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L^p-convergence of Bernstein–Kantorovich-type operators

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Abstract. We study a Kantorovich-type modification of the operators introduced in [1] and we characterize their convergence in the L^p-norm. We also furnish a quantitative estimate of the convergence.

In [1] and [2] we introduced a modification of classical Bernstein operators in $\mathcal{C}([0,1])$ which we used to approximate the solutions of suitable parabolic problems. These operators are defined by

$$(1)\quad A_n(f)(x) := \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} f(k/n), \qquad f \in \mathcal{C}([0,1]),\ x \in [0,1],$$

where the coefficients $\alpha_{n,k}$ satisfy the recursive formulas

$$(2)\qquad \alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k-1}, \qquad k = 1, \ldots, n,$$

$$(3)\qquad \alpha_{n,0} = \lambda_n, \qquad \alpha_{n,n} = \varrho_n,$$

and $(\lambda_n)_{n\in\mathbb{N}}, (\varrho_n)_{n\in\mathbb{N}}$ are fixed sequences of real numbers.

In [1], we investigated convergence and regularity properties of these operators; in particular, we found that $(A_n)_{n\in\mathbb{N}}$ converges strongly in $\mathcal{C}([0,1])$ if and only if $(\lambda_n)_{n\in\mathbb{N}}$ and $(\varrho_n)_{n\in\mathbb{N}}$ converge. In this case $A_n(f) \to w \cdot f$, for every $f \in \mathcal{C}([0,1])$, where

$$(4)\qquad w(x) = \sum_{m=1}^{\infty} (\lambda_m x(1-x)^m + \varrho_m x^m (1-x))$$

is continuous in [0,1] and analytic in]0,1[.

Connections with semigroup theory and evolution equations, via a Voronovskaya-type formula, have been explored in [2].

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In this paper we deal with a Kantorovich-type version of the operators (1) (see [3, p. 30]) and characterize the convergence in the L^p-norm giving also a quantitative estimate.

Let $1 \le p < \infty$ and define an operator $K_n : L^p([0,1]) \to L^p([0,1])$ by

$$K_n(f)(x) := \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt, \tag{5}$$

for every $f \in L^p([0,1])$ and $x \in [0,1]$, where the coefficients $\alpha_{n,k}$ satisfy (2) and (3).

If $\lambda_m = \varrho_m = 1$ for every $m = 1, \dots, n$, then $\alpha_{n,k} = \binom{n}{k}$, $k = 0, \dots, n$, whence the operator K_n becomes the well-known nth *Bernstein–Kantorovich operator* on $L^p([0,1])$ (see, e.g., [3, p. 31]), in the sequel denoted by U_n.

We define

$$s(n) := \max_{m \le n} \{|\lambda_m|, |\varrho_m|\}, \qquad M := \sup_{n \ge 1} s(n) \le \infty. \tag{6}$$

Note that $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$ and $x^k(1-x)^{n-k} \ge 0$ for every $x \in [0,1]$. Hence by the convexity of the function $t \to t^p$ $(p \ge 1)$ and Jensen's inequality applied to the measure $(n+1)dt$, we get, for every $f \in L^p([0,1])$,

$$|K_n(f)(x)|^p \le s(n)^p \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} |f(t)|^p\, dt.$$

Consequently, the equality

$$\int_0^1 x^k (1-x)^{n-k}\, dx = \frac{1}{n+1} \binom{n}{k}^{-1}, \qquad k = 0, \dots, n,$$

yields $\|K_n(f)\|_p \le s(n) \|f\|_p$ and hence

$$\|K_n(f)\|_p \le s(n). \tag{7}$$

On the other hand, if we take the function $f = \operatorname{sign}(\lambda_n) \cdot \chi_{[0,1/(n+1)]}$, then $\|f\|_p = 1/(n+1)^{1/p}$ and

$$\|K_n\|_p \ge \frac{\|K_n(f)\|_p}{\|f\|_p} = |\lambda_n| \left(\frac{n+1}{np+1} \right)^{1/p} \ge p^{-1/p} |\lambda_n|,$$

from which

$$|\lambda_n| \le p^{1/p} \|K_n\|_p.$$

Analogously,

$$|\varrho_n| \le p^{1/p} \|K_n\|_p.$$

These last inequalities together with (7) lead us to the following result.

