Convergence of iterates of Lasota–Mackey–Tyrcha operators
by Wojciech Bartoszek (Pretoria)

Abstract. We provide sufficient and necessary conditions for asymptotic periodicity of iterates of strong Feller stochastic operators.

1. Let $(X,d)$ be a locally compact, metric, Polish space and $\mathcal{B}$ denote the $\sigma$-algebra of Borel subsets in $X$. Given a $\sigma$-finite measure $\mu$ on $(X,\mathcal{B})$ we denote by $(L^1(\mu),\|\cdot\|)$ the Banach lattice of $\mu$-integrable functions on $X$. Functions which are equal $\mu$-almost everywhere are identified. A linear operator $P$ on $L^1(\mu)$ is called stochastic (or Markov according to Lasota’s terminology) if

$$ Pf \geq 0 \quad \text{and} \quad \int_X Pf \, d\mu = 1 $$

for all nonnegative and normalized (densities) $f \in L^1(\mu)$. The convex set of all densities is denoted by $D_\mu$ (simply $D$ if $X = \mathbb{R}_+$ and $\mu$ is the Lebesgue measure on $\mathbb{R}_+$). If there exists a Borel measurable function $k : X \times X \to \mathbb{R}_+$ such that

$$ Pf(x) = \int_X k(x,y)f(y) \, d\mu(y) $$

then $P$ is called a kernel operator.

We notice that each kernel stochastic operator may be extended to the Banach lattice $M(X)$ of all bounded signed Borel measures on $(X,\mathcal{B})$. Namely, if $\nu \in M(X)$ and $A$ is Borel we define
\[ P\nu(A) = \int_X \int_X k(x, y) 1_A(x) \, d\mu(x) \, d\nu(y). \]

Obviously \( P\nu \in L^1(\mu). \)

The paper is particularly devoted to stochastic kernel operators on \( L^1(\mathbb{R}_+) \) with kernels
\[
(*) \quad k(x, y) = \begin{cases} 
\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) & \text{if } 0 \leq y \leq \lambda(x) \\
0 & \text{otherwise.}
\end{cases}
\]

They appear in mathematical modelling of the cell cycle. A systematic study of the asymptotic properties of iterates of \( (*) \) is being continued by Lasota and his collaborators. The reader is referred to [6] for a comprehensive and updated review of the subject. Here we shall concentrate on the mathematical side rather than on biological applications. Our paper often refers to [1]. Following it we shall assume:

\begin{align*}
(H) & \quad H : [0, \infty) \rightarrow [0, \infty) \text{ is nonincreasing and absolutely continuous, } H(0) = 1 \text{ and } \lim_{x \rightarrow \infty} H(x) = 0, \\
(Q\lambda) & \quad Q : [0, \infty) \rightarrow [0, \infty) \text{ and } \lambda : [0, \infty) \rightarrow [0, \infty) \text{ are nondecreasing, absolutely continuous, } Q(0) = \lambda(0) = 0 \text{ and } \lim_{x \rightarrow \infty} Q(x) = \lim_{x \rightarrow \infty} \lambda(x) = \infty.
\end{align*}

The class of stochastic operators \( P \) with kernels \( (*) \) satisfying \( (H) \) and \( (Q\lambda) \) is denoted by LMT (Lasota, Mackey, Tyrcha (cf. [7]) who contributed much to the discussed matters). It has recently been proved in [1] that if a LMT stochastic operator \( P \) additionally satisfies:

\begin{align*}
(\alpha) & \quad \int_0^\infty x^\alpha h(x) \, dx < \lim \inf_{x \rightarrow \infty} Q(\lambda(x))^\alpha - Q(x)^\alpha \text{ for some } 0 < \alpha \leq 1, \\
& \quad \text{where } h(x) = -dH(x)/dx \text{ for almost all } x, \text{ and}
\end{align*}

\begin{align*}
(c) & \quad \text{there exists a nonnegative } c \text{ such that } h(x) > 0 \text{ for almost all } x \geq c,
\end{align*}

then there exists a unique \( f_* \in \mathcal{D} \) such that

\[ \lim_{n \rightarrow \infty} \| P^n f - f_* \| = 0 \quad \text{for all } f \in \mathcal{D} \]

