = n_k \nearrow \infty$ we denote the by $\Omega_{\underline{n}}$ the closed sublattice of Ω_j consisting of all vectors f which are recurrent along the sequence n_k (i.e. $\|P^{n_k} f - f\| \to 0$ as $k \to \infty$). We notice that regardless of the dimension of $\Omega_{\underline{n}}$, for every compact $C \subseteq X$ the restricted sublattice $\Omega_{\underline{n}}|_C$ is finite-dimensional. In fact, $\dim \Omega_{\underline{n}}|_C \le r_C$, where r_C denotes the largest j such that there are $0 \le h_1, \ldots, h_j \le 1$, $h_l \in P^* B_1$, with

$$\sup_{x\in C} |h_l(x) - h_{\widetilde{l}}(x)| = 1$$

for distinct $l, \widetilde{l}$ (it follows from (SFS) that r_C is finite). Let $\widetilde{g}_1 = \beta_1 g_1|_C, \ldots, \widetilde{g}_{r_C} = \beta_{r_C} g_{r_C}|_C$ form a normalized, positive and orthogonal basis in $\Omega_{\underline{n}}|_C$ (for some $\beta_l \ge 1$ and $g_l \in \Omega_{\underline{n}}$). Given $\varepsilon > 0$ we find a compact set $C = C_\varepsilon \subseteq X$ such that

$$\int\limits_C f_j \, d\mu > 1 - \varepsilon.$$

It follows from the ergodicity of f_j that for each density $g \in \Omega_{\underline{n}}$ we have $A_n g \to f_j$ in $L^1(F_j)$. Hence there exists n such that

$$\int\limits_C P^n g \, dx > 1 - \varepsilon.$$

We have

$$P^n g|_C = \sum_{l=1}^{r_C} \alpha_l \widetilde{g}_l, \quad \text{where} \quad \alpha_l \ge 0, \text{ and } 1 \ge \sum_{l=1}^{r_C} \alpha_l > 1 - \varepsilon.$$

Equivalently, for each $g \in \Omega_{\underline{n}}$ there is a natural n so that

$$\operatorname{dist}(P^n g, \operatorname{conv}\{\widetilde{g}_1, \ldots, \widetilde{g}_{r_C}, 0\}) < \varepsilon.$$

Therefore,

$$\operatorname{dist}(P^{n+k}g, \mathcal{F}_{\varepsilon,\underline{n}}) \le \varepsilon \quad \text{for all } k \ge 0,$$

where $\mathcal{F}_{\varepsilon,\underline{n}}$ denotes the L^1-norm closure of the set

$$\Big\{ \sum_{l=1}^{r_C} \alpha_l P^k \widetilde{g}_l : k = 0,1,2,\ldots,\ \sum_{l=1}^{r_C} \alpha_l \le 1,\ \alpha_l \ge 0 \Big\}.$$

As all trajectories in $L^1(F_j)$ are norm relatively compact the set $\mathcal{F}_{\varepsilon,\underline{n}}$ is compact. Clearly it is P-invariant. Hence by recurrence of $P^n g$ we obtain

$$\operatorname{dist}(g, \mathcal{F}_{\varepsilon,\underline{n}}) \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that the set of all densities from $\Omega_{\underline{n}}$ is relatively compact, and $\Omega_{\underline{n}}$ is finite-dimensional with $\dim \Omega_{\underline{n}} \le r_C$. Moreover, P has a positive inverse on $\Omega_{\underline{n}}$, so from the general theory of Markov operators P permutes vectors of a unique, positive, normalized and orthogonal basis in $\Omega_{\underline{n}}$. In particular, P is periodic (i.e. $P^{\mathrm{d}} = \mathrm{Id}$, where $\mathrm{d} = \mathrm{d}(\underline{n})$ depends on $\underline{n}$) on $\Omega_{\underline{n}}$.