Proposition 1. *The sequence* $(\|K_n\|_p)_{n\in\mathbb{N}}$ *is bounded if and only if the sequences* $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ *and* $\varrho = (\varrho_n)_{n\in\mathbb{N}}$ *are bounded. In this case*

$$\sup_{n\geq 1} \|K_n\|_p \leq M. \tag{8}$$

In the following, we assume that the sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\varrho = (\varrho_n)_{n\in\mathbb{N}}$ are bounded. Consequently, the function w defined by (4) satisfies

$$\|w\|_\infty \leq M. \tag{9}$$

Observe that w is not necessarily continuous on [0,1]. More precisely, if $\lambda_n \geq 0$ and $\varrho_n \geq 0$ for every $n \geq 1$, then the existence of the limit

$$\lim_{x\to 0^+} w(x) \qquad (\lim_{x\to 1^-} w(x), \text{ respectively})$$

is equivalent to the existence of the limit

$$\lim_{n\to\infty} \frac{\lambda_1 + \ldots + \lambda_n}{n} \qquad \left(\lim_{n\to\infty} \frac{\varrho_1 + \ldots + \varrho_n}{n}, \text{ respectively}\right),$$

and these two limits coincide (see, e.g. [5, Ch. 7, §5, pp. 226–229]).

However, by (9), the operator $K : L^p([0,1]) \to L^p([0,1])$ defined by $K(f) = w\cdot f$ for every $f \in L^p([0,1])$ is continuous in the L^p-norm and satisfies

$$\|K\|_p = \|w\|_\infty.$$

Before stating our convergence results, we need some elementary formulas for Kantorovich operators. Using the following identities for Bernstein operators:

$$B_n(\mathbf{1}) = \mathbf{1}, \quad B_n(\mathrm{id}) = \mathrm{id}, \quad B_n(\mathrm{id}^2) = \frac{n-1}{n}\,\mathrm{id}^2 + \frac{1}{n}\,\mathrm{id},$$

we obtain by direct computation

$$U_n(\mathbf{1}) = \mathbf{1}, \quad U_n(\mathrm{id}) = \frac{n}{n+1}\,\mathrm{id} + \frac{1}{2(n+1)},$$
$$U_n(\mathrm{id}^2) = \frac{n(n-1)}{(n+1)^2}\,\mathrm{id}^2 + \frac{2n}{(n+1)^2}\,\mathrm{id} + \frac{1}{3(n+1)^2}$$

and consequently, for fixed $x \in [0,1]$,

$$\begin{aligned} &U_n((\mathrm{id} - x\cdot\mathbf{1})^2)(x) \\ &\quad = \frac{n-1}{(n+1)^2}x(1-x) + \frac{1}{3(n+1)^2} \leq \frac{3n+1}{12(n+1)^2} \leq \frac{1}{4(n+1)}. \end{aligned} \tag{10}$$

Moreover, we give an explicit expression of the function $K_n(\mathbf{1})$ in terms of the assigned sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\varrho_n)_{n\in\mathbb{N}}$.

Lemma 2. *We have*

$$K_n(\mathbf{1}) = \sum_{m=1}^{n-1} (\lambda_m x(1-x)^m + \varrho_m x^m(1-x)) + \lambda_n(1-x)^n + \varrho_n x^n. \tag{11}$$

Proof. We proceed by induction on $n \geq 1$; if $n = 1$, (11) is obviously true. Supposing (11) true for $n \geq 1$, we have by (2) and (3),

