ANNALES
POLONICI MATHEMATICI
LXIII.3 (1996)

Representation formulae for (C_0) m-parameter operator semigroups

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Abstract. Some general representation formulae for (C_0) m-parameter operator semigroups with rates of convergence are obtained by the probabilistic approach and multiplier enlargement method. These cover all known representation formulae for (C_0) one- and m-parameter operator semigroups as special cases. When we consider special semigroups we recover well-known convergence theorems for multivariate approximation operators.

1. Introduction. Recently the study of representation formulae for (C_0) operator semigroups has attracted much attention (Shaw [17, 18], Butzer–Hahn [2], Pfeifer [13–15] and Chen–Zhou [3]). They gave some general formulae that include earlier (Post–Widder, Hille–Phillips [8] and Chung [4]) concrete representation formulae. But most of the work done so far is confined to the one-parameter case, while Shaw's method for the multi-parameter case is not an easy one to get new formulae and the results are without rates of convergence. In this article we try to give some general representation formulae for (C_0) m-parameter operator semigroups. The main idea is the use of a probabilistic setting in the representation of operator semigroups, initiated by Chung [4] and developed by Butzer–Hahn [2] and Pfeifer [13], and the so-called multiplier enlargement method of Hsu–Wang [9, 19] and Shaw [17, 18]. At the same time, by introducing a modified second modulus of continuity of an operator semigroup and a Steklov-type element we establish quantitative estimates of the obtained formulae.

To the best of our knowledge, all existing representation formulae for (C_0) one- and multi-parameter operator semigroups are special cases of our results. In particular, Shaw's formulae [17, 18] for m-parameter operator

1991 *Mathematics Subject Classification*: Primary 47D03, 41A36, 41A63, 60G50; Secondary 41A65, 60H30, 41A17, 41A80, 47A56.

Key words and phrases: multi-parameter operator semigroups, representation formulae, multivariate approximation, sum of random vectors, Banach space, multiplier enlargement method, rate of convergence, second modulus of continuity, inequalities.

semigroups are special cases of our results when specifying the random vectors considered. Also with our method it is easier to obtain new formulae.

At the end we give examples to show the application of our results in multivariate operator approximation theory when we consider particular operator semigroups.

2. Preliminaries. Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$, and $\mathcal{E}(\mathcal{X})$ be the Banach algebra of endomorphisms of $\mathcal{X}$. If $T \in \mathcal{E}(\mathcal{X})$, $\|T\|$ also denotes the norm of T. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space supplied with the usual definition of arithmetical operations and metric. We write $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, $\bar{t} = t_1 + \ldots + t_m$, $\overline{|t|} = |t_1| + \ldots + |t_m|$ and denote the unit vectors by $e_1, \ldots, e_m$, where $e_k = (0, \ldots, 1, \ldots, 0)$ with 1 in the kth place and 0 elsewhere. Further, let

$$\mathbb{R}^m_+ = \{t \in \mathbb{R}^m : t_k \ge 0,\ k = 1, \ldots, m\},$$

the first closed 2^m-ant in $\mathbb{R}^m$. $\mathbb{Z}_+$ denotes the set of all non-negative integers and

$$\mathbb{Z}^m_+ = \{n = (n_1, \ldots, n_m) : n_k \in \mathbb{Z}_+,\ k = 1, \ldots, m\},$$

while $\mathbb{N}$ is the set of all positive integers.

A family of bounded linear operators $\{T(t) : t \in \mathcal{R}^m_+\}$ on $\mathcal{X}$ is called a (C_0) *m-parameter operator semigroup* in $\mathcal{E}(\mathcal{X})$ when the following three conditions are satisfied:

$$T(t+s) = T(t)T(s), \qquad t, s \in \mathbb{R}^m_+; \tag{1}$$

$$T(0) = I \qquad \text{(identity operator)}; \tag{2}$$

$$\operatorname*{s-lim}_{t\in\mathbb{R}^m_+,\, t\to 0} T(t)f = f, \qquad f \in \mathcal{X}. \tag{3}$$

It is known that $\{T(t) : t \in \mathbb{R}^m_+\}$ is then the direct product of m (C_0) one-parameter operator semigroups in $\mathcal{E}(\mathcal{X})$:

$$T(t) = \prod_{k=1}^{m} T_k(t_k), \tag{4}$$

where $T_k(t_k) = T(t_k e_k)$. The operators $\{T_k(t_k) : 0 \le t_k < \infty\}$ $(k = 1, \ldots, m)$ commute with each other.