\((P \text{ is asymptotically stable})\). In this paper we drop condition \((c)\) and prove that the iterates of a LMT operator with \( (H), (Q\lambda) \) and \((\alpha)\) are strong operator topology convergent to a finite-dimensional projection (with a slight abuse of the terminology such operators are also called stable (cf. [9])).

We begin with considering a general case. Let us recall (cf. [10]) that a kernel stochastic operator \( P \) on \( L^1(\mu) \) is called \textit{strong Feller in the strict sense} if
Convergence of iterates

(SFS) \( X \ni y \rightarrow k(\cdot, y) \in D_\mu \) is \( L^1 \)-norm continuous.

Note that (SFS) implies the continuity of \( P^* h \), where \( h \in L^\infty(\mu) \) and \( P^* \) stands for the adjoint operator. This easily follows from \( P^* h(y) = \int_X k(x, y) h(x) \, d\mu(x) \). It is also well known that if \( X \) is compact then (SFS) kernel stochastic operators are compact (see [10]). More details concerning asymptotic properties of iterates of compact (or quasi-compact) positive contractions on Banach lattices can be found in [3] and [4].

If \( X \) is not compact then (SFS) does not guarantee automatically any regularity of the trajectories \( P^nf \). For instance, it may happen that for some \( f \in D_\mu \) the sequence \( P^nf \) converges to a density, while for other \( f \) we have \( \int_K P^nf \, d\mu \to 0 \) for every compact \( K \subset X \). Roughly speaking, starting from “good” states the process is rather concentrated, but starting from “bad” states it escapes to “infinity”. Also all mixed situations may occur. The so-called Doeblin condition is never satisfied if the transition kernels \( k(\cdot, \cdot) \) do not allow “long jumps” (i.e. if \( d(y, z) \to \infty \) implies \( \int_X |k(x, y) - k(x, z)| \, d\mu(x) \to 0 \)).

For noncompact \( X \), in order to obtain asymptotic regularity of iterates of (SFS) stochastic operators, we must impose some extra assumptions. Following [5] we say that a stochastic operator \( P \) on \( L^1(\mu) \) is asymptotically periodic if there exist densities \( g_1, \ldots, g_r \in L^1(\mu) \) with disjoint supports, functionals \( A_1, \ldots, A_r \) on \( L^1(\mu) \) and a permutation \( \alpha \) of \( \{1, \ldots, r\} \) so that for all \( f \in L^1(\mu) \) we have

\[
\lim_{n \to \infty} \left\| P^n f - \sum_{j=1}^r A_j(f) g_{\alpha^n(j)} \right\| = 0.
\]

\( P \) is said to be constrictive if there exists an \( L^1 \)-norm compact set \( \mathcal{F} \subseteq D_\mu \) such that \( \text{dist}(P^n f, \mathcal{F}) \to 0 \) for all \( f \in D_\mu \). It has been proved in [5] that each constrictive stochastic operator \( P \) on \( L^1(\mu) \) is asymptotically periodic.

Given a (SFS) stochastic operator \( P \) on \( L^1(\mu) \) we identify here an invariant sublattice on which \( P \) is asymptotically periodic. This sublattice appears to be trivial exactly when for each compact \( K \subset X \) there exists \( f \in D_\mu \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N-1} \int_K P^j f \, d\mu = 0.
\]

Condition (SFS) is usually easy to verify. We remark that many important kernels used in mathematical modelling of biological systems have this property. For instance, using [8], Theorem 7.4.8, we easily check that if \( y_n \to y \) then
\[
\int_0^\infty \left| \frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) - \frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y_n)) \right| \, dx \\
= \int_0^\infty |h(Q(\lambda(x)) - Q(y)) - h(Q(\lambda(x)) - Q(y_n))|(Q \circ \lambda)'(x) \, dx \\
= \int_0^\infty |h(t - Q(y)) - h(t - Q(y_n))| \, dt \xrightarrow{n \to \infty} 0,
\]
with our convention that \( h(x) \equiv 0 \) if \( x \leq 0 \). Hence LMT operators satisfy (SFS).