$$\begin{aligned}
K_{n+1}(\mathbf{1}) &= \sum_{k=0}^{n+1} \alpha_{n+1,k} x^k (1-x)^{n+1-k} \\
&= \lambda_{n+1}(1-x)^{n+1} + \varrho_{n+1} x^{n+1} \\
&\quad + \sum_{k=1}^{n} (\alpha_{n,k} + \alpha_{n,k-1}) x^k (1-x)^{n+1-k} \\
&= \lambda_{n+1}(1-x)^{n+1} + \varrho_{n+1} x^{n+1} \\
&\quad + (1-x) \sum_{k=1}^{n} \alpha_{n,k} x^k (1-x)^{n-k} + x \sum_{k=0}^{n-1} \alpha_{n,k} x^k (1-x)^{n-k} \\
&= (\lambda_{n+1} - \lambda_n)(1-x)^{n+1} + (\varrho_{n+1} - \varrho_n) x^{n+1} \\
&\quad + (1-x) \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} + x \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} \\
&= (\lambda_{n+1} - \lambda_n)(1-x)^{n+1} + (\varrho_{n+1} - \varrho_n) x^{n+1} + K_n(\mathbf{1}) \\
&= (\lambda_{n+1} - \lambda_n)(1-x)^{n+1} + (\varrho_{n+1} - \varrho_n) x^{n+1} \\
&\quad + \sum_{m=1}^{n-1} (\lambda_m x(1-x)^m + \varrho_m x^m (1-x)) + \lambda_n (1-x)^n + \varrho_n x^n \\
&= \lambda_{n+1}(1-x)^{n+1} + \varrho_{n+1} x^{n+1} + \sum_{m=1}^{n-1} (\lambda_m x(1-x)^m \\
&\quad + \varrho_m x^m (1-x)) + \lambda_n (1-x)^n x + \varrho_n x^n (1-x) \\
&= \lambda_{n+1}(1-x)^{n+1} + \varrho_{n+1} x^{n+1} \\
&\quad + \sum_{m=1}^{n} (\lambda_m x(1-x)^m + \varrho_m x^m (1-x))
\end{aligned}$$

and hence (11) holds for $n+1$. ∎

Theorem 3. *The following statements are equivalent*:

(a) *For every* $f \in L^p([0,1])$, *the sequence* $(K_n(f))_{n\in\mathbb{N}}$ *converges in the* L^p*-norm*;

(b) *The sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(\varrho_n)_{n\in\mathbb{N}}$ *are bounded.*

Moreover, *if statement* (a) *or equivalently* (b) *is satisfied*, *then*

$$\lim_{n\to\infty} \|K_n(f) - w\cdot f\|_p = 0 \tag{12}$$

for every $f \in L^p([0,1])$.

Proof. By the Banach–Steinhaus theorem and Proposition 1, we only have to prove the implication (b)⇒(a). By Proposition 1 again, the sequence $(K_n)_{n\in\mathbb{N}}$ is equibounded in the L^p-norm, and therefore it is sufficient to show that $\lim_{n\to\infty} \|K_n(f) - w\cdot f\|_p = 0$ for every $f \in \mathcal{C}([0,1])$.

If $f \in \mathcal{C}([0,1])$, then

$$\text{(i)}\quad \|K_n(f) - w\cdot f\|_p \le \|K_n(f) - f\cdot K_n(\mathbf{1})\|_\infty + \|f\|_\infty \cdot \|K_n(\mathbf{1}) - w\|_p.$$

By (10) and the inequality $|f(t)-f(x)| \le (1+\delta^{-2}(t-x)^2)\omega(f,\delta)$, where $\omega(f,\delta)$ is the modulus of continuity of f, we get

$$\begin{aligned}
&|K_n(f)(x) - f(x)\cdot K_n(\mathbf{1})(x)| \\
&\quad\le M\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(n+1)\int\limits_{k/(n+1)}^{(k+1)/(n+1)} |f(t)-f(x)|\,dt \\
&\quad\le M\omega(f,\delta)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(n+1)\int\limits_{k/(n+1)}^{(k+1)/(n+1)}\left(1+\frac{(t-x)^2}{\delta^2}\right)dt \\
&\quad\le M\omega(f,\delta)\left(1+\frac{1}{\delta^2}\cdot\frac{1}{4(n+1)}\right).
\end{aligned}$$

Taking $\delta = 1/\sqrt{n+1}$, we obtain

$$\|K_n(f) - f\cdot K_n(\mathbf{1})\|_\infty \le \frac{5}{4}M\omega\left(f,\frac{1}{\sqrt{n+1}}\right).$$