Let A_k be the infinitesimal generator of $\{T_k(t_k) : 0 \le t_k < \infty\}$ with domain $D(A_k)$, $k = 1, \ldots, m$. Then if f is in $D(A_k)$ so is $T(t)f$ for each $t \in \mathbb{R}^m_+$ and

$$A_k T(t) f = T(t) A_k f.$$

Further, if $f \in D(A_j)$ and $f \in D(A_j A_k)$ then $f \in D(A_k A_j)$ and $A_k A_j f = A_j A_k f$ $(j, k = 1, \ldots, m)$. In the following we use the notation

$$D^2 := \bigcap_{k,j=1}^{m} D(A_k A_j).$$

D^2 is a linear subspace of $\mathcal{X}$.

To each $k = 1, \ldots, m$, there correspond two numbers $M_k \geq 1$ and $\omega_k \geq 0$ such that

$$\|T_k(t_k)\| \leq M_k e^{\omega_k t_k}, \qquad 0 \leq t_k < \infty.$$

Hence we have the inequality

$$\|T(t)\| \leq M \exp(\omega(t_1 + \ldots + t_m)) = M e^{\omega \bar{t}}, \qquad t \in \mathbb{R}_+^m, \tag{5}$$

where $M = M_1 \ldots M_m$ and $\omega = \max\{\omega_k : 1 \leq k \leq m\}$.

In the following it is always understood that $\{T(t) : t \in \mathbb{R}_+^m\}$ satisfies (5), unless otherwise specified.

For the above definitions and properties of operator semigroups we refer to Butzer–Berens [1], Hille–Phillips [8] or W. Köhnen [11].

Let $(\Omega, \mathcal{A}, P)$ be a probability space. For every real-valued random variable X defined on $(\Omega, \mathcal{A}, P)$, $E(X)$ denotes its expectation. If $\xi = E(X)$ exists then $\sigma^2 = \sigma^2(X) = E[(X - \xi)^2]$ is the variance of X. Let further $\Psi_X(u) = E(u^X)$, $u \geq 0$, and $\Psi_X^*(u) = E(e^{uX})$, $u \in \mathbb{R}$, denote the probability-generating function and the moment-generating function of X respectively.

We need to consider m-dimensional random vectors, also denoted by $X, Y, \ldots$, on $(\Omega, \mathcal{A}, P)$. For an m-dimensional random vector $X = (X_{01}, \ldots, X_{0m})$, we also use $E(X)$ to denote its expectation:

$$E(X) := (E(X_{01}), \ldots, E(X_{0m}))$$

and define

$$\sigma_i^2(X) := \sigma^2(X_{0i}).$$

It is not difficult to extend the theory of the extended Pettis integral developed in [13] to the multivariate case.

Let $\{T(t) : t \in \mathbb{R}_+^m\}$ be as above and X be an $\mathbb{R}_+^m$-valued random vector such that

$$\Psi_{\overline{X}}^*(\omega) < \infty, \qquad \overline{X} = X_{01} + \ldots + X_{0m}.$$

Then for every $f \in \mathcal{X}$ define

$$E[T(X)f] := \int_{\Omega} T(X)f \, dP,$$

which exists in the Bochner sense in $\mathcal{X}$ by the strong continuity of $\{T(t) : t \in \mathbb{R}_+^m\}$ and (5). Moreover, the map $E[T(X)] : f \to E[T(X)f]$ on $\mathcal{X}$ defines a bounded linear operator $E[T(X)] \in \mathcal{E}(\mathcal{X})$ with

$$\|E[T(X)]\| \leq M \Psi_{\overline{X}}^*(\omega).$$

$E[T(X)]$ is called the expectation of $T(X)$ and is understood as an extended Pettis integral following [13].

If X and Y are independent $\mathbb{R}_+^m$-valued random vectors such that $\Psi_{\overline{X}}^*(\omega) < \infty$ and $\Psi_{\overline{Y}}^*(\omega) < \infty$ then $E[T(X)], E[T(Y)]$ and $E[T(X+Y)]$ exist in $\mathcal{E}(\mathcal{X})$ and

$$E[T(X) \circ T(Y)] = E[T(X+Y)] = E[T(X)] \circ E[T(Y)],$$

where "$\circ$" denotes composition.

For the above see [13], [15], [16] and the references cited there.