For arbitrary $\Omega_{\underline{n}}$, $\Omega_{\underline{m}}$ we may find d (for instance $\mathrm{d} = \mathrm{d}(\underline{n}) \cdot \mathrm{d}(\underline{m})$) such that $\Omega_{\underline{n}}, \Omega_{\underline{m}} \subseteq \Omega_{\{k\mathrm{d}\}}$. Hence,

$$\dim \Omega_j|_C = \dim\{f|_C : f \in \Omega_j\} \le r_C.$$

Repeating the arguments applied to $\Omega_{\underline{n}}|_C$, we construct a compact set $\mathcal{F}_\varepsilon$ such that

$$\operatorname{dist}(g, \mathcal{F}_\varepsilon) \le \varepsilon \quad \text{for all densities } g \in \Omega_j.$$

Since ε may be taken as small as we wish, Ω_j is finite-dimensional. For each density $f \in L^1(F_j)$ the iterates $P^n f$ are attracted to the set $\mathcal{D}_\mu \cap \Omega_j$, which obviously is norm compact. In particular, P is constrictive. By [5] (see also [2]–[4]), P is asymptotically periodic on $L^1(F_j)$. We easily extend this property to $L^1(F)$ where $F = \bigcup_{j=1}^r F_j$. ∎

We want to emphasize that if P satisfies (SFS) and P^* preserves $C_0(X)$, and F is nontrivial, then for each $f \in L^1(F)$ and $\varepsilon > 0$ there exists f_ε such that $\|f - f_\varepsilon\| \le \varepsilon$ and the trajectory $\gamma(f_\varepsilon)$ asymptotically becomes periodic (i.e. $\omega(f_\varepsilon) = \{\widetilde{f} : \underline{\lim}_{n\to\infty} \|P^n f_\varepsilon - \widetilde{f}\| = 0\}$ is finite, and P permutes $\omega(f_\varepsilon)$). Then we may say that P is *almost asymptotically periodic* on $L^1(F)$. In contrast to this, one can show that the substochastic operator $\widetilde{P}$ defined on $L^1(F^{\mathrm{c}})$ by $\widetilde{P} f = (Pf)|_{F^{\mathrm{c}}}$ (where $F^c = X \setminus F$) is Cesàro sweeping (consult [6] for the terminology). For general $f \in \mathcal{D}_\mu$ the asymptotic properties of the trajectory $\gamma(f)$ depend on

$$\delta(f) = \lim_{n\to\infty} \int_F P^n f \, d\mu.$$

If $\delta(f) > 0$ then an asymptotically nontrivial portion of $P^n f$ behaves periodically. The case when the quantity $\delta(f)$ is uniformly separated from 0, for all $f \in \mathcal{D}$, is discussed below.

Corollary 1. *Let P be a kernel stochastic operator on $L^1(\mu)$ satisfying (SFS) and such that P^* preserves $C_0(X)$. Then the following conditions are equivalent*:

(i) *P is asymptotically periodic on $L^1(\mu)$*,

(ii) *there exist a compact set $K \subseteq X$ and $\delta > 0$ such that*

$$\overline{\lim_{n\to\infty}} \int_K \frac{f + Pf + \ldots + P^{n-1}f}{n} d\mu > \delta \qquad \textit{for all } f \in \mathcal{D}_\mu.$$

Proof. Only (ii) implies (i) needs to be proved. By Theorem 1 it is enough to show that for each $f \in \mathcal{D}_\mu$ we have

$$\lim_{n\to\infty} \int_F P^n f d\mu = 1$$

(here we may repeat essentially the same arguments as in the proof of Theorem 1.3 in [1], but for the sake of completeness we provide a full proof). Choosing a subsequence if necessary we may assume that

$$\left(\frac{1}{n_k}\sum_{j=0}^{n_k-1} P^j f\right)\bigg|_K \xrightarrow[k\to\infty]{} f_*|_K$$

in the L^1-norm, where f_* is P-invariant. By (ii) we easily get $\delta < \|f_*|_K\|$. As a result, for every $f \in \mathcal{D}_\mu$ there is a natural n so that

$$\int_F P^n f \, d\mu > \delta.$$

Suppose that there exists $f \in \mathcal{D}_\mu$ with $\delta(f) < 1$. If m is large enough then

$$\int_F P^m f \, d\mu > \delta(f) - \frac{(1-\delta(f))\delta}{2}.$$

Consider

$$f_1 = \frac{\mathbf{1}_{F^c} P^m f}{\int_{F^c} P^m f \, d\mu}.$$

There is n such that

$$\int_F P^n f_1 \, d\mu = \frac{1}{\int_{F^c} P^m f \, d\mu} \int_F P^n(\mathbf{1}_{F^c} P^m f) \, d\mu > \delta.$$