2. The purpose of this section is to show asymptotic periodicity of (SFS) operators. The reader can view it as a generalization of [2].

We denote by \( C_0(X) \) the Banach lattice of all continuous functions \( h \) on \( X \) such that for every \( \varepsilon > 0 \) there exists a compact set \( E_\varepsilon \subseteq X \) such that \( |h(x)| \leq \varepsilon \) for all \( x \notin E_\varepsilon \) (endowed with the ordinary sup-norm \( \| \cdot \|_{\text{sup}} \)). Given a stochastic operator \( P \) we denote by \( F \) the minimal (modulo sets of measure zero) measurable set which carries supports of all \( P \)-invariant densities (its existence follows from separability of the \( L^1(\mu) \)). Obviously \( L^1(F) \) is \( P \)-invariant.

The next result, which will be the main ingredient of the proofs in Section 3, is also of some independent interest.

**Theorem 1.** Let \( P \) be a (SFS) stochastic operator on \( L^1(\mu) \) such that \( P^* \) preserves \( C_0(X) \). If

(i) there exists a compact set \( K \subseteq X \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \int_K \sum_{j=0}^{N-1} P^j f \, d\mu > 0 \quad \text{for all } f \in D_\mu,
\]

then \( F \) is nontrivial and \( P \) is asymptotically periodic on \( L^1(F) \). In particular, there are only finitely many \( P \)-invariant ergodic densities.

**Proof.** The set of all subprobabilistic positive measures on \( X \) is a compact convex set with respect to the vague topology (we say that a variation norm bounded sequence of measures \( \nu_n \) is ***vaguely convergent*** to \( \nu \) if \( \lim_{n \to \infty} \int_X h \, d\nu_n = \int_X h \, d\nu \) for all \( h \in C_0(X) \)). Given \( f \in D_\mu \) we may choose a sequence \( n_k \not\to \infty \) so that the measures with densities

\[
\frac{1}{n_k} \sum_{j=0}^{n_k-1} P^j f = A_{n_k} f
\]

are vaguely convergent. By (i) the limit \( \nu \) is nonzero and \( PA_{n_k} f \) tends to
Convergence of iterates

$P\nu$ vaguely. Since
\[
\|A_n f - P A_n f\| = \left\| \frac{P^n f - f}{n_k} \right\| \to 0
\]
we conclude that $\nu = P\nu \in L^1(\mu)$ is a fixed point of $P$. Normalizing $\nu$ if necessary we obtain a $P$-invariant density.

Now we show that the linear subspace (sublattice) $\text{Fix}(P)$ of all $P$-invariant functions is finite-dimensional. Assume we are given pairwise orthogonal $P$-invariant densities $f_1, \ldots, f_k$. By (i) we have $\int K f_j d\mu > 0$. Consider the following family of (restricted to $K$) continuous functions:
\[
g_j = (P^* 1_{F_j})|_K, \quad \text{where} \quad F_j = \text{supp}(f_j).
\]
Clearly $g_j(x) = 1$ for all $x \in F_j \cap K$, and $g_j(x) = 0$ if $x \in \bigcup_{l \neq j} F_l \cap K$.