Finally, we estimate the second term on the right-hand side of (i). By Lemma 2, we have

$$\begin{aligned}
&|K_n(\mathbf{1})(x) - w(x)| \\
&\quad= \left|\lambda_n(1-x)^n + \varrho_n x^n - \sum_{m=n}^{\infty}(\lambda_m x(1-x)^m + \varrho_m x^m(1-x))\right| \\
&\quad= \left|(1-x)^n\sum_{m=0}^{\infty}(\lambda_n-\lambda_{n+m})x(1-x)^m + x^n\sum_{m=0}^{\infty}(\varrho_n-\varrho_{n+m})x^m(1-x)\right| \\
&\quad\le 2M((1-x)^n + x^n);
\end{aligned}$$

this yields

$$\|f\cdot K_n(\mathbf{1})-w\cdot f\|_p \le 2M\Big(\int_0^1 ((1-x)^n+x^n)^p\,dx\Big)^{1/p}\|f\|_\infty$$
$$\le 4M\Big(\int_0^1 x^{np}\,dx\Big)^{1/p}\|f\|_\infty$$
$$=4M\Big(\frac{1}{np+1}\Big)^{1/p}\|f\|_\infty$$

and the proof is complete. ∎

It is well known that if $f\in L^p([0,1])$ and $x\in[0,1]$ is a Lebesgue point for f, i.e.,

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta |f(x+t)-f(x)|\,dt=0,$$

then (see [3, p. 30])

$$\lim_{n\to\infty} U_n(f)(x)=f(x). \tag{13}$$

In particular,

$$\lim_{n\to\infty} U_n(f)=f \quad \text{a.e.}$$

Next we prove an analogous result for the operators K_n.

PROPOSITION 4. *If* $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ *and* $\varrho=(\varrho_n)_{n\in\mathbb{N}}$ *are bounded sequences and* $f\in L^p([0,1])$, *then*

$$\lim_{n\to\infty} K_n(f)(x)=w(x)\cdot f(x) \tag{14}$$

at every Lebesgue point $x\in\,]0,1[$. *Consequently*, $\lim_{n\to\infty} K_n(f)=w\cdot f$ *a.e.*

Proof. Let $x\in\,]0,1[$ be a Lebesgue point for f. Then

$$|K_n(f)(x)-f(x)\cdot K_n(\mathbf{1})(x)|$$
$$\le M\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)}|f(t)-f(x)|\,dt$$
$$=MU_n(u_x)(x),$$

where $u_x(t):=|f(t)-f(x)|$.

Since x is a Lebesgue point for u_x and $u_x(x)=0$, by (13) it follows that $\lim_{n\to\infty}U_n(u_x)(x)=0$ and hence $\lim_{n\to\infty}|K_n(f)(x)-f(x)\cdot K_n(\mathbf{1})(x)|=0$.

Moreover, $\lim_{n\to\infty}|f(x)|\cdot|K_n(\mathbf{1})(x)-w(x)|=0$ since $x\in\,]0,1[$ and therefore (14) follows. ∎

Finally, we state a quantitative estimate of the convergence in terms of the *averaged modulus of smoothness* $\tau(f,\delta)_p$ defined by

$$\tau(f,\delta)_p := \Big(\int_0^1 \omega(f,x,\delta)^p\,dx\Big)^{1/p} \tag{15}$$

for every $f \in L^p([0,1])$, $1 \le p < \infty$, and $\delta > 0$, where

$$\omega(f,x,\delta) := \sup\{|f(t+h)-f(t)| \mid t,t+h \in [x-\delta/2, x+\delta/2]\cap[0,1]\}. \tag{16}$$

Denote by $\mathcal{M}([0,1])$ the space of all bounded measurable real functions on [0,1].

If $L : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ is a positive operator satisfying $L(\mathbf{1}) = \mathbf{1}$ and

$$d = \|\,\mathrm{id}^2 + L(\mathrm{id}^2) - 2\,\mathrm{id}\cdot L(\mathrm{id})\|_\infty, \tag{17}$$

it is well known that

$$\|L(f)-f\|_p \le 748\tau(f,\sqrt{d})_p \tag{18}$$

for every $f \in \mathcal{M}([0,1])$ and $1 \le p < \infty$ (see, e.g., [4, Theorem 4.3]).