Thus,

$$\begin{aligned}\int_F P^{n+m} f\,d\mu &= \int_F P^n(\mathbf{1}_F P^m f + \mathbf{1}_{F^c} P^m f)\,d\mu\\ &= \int_F P^n(\mathbf{1}_F P^m f)\,d\mu + \int_F P^n(\mathbf{1}_{F^c} P^m f)\,d\mu\\ &> \int_F P^m f\,d\mu + \delta \int_{F^c} P^m f\,d\mu\\ &\ge \delta(f) - \frac{(1-\delta(f))\delta}{2} + (1-\delta(f))\delta\\ &= \delta(f) + \frac{(1-\delta(f))\delta}{2} > \delta(f),\end{aligned}$$

contradicting the definition of $\delta(f)$. ■

Comment. We remark that all the above results remain valid for P being *strongly Feller* (i.e. P^*h is continuous for all bounded measurable h). In fact, it is well known (see Theorem 5.9 on p. 37 of [10]) that strong Feller implies (SFS) for P^2.

3. In this section we study asymptotic properties of the iterates of LMT operators. It has been just noticed that they are strong Feller in the strict sense. Since

$$k(x,y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) = 0$$

if

$$x \le \lambda_*^{-1}(y) = \inf\{0 \le z : \lambda(z) = y\},$$

and $\lambda_*^{-1}(y)$ tends to ∞ with y, it follows that P^* preserves $C_0(\mathbb{R}_+)$. Therefore the results of Section 2 are applicable.

THEOREM 2. *Let P be a LMT stochastic operator associated with H, Q, λ. Assume that there exist $a > 0$ and $\delta > 0$ so that*

$$\overline{\lim_{n\to\infty}} \int_0^a \frac{f + Pf + \ldots + P^{n-1}f}{n}\,dx > \delta \qquad \textit{for all } f \in \mathcal{D}.$$

Then

(a) $a_* = \sup\{x \ge 0 : \lambda(x) \le x\} < a$,

(b) $\mathrm{Fix}(P)$ *is finite-dimensional and* $\lim_{n\to\infty} \|P^n f - Sf\| = 0$ *for all* $f \in L^1(\mathbb{R}_+)$, *where S is a stochastic projection onto* $\mathrm{Fix}(P)$,

(c) $\dim \mathrm{Fix}(P) \le \lceil a/T(P,a)\rceil$, *where* $T(P,r) = \sup\{t > 0 :$ *if* $0 \le y, \widetilde{y} \le r$ *and* $|y-\widetilde{y}| \le t$ *then* $\|k(\cdot,y) - k(\cdot,\widetilde{y})\| < 2\}$ *and* $\lceil z\rceil$ *stands for the smallest natural number greater than or equal to z. In particular, P is asymptotically stable if* $T(P,a) \ge a$.

P r o o f. By Corollary 1 the operator P is asymptotically periodic. If $\lambda(x) \le x$ then the space $L^1([x,\infty))$ is P-invariant. By easy calculations,

$$P^*\mathbf{1}_{[c,d)}(y)$$
$$= \begin{cases} H(Q(\lambda(c)) - Q(y)) - H(Q(\lambda(d)) - Q(y)) & \text{if } 0 \le y < \lambda(c), \\ 1 - H(Q(\lambda(d)) - Q(y)) & \text{if } \lambda(c) \le y < \lambda(d), \\ 0 & \text{if } \lambda(d) \le y. \end{cases}$$

If $\lambda(c) \le c$ then substituting $d = \infty$ we get

$$P^*\mathbf{1}_{[c,\infty)}(y) \ge \mathbf{1}_{[c,\infty)}(y) \quad \text{ for all } y.$$