As a result, $\|g_j - g_l\|_{\text{sup}} = 1$ for $j \neq l$. The condition (SFS) combined with the Arzelà theorem easily gives $\|\cdot\|_{\text{sup}}$-compactness of $P^* B_1|_K$, where $B_1$ stands for the unit ball of $L^\infty(\mu)$. Hence, $k$ is bounded and there are only finitely many ergodic $P$-invariant densities $f_1, \ldots, f_r$.

For fixed $1 \leq j \leq r$ we show that $P$ is asymptotically periodic on $L^1(F_j)$. First we notice that each trajectory
\[
\gamma(f) = \{P^n f\}_{n \geq 0}, \quad \text{where} \quad f \in L^1(F_j),
\]
is $L^1$-norm relatively compact. We may confine discussion to $0 \leq f \leq f_j$. Clearly $\gamma(f)$ is weakly compact, which follows from invariance and weak compactness of the order interval $[0, f_j] = \{f \in L^1(F_j) : 0 \leq f \leq f_j\}$ (see [11], II.5.10). Let $P^{n_i} f$ be an arbitrary sequence. We choose a subsequence $P^{n_{l_m}} f$ which is weakly convergent to $\tilde{f}$. Suppose $P^{n_{l_m}} f$ is not norm relatively compact. Choosing a further subsequence if necessary we may assume that
\[
\|P^{n_{l_m} + 1} f - P^{n_{l_{m+1}}} f\| > \epsilon
\]
for some $\epsilon > 0$ and all $m$. By Prokhorov’s theorem the sequence of densities $P^n f$ is tight. Hence there exists a compact set $K_\epsilon \subseteq X$ such that for all $n$,
\[
\int_{X \setminus K_\epsilon} P^n f d\mu \leq \epsilon / 4.
\]
Now we find $h_m \in L^\infty(F_j)$ with $|h_m| \leq 1$ so that
\[
\int_X P(P^{n_{l_{m+1}}} f - P^{n_{l_m}} f) h_m d\mu > \epsilon.
\]
Then
\[ \int_{K_\varepsilon} (P^{m_{m+1}} f - P^{m_m} f) P^* h_m \, d\mu \geq \varepsilon/2. \]

As before \( \{P^* h_m|_{K_\varepsilon}\}_{m=1}^\infty \) is relatively compact for the uniform convergence on \( K_\varepsilon \). Choosing again a subsequence we may assume that \( P^* h_m \to h \) uniformly on \( K_\varepsilon \). This leads to a contradiction as
\[ \frac{\varepsilon}{2} \leq \lim_{m \to \infty} \int_{K_\varepsilon} (P^{m_{m+1}} f - P^{m_m} f) P^* h_m \, d\mu = \lim_{m \to \infty} \int_{K_\varepsilon} (P^{m_{m+1}} f - P^{m_m} f) h \, d\mu = 0. \]

We denote by \( \Omega_j \) the subspace of all \( L^1 \)-norm recurrent \( f \in L^1(F_j) \). It is well known that \( \Omega_j \) consists of all limit vectors in \( L^1(F_j) \) (see [3], [4] for all details). Given a sequence \( n = n_k \not\to \infty \) we denote the by \( \Omega_n \) the closed sublattice of \( \Omega_j \) consisting of all vectors \( f \) which are recurrent along the sequence \( n_k \) (i.e. \( \|P^{n_k} f - f\| \to 0 \) as \( k \to \infty \)). We notice that regardless of the dimension of \( \Omega_n \), for every compact \( C \subseteq X \) the restricted sublattice \( \Omega_n|_C \) is finite-dimensional. In fact, \( \text{dim} \Omega_n|_C \leq r_C \), where \( r_C \) denotes the largest \( j \) such that there are \( 0 \leq h_1, \ldots, h_j \leq 1, h_l \in P^* B_1 \), with
\[ \sup_{x \in C} |h_l(x) - h_{\tilde{l}}(x)| = 1 \]
for distinct \( l, \tilde{l} \) (it follows from (SFS) that \( r_C \) is finite). Let \( \tilde{g}_1 = \beta_1 g_{1|C}, \ldots, \tilde{g}_{r_C} = \beta_{r_C} g_{r_C|C} \) form a normalized, positive and orthogonal basis in \( \Omega_{2|C} \) (for some \( \beta_l \geq 1 \) and \( g_l \in \Omega_n \)). Given \( \varepsilon > 0 \) we find a compact set \( C = C_\varepsilon \subseteq X \) such that
\[ \int_C f_j \, d\mu > 1 - \varepsilon. \]