In the case of Bernstein–Kantorovich operators, the preceding inequality yields

$$\|U_n(f)-f\|_p \le 748\tau\Big(f,\frac{1}{\sqrt{n+1}}\Big)_p. \tag{19}$$

If $L(\mathbf{1})$ is strictly positive, we may apply (18) to the operator $L/L(\mathbf{1})$ and we have

$$\|L(f)-f\cdot L(\mathbf{1})\|_p \le \|L(\mathbf{1})\|\,\Big\|\frac{L(f)}{L(\mathbf{1})} - f\Big\|_p \le 748\|L(\mathbf{1})\|\tau(f,\sqrt{\delta})_p, \tag{20}$$

where

$$\delta = \Big\|\frac{\mathrm{id}^2\cdot L(\mathbf{1}) + L(\mathrm{id}^2) - 2\,\mathrm{id}\cdot L(\mathrm{id})}{L(\mathbf{1})}\Big\|_\infty. \tag{21}$$

THEOREM 5. *Assume that the sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\varrho = (\varrho_n)_{n\in\mathbb{N}}$ are bounded. Then, for every $n \ge 1$ and $f \in \mathcal{M}([0,1])$,*

$$\|K_n(f) - w\cdot f\|_p \le C\tau\Big(f,\frac{1}{\sqrt{n+1}}\Big)_p + 4M\Big(\frac{1}{np+1}\Big)^{1/p}\|f\|_\infty, \tag{22}$$

where the constant C depends only on λ and ϱ (e.g., $C = 1683M$).

P r o o f. For every $f \in \mathcal{M}([0,1])$, we write

$$\|K_n(f) - w\cdot f\|_p \le \|K_n(f) - f\cdot K_n(\mathbf{1})\|_p + \|f\|_\infty \cdot \|K_n(\mathbf{1}) - w\|_p \tag{i}$$

and we estimate separately the two right-hand terms.

If $c > M$, then we consider $K_{n,c} = K_n + c \cdot I$ which satisfies $K_{n,c}(\mathbf{1}) > 0$ and

(ii) $$K_{n,c}(f) - f \cdot K_{n,c}(\mathbf{1}) = K_n(f) - f \cdot K_n(\mathbf{1}).$$

By (19) and (20) we have

(iii) $$\|K_{n,c}(f) - f \cdot K_{n,c}(\mathbf{1})\|_p \le 748 \|K_{n,c}(\mathbf{1})\| \tau(f, \sqrt{\delta})_p,$$

where, by (10),

$$\begin{aligned}\delta &= \left\| \frac{\mathrm{id}^2 \cdot K_{n,c}(\mathbf{1}) + K_{n,c}(\mathrm{id}^2) - 2\,\mathrm{id} \cdot K_{n,c}(\mathrm{id})}{K_{n,c}(\mathbf{1})} \right\|_\infty \\ &= \left\| \frac{\mathrm{id}^2 \cdot K_n(\mathbf{1}) + K_n(\mathrm{id}^2) - 2\,\mathrm{id} \cdot K_n(\mathrm{id})}{K_{n,c}(\mathbf{1})} \right\|_\infty \\ &= \sup_{0 \le x \le 1} \left| \frac{K_n((\mathrm{id} - x \cdot \mathbf{1})^2)(x)}{K_{n,c}(\mathbf{1})(x)} \right| \\ &\le \frac{M}{c - M} \sup_{0 \le x \le 1} |U_n((\mathrm{id} - x \cdot \mathbf{1})^2)(x)| \le \frac{M}{4(c - M)(n+1)}.\end{aligned}$$

Choosing $c = \frac{5}{4}M$, we obtain $\delta \le 1/(n+1)$ and $\|K_{n,c}(\mathbf{1})\|_\infty \le \frac{9}{4}M$. Consequently, by (ii) and (iii), it follows that

$$\|K_n(f) - f \cdot K_n(\mathbf{1})\|_p \le 1683 M \tau\left(f, \frac{1}{\sqrt{n+1}}\right)_p.$$

The second term on the right-hand side of the inequality (i) has been already estimated in the proof of Theorem 3. ∎

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