Hence the set $\{x : \lambda(x) \le x\}$ must be bounded and a_* is finite. Now it is clear that

$$\lambda(a_*) = a_* \quad \text{and} \quad a_* < a.$$

Let $g_1, \ldots, g_r$ be a basis of positive, normalized and pairwise orthogonal functions in the space Ω of all recurrent elements and $g_1, \ldots, g_l$ be a cycle (i.e. $Pg_j = g_{j+1}$ for $1 \le j \le l$, where $j+1$ is understood modulo l). Define $D_j = \operatorname{supp} g_j$ and $c_j = \operatorname{ess\,inf} D_j$. Then we have

$$(\beta) \qquad P^*\mathbf{1}_{D_j}(y) = \begin{cases} 1 & \text{if } y \in \overline{D}_{j-1}, \\ 0 & \text{for all } y \in \overline{D}_s \text{ if } s \ne j-1. \end{cases}$$

We may assume that $\max\{c_1, \ldots, c_l\} = c_l$. Thus,

$$P^*\mathbf{1}_{[c_l,\infty)} \ge P^*\mathbf{1}_{D_l} \ge \mathbf{1}_{D_{l-1}}.$$

By continuity,

$$P^*\mathbf{1}_{[c_l,\infty)}(c_{l-1}) = P^*\mathbf{1}_{D_l}(c_{l-1}) = 1.$$

Since

$$P^*\mathbf{1}_{[c_l,\infty)}(y) = \begin{cases} H(Q(\lambda(c_l)) - Q(y)), & 0 \le y \le \lambda(c_l), \\ 1 & \text{otherwise}, \end{cases}$$

we conclude that

$$H(Q(\lambda(c_l)) - Q(y)) = 1 \quad \text{ for all } c_{l-1} \le y \le \lambda(c_l).$$

Therefore

$$P^*\mathbf{1}_{[c_l,\infty)} \ge \mathbf{1}_{[c_{l-1},\infty)} \ge \mathbf{1}_{[c_l,\infty)}.$$

This implies that $L^1([c_l,\infty))$ is P-invariant. Since $g_1, \ldots, g_l$ form a cycle it is possible only if $c_1 = c_2 = \ldots = c_l$. Hence $l = 1$, since by (β) the continuous functions $P^*\mathbf{1}_{D_j}$ would take values 0 and 1 arbitrary close to c_l. Repeating the previous discussion for other cycles, one shows that each of them reduces to a singleton, and the convergence

$$\lim_{n\to\infty} \|P^n f - Sf\| = 0$$

follows. Clearly S is a finite-dimensional stochastic projection onto $\Omega = \mathrm{Fix}(P)$. Let $F_1, \ldots, F_r$ be the supports of ergodic densities. We have

$$\|k(\cdot, y) - k(\cdot, \widetilde{y})\| = 2$$

if $y, \widetilde{y}$ are taken from distinct sets $F_j \cap [0, a]$. This yields the estimate

$$\dim(S) \le \lceil 1/T(P, a)\rceil. \ \blacksquare$$

Combining [1], Theorem 2.1, with our Theorem 2 we immediately get

COROLLARY 2. *Let P be a LMT stochastic operator and suppose there exist positive ε, ϱ, a and $0 < \alpha \le 1$ such that*

$$\varepsilon + \int_0^\infty x^\alpha h(x)dx < \varrho < Q(\lambda(t))^\alpha - Q(t)^\alpha \quad \text{for all } t \ge a.$$

Then there exists a finite-dimensional stochastic projection S such that

$$\lim_{n\to\infty} \|P^n f - Sf\| = 0 \quad \text{for all } f \in L^1(\mathbb{R}_+).$$

Moreover, $\dim(S) \le \lceil a/T(P, a)\rceil$.

Proof. By [1] (see the proof of Theorem 2.1) for every $f \in \mathcal{D}$ there exists a natural $n_0(f)$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\int_0^a P^j f dx \ge \frac{\varepsilon}{2M} \quad \text{for all } n \ge n_0(f),$$

where $M = \sup\{|Q(\lambda(x))^\alpha - Q(x)^\alpha - \varrho| : 0 \le x \le a\}$. Now we can apply Theorem 2. $\blacksquare$

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