It follows from the ergodicity of \( f_j \) that for each density \( g \in \Omega_n \) we have \( A_n g \to f_j \) in \( L^1(F_j) \). Hence there exists \( n \) such that
\[ \int_C P^n g \, dx > 1 - \varepsilon. \]

We have
\[ P^n g|_C = \sum_{l=1}^{r_C} \alpha_l \tilde{g}_l, \quad \text{where} \quad \alpha_l \geq 0, \quad \text{and} \quad 1 \geq \sum_{l=1}^{r_C} \alpha_l > 1 - \varepsilon. \]

Equivalently, for each \( g \in \Omega_n \) there is a natural \( n \) so that
\[ \text{dist}(P^n g, \text{conv}\{\tilde{g}_1, \ldots, \tilde{g}_{r_C}, 0\}) < \varepsilon. \]
Therefore,
\[ \text{dist}(P^{n+k} g, \mathcal{F}_{\varepsilon,n}) \leq \varepsilon \quad \text{for all } k \geq 0, \]
where \( \mathcal{F}_{\varepsilon,n} \) denotes the \( L^1 \)-norm closure of the set
\[ \left\{ \sum_{i=1}^{r_C} \alpha_i P^k \tilde{g}_i : k = 0, 1, 2, \ldots, \sum_{i=1}^{r_C} \alpha_i \leq 1, \alpha_i \geq 0 \right\}. \]
As all trajectories in \( L^1(F_j) \) are norm relatively compact the set \( \mathcal{F}_{\varepsilon,n} \) is compact. Clearly it is \( P \)-invariant. Hence by recurrence of \( P^n g \) we obtain
\[ \text{dist}(g, \mathcal{F}_{\varepsilon,n}) \leq \varepsilon. \]
Since \( \varepsilon > 0 \) is arbitrary, this implies that the set of all densities from \( \Omega_n \) is relatively compact, and \( \Omega_n \) is finite-dimensional with \( \dim \Omega_n \leq r_C \). Moreover, \( P \) has a positive inverse on \( \Omega_n \), so from the general theory of Markov operators \( P \) permutes vectors of a unique, positive, normalized and orthogonal basis in \( \Omega_n \). In particular, \( P \) is periodic (i.e. \( P^d = \text{Id} \), where \( d = d(n) \) depends on \( n \)) on \( \Omega_n \).

For arbitrary \( \Omega_n, \Omega_m \) we may find \( d \) (for instance \( d = d(n) \cdot d(m) \)) such that \( \Omega_n, \Omega_m \subseteq \Omega_{(kd)} \). Hence,
\[ \dim \Omega_j|C = \dim \{ f|C : f \in \Omega_j \} \leq r_C. \]
Repeating the arguments applied to \( \Omega_n|C \), we construct a compact set \( \mathcal{F}_\varepsilon \) such that
\[ \text{dist}(g, \mathcal{F}_\varepsilon) \leq \varepsilon \quad \text{for all densities } g \in \Omega_j. \]
Since \( \varepsilon \) may be taken as small as we wish, \( \Omega_j \) is finite-dimensional. For each density \( f \in L^1(F_j) \) the iterates \( P^n f \) are attracted to the set \( D_\mu \cap \Omega_j \), which obviously is norm compact. In particular, \( P \) is constrictive. By [5] (see also [2]–[4]), \( P \) is asymptotically periodic on \( L^1(F_j) \). We easily extend this property to \( L^1(F) \) where \( F = \bigcup_{j=1}^{r_C} F_j. \)

We want to emphasize that if \( P \) satisfies (SFS) and \( P^* \) preserves \( C_0(X) \), and \( F \) is nontrivial, then for each \( f \in L^1(F_j) \) the iterates \( P^n f \) are attracted to the set \( D_\mu \cap \Omega_j \), which obviously is norm compact. In particular, \( P \) is constrictive. By [5] (see also [2]–[4]), \( P \) is asymptotically periodic on \( L^1(F_j) \). We easily extend this property to \( L^1(F) \) where \( F = \bigcup_{j=1}^{r_C} F_j. \)

In contrast to this, one can show that the substochastic operator \( \tilde{P} \) defined on \( L^1(F^c) \) by \( \tilde{P} f = (P f)|_{F^c} \) (where \( F^c = X \setminus F \)) is Cesàro sweeping (consult [6] for the terminology). For general \( f \in D_\mu \) the asymptotic properties of the trajectory \( \gamma(f) \) depend on
\[ \delta(f) = \lim_{n \to \infty} \int_F P^n f \, d\mu. \]
If $\delta(f) > 0$ then an asymptotically nontrivial portion of $P^n f$ behaves periodically. The case when the quantity $\delta(f)$ is uniformly separated from 0, for all $f \in \mathcal{D}$, is discussed below.

**Corollary 1.** Let $P$ be a kernel stochastic operator on $L^1(\mu)$ satisfying (SFS) and such that $P^*$ preserves $C_0(X)$. Then the following conditions are equivalent:

1. $P$ is asymptotically periodic on $L^1(\mu)$,
2. there exist a compact set $K \subseteq X$ and $\delta > 0$ such that
   \[
   \lim_{n \to \infty} \int_K \frac{f + Pf + \ldots + P^{n-1}f}{n} \, d\mu > \delta \quad \text{for all } f \in \mathcal{D}_\mu.
   \]

**Proof.** Only (ii) implies (i) needs to be proved. By Theorem 1 it is enough to show that for each $f \in \mathcal{D}_\mu$ we have
\[
\lim_{n \to \infty} \int_F P^n f \, d\mu = 1
\]
(here we may repeat essentially the same arguments as in the proof of Theorem 1.3 in [1], but for the sake of completeness we provide a full proof). Choosing a subsequence if necessary we may assume that
\[
\left( \frac{1}{n_k} \sum_{j=0}^{n_k-1} P^j f \right) \biggm|_{K} \xrightarrow{k \to \infty} f_*|_K
\]
in the $L^1$-norm, where $f_*$ is $P$-invariant. By (ii) we easily get $\delta < \|f_*|_K\|$. As a result, for every $f \in \mathcal{D}_\mu$ there is a natural $n$ so that
\[
\int_F P^n f \, d\mu > \delta.
\]
Suppose that there exists $f \in \mathcal{D}_\mu$ with $\delta(f) < 1$. If $m$ is large enough then
\[
\int_F P^m f \, d\mu > \delta(f) - \frac{1 - \delta(f))\delta}{2}.
\]
Consider
\[
f_1 = \frac{1_{F^c} P^m f}{\int_{F^c} P^m f \, d\mu}.
\]
There is $n$ such that
\[
\int_F P^n f_1 \, d\mu = \frac{1}{\int_{F^c} P^m f \, d\mu} \int_F P^n (1_{F^c} P^m f) \, d\mu > \delta.
\]
Thus,
\[ \int_F P^{n+m} f \, d\mu = \int_F P^n (1_F P^m f + 1_{F^c} P^m f) \, d\mu \]
\[ = \int_F P^n (1_F P^m f) \, d\mu + \int_F P^n (1_{F^c} P^m f) \, d\mu \]
\[ > \int_F P^m f \, d\mu + \delta \int_F P^m f \, d\mu \]
\[ \geq \delta(f) - \frac{(1 - \delta(f))\delta}{2} + (1 - \delta(f))\delta \]
\[ = \delta(f) + \frac{(1 - \delta(f))\delta}{2} > \delta(f), \]
contradicting the definition of \( \delta(f) \).

**Comment.** We remark that all the above results remain valid for \( P \) being strongly Feller (i.e. \( P^* h \) is continuous for all bounded measurable \( h \)). In fact, it is well known (see Theorem 5.9 on p. 37 of [10]) that strong Feller implies (SFS) for \( P^* \).

3. In this section we study asymptotic properties of the iterates of LMT operators. It has been just noticed that they are strong Feller in the strict sense. Since
\[ k(x, y) = \frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) = 0 \]
if
\[ x \leq \lambda^{-1}_*(y) = \inf \{ 0 \leq z : \lambda(z) = y \}, \]
and \( \lambda^{-1}_*(y) \) tends to \( \infty \) with \( y \), it follows that \( P^* \) preserves \( C_0(\mathbb{R}_+) \). Therefore the results of Section 2 are applicable.

**Theorem 2.** Let \( P \) be a LMT stochastic operator associated with \( H, Q, \lambda \). Assume that there exist \( a > 0 \) and \( \delta > 0 \) so that
\[ \lim_{n \to \infty} \int_0^a \frac{f + Pf + \ldots + P^{n-1}f}{n} \, dx \geq \delta \quad \text{for all } f \in \mathcal{D}. \]
Then
(a) \( a_* = \sup \{ x \geq 0 : \lambda(x) \leq x \} < a \),
(b) \( \text{Fix}(P) \) is finite-dimensional and \( \lim_{n \to -\infty} \| P^n f - Sf \| = 0 \) for all \( f \in L^1(\mathbb{R}_+) \), where \( S \) is a stochastic projection onto \( \text{Fix}(P) \),
(c) \( \dim \text{Fix}(P) \leq \lceil a/T(P, a) \rceil \), where \( T(P, r) = \sup \{ t > 0 : 0 \leq y, \tilde{y} \leq r \text{ and } |y - \tilde{y}| \leq t \text{ then } \| k(\cdot, y) - k(\cdot, \tilde{y}) \| < 2 \} \) and \( \lceil z \rceil \) stands for the smallest natural number greater than or equal to \( z \). In particular, \( P \) is asymptotically stable if \( T(P, a) \geq a \).
Proof. By Corollary 1 the operator $P$ is asymptotically periodic. If \( \lambda(x) \leq x \) then the space \( L^1([x, \infty)) \) is \( P \)-invariant. By easy calculations,

\[
P^* 1_{[c,d)}(y) = \begin{cases} 
H(Q(\lambda(c)) - Q(y)) - H(Q(\lambda(d)) - Q(y)) & \text{if } 0 \leq y < \lambda(c), \\
1 - H(Q(\lambda(d)) - Q(y)) & \text{if } \lambda(c) \leq y < \lambda(d), \\
0 & \text{if } \lambda(d) \leq y.
\end{cases}
\]

If \( \lambda(c) \leq c \) then substituting \( d = \infty \) we get

\[P^* 1_{[c,\infty)}(y) \geq 1_{[c,\infty)}(y) \text{ for all } y.\]

Hence the set \( \{ x : \lambda(x) \leq x \} \) must be bounded and \( a_+ \) is finite. Now it is clear that

\[\lambda(a_+) = a_+ \text{ and } a_+ < a.\]

Let \( g_1, \ldots, g_r \) be a basis of positive, normalized and pairwise orthogonal functions in the space \( \Omega \) of all recurrent elements and \( g_1, \ldots, g_l \) be a cycle (i.e. \( Pg_j = g_{j+1} \) for \( 1 \leq j \leq l \), where \( j+1 \) is understood modulo \( l \)). Define \( D_j = \text{supp } g_j \) and \( c_j = \text{ess inf } D_j \). Then we have

\[(\beta) \quad P^* 1_{D_j}(y) = \begin{cases} 
1 & \text{if } y \in D_{j-1}, \\
0 & \text{for all } y \in D_s \text{ if } s \neq j - 1.
\end{cases}\]

We may assume that \( \max \{c_1, \ldots, c_l\} = c_l \). Thus,

\[P^* 1_{[c_l,\infty)} \geq P^* 1_{D_l} \geq 1_{D_{l-1}}.\]

By continuity,

\[P^* 1_{[c_l,\infty)}(c_l-1) = P^* 1_{D_l}(c_l-1) = 1.\]

Since

\[P^* 1_{[c_l,\infty)}(y) = \begin{cases} 
H(Q(\lambda(c_l)) - Q(y)), & 0 \leq y \leq \lambda(c_l), \\
1 & \text{otherwise},
\end{cases}\]

we conclude that

\[H(Q(\lambda(c_l)) - Q(y)) = 1 \text{ for all } c_{l-1} \leq y \leq \lambda(c_l).\]

Therefore

\[P^* 1_{[c_l,\infty)} \geq 1_{[c_{l-1},\infty)} \geq 1_{[c_1,\infty)}.\]

This implies that \( L^1([c_1, \infty)) \) is \( P \)-invariant. Since \( g_1, \ldots, g_l \) form a cycle it is possible only if \( c_1 = c_2 = \ldots = c_l \). Hence \( l = 1 \), since by \( (\beta) \) the continuous functions \( P^* 1_{D_j} \) would take values 0 and 1 arbitrary close to \( c_l \). Repeating the previous discussion for other cycles, one shows that each of them reduces to a singleton, and the convergence

\[\lim_{n \to \infty} \|P^n f - Sf\| = 0\]
follows. Clearly $S$ is a finite-dimensional stochastic projection onto $\Omega = \text{Fix}(P)$. Let $F_1, \ldots, F_r$ be the supports of ergodic densities. We have
$$\|k(\cdot, y) - k(\cdot, \tilde{y})\| = 2$$
if $y, \tilde{y}$ are taken from distinct sets $F_j \cap [0, a]$. This yields the estimate
$$\dim(S) \leq \lceil 1/T(P, a) \rceil.$$ Combining [1], Theorem 2.1, with our Theorem 2 we immediately get

**Corollary 2.** Let $P$ be a LMT stochastic operator and suppose there exist positive $\varepsilon, \varrho, a$ and $0 < \alpha \leq 1$ such that
$$\varepsilon + \int_0^a x^\alpha h(x)dx < \varrho < Q(\lambda(t))^\alpha - Q(t)^\alpha \quad \text{for all } t \geq a.$$ Then there exists a finite-dimensional stochastic projection $S$ such that
$$\lim_{n \to \infty} \|P^nf - Sf\| = 0 \quad \text{for all } f \in L^1(\mathbb{R}_+).$$ Moreover, $\dim(S) \leq \lfloor a/T(P, a) \rfloor$.

**Proof.** By [1] (see the proof of Theorem 2.1) for every $f \in D$ there exists a natural $n_0(f)$ such that
$$\frac{1}{n} \sum_{j=0}^{n-1} \int_0^a P^j f dx \geq \frac{\varepsilon}{2M} \quad \text{for all } n \geq n_0(f),$$
where $M = \sup\{|Q(\lambda(x))^\alpha - Q(x)^\alpha - \varrho| : 0 \leq x \leq a\}$. Now we can apply Theorem 2. \hfill \blacksquare

**References**


Department of Mathematics
University of South Africa
P.O. Box 392
Pretoria 0001, South Africa
E-mail: bartowk@risc5.unisa.ac.za

*Reçu par la Rédaction le 20.11.1994
Révisé le 26.4.1995*