3. Auxiliary results. We need a Taylor expansion integral formula for (C_0) m-parameter operator semigroups.

LEMMA 1. *Assume* $\{T(t) : t \in \mathbb{R}_+^m\}$ *is a* (C_0) *m-parameter operator semigroup satisfying* (5). *Then for every* $g \in D^2$ *and* $s, t \in \mathbb{R}_+^m$,

$$\begin{aligned} (6) \quad & T(t)g - T(s)g \\ &= T(s)[(t_1 - s_1)A_1 g + \ldots + (t_m - s_m)A_m g] \\ &\quad + \int_0^1 (1-u)T(s + u(t-s))((t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m)^2 g\, du. \end{aligned}$$

Proof. Let $G(u) = T(s + u(t-s))g \in \mathcal{X}$, $u \in [0,1]$. Then

$$\begin{aligned} G'(u) &:= \frac{dG(u)}{du} \\ &= T(s + u(t-s))[(t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m]g \end{aligned}$$

and

$$G''(u) = T(s + u(t-s))[(t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m]^2 g.$$

Now (6) follows from the Taylor formula with integral remainder for Banach space valued functions (see, e.g., [5, Theorem 8.14.3]). ∎

For our purpose we need a second modulus of continuity $\omega_2(Tf, \delta)$ and the Steklov operator $J_h(f)$ $(h > 0)$ for a (C_0) m-parameter operator semigroup $\{T(t) : t \in \mathbb{R}_+^m\}$ and $f \in \mathcal{X}$.

DEFINITION 1.

$$\omega_2(Tf, \delta) = \sup_{\substack{t=(t_1,\ldots,t_m) \\ 0 \le t_i,\, t_j \le \delta}} \{\|(T(t) - I)^2 f\|, \|(T_i(t_i) - I)(T_j(t_j) - I)f\|\}.$$

We have $\omega_2(Tf, \delta) \to 0$ as $\delta \to 0$, by the strong continuity of $\{T(t) : t \in \mathbb{R}_+^m\}$.

DEFINITION 2.

$$J_h(f) = \left(\frac{2}{h}\right)^{2m} \underbrace{\int_0^{h/2} \ldots \int_0^{h/2}}_{2m} [2T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m) - T(2\xi_1+2\eta_1,\ldots,2\xi_m+2\eta_m)] f\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m.$$

The integral may be considered as a multi-$\mathcal{X}$-valued Riemann integral.

We have the following lemma:

LEMMA 2. (i) $J_h(f) \in D^2$ *for all* $f \in \mathcal{X}$;
(ii) $\|f - J_h(f)\| \le \omega_2(Tf, h)$;
(iii) $\|A_i A_j J_h(f)\| \le 9Me^{2(m-1)h\omega}\omega_2(Tf,h)/h^2$, $1 \le i, j \le m$.

Proof. (i) Let

$$J_1 = \underbrace{\int_0^{h/2} \ldots \int_0^{h/2}}_{2m} T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m) f\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m, \tag{7}$$

$$J_2 = \underbrace{\int_0^{h/2} \ldots \int_0^{h/2}}_{2m} T(2\xi_1+2\eta_1,\ldots,2\xi_m+2\eta_m) f\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m = \left(\frac{1}{2}\right)^{2m} \underbrace{\int_0^{h} \ldots \int_0^{h}}_{2m} T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m) f\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m. \tag{8}$$

It is not difficult to show that $J_1 \in D^2$, $J_2 \in D^2$ (cf. [1, p. 10]) and hence (i) holds.

(ii) We have

$$\|f - J_h(f)\| = \left\| \left(\frac{2}{h}\right)^{2m} \underbrace{\int_0^{h/2} \ldots \int_0^{h/2}}_{2m} [f - 2T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m) + T(2\xi_1+2\eta_1,\ldots,2\xi_m+2\eta_m)] f\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m \right\|$$
$$= \left\| \left(\frac{2}{h}\right)^{2m} \underbrace{\int_0^{h/2} \ldots \int_0^{h/2}}_{2m} [T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m) - I]^2 f \times d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m \right\|$$

$$\leq \left(\frac{2}{h}\right)^{2m} \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m} \|[T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m)-I]^2 f\|$$
$$\times\, d\xi_1\, d\eta_1 \ldots d\xi_m\, d\eta_m$$
$$\leq \omega_2(Tf,h).$$

(iii) When $i \neq j$, similar to the one-parameter operator semigroup case (ibid.), we can show

$$A_iA_jJ_1 = \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m-2} \prod_{k\neq i,j} T_k(\xi_k+\eta_k)T_i(\eta_i)T_j(\eta_j)$$
$$\times (T_i(h/2)-I)(T_j(h/2)-I)f \prod_{k\neq i,j} d\xi_k\, d\eta_k\, d\eta_i\, d\eta_j$$

and

$$A_iA_jJ_2 = \left(\frac{1}{2}\right)^{2m} \underbrace{\int_0^{h} \cdots \int_0^{h}}_{2m-2} \prod_{k\neq i,j} T_k(\xi_k+\eta_k)T_i(\eta_i)T_j(\eta_j)$$
$$\times (T_i(h)-I)(T_j(h)-I)f \prod_{k\neq i,j} d\xi_k\, d\eta_k\, d\eta_i\, d\eta_j.$$

So

$$\|A_iA_jJ_h(f)\| = \|(2/h)^{2m}[2A_iA_jJ_1 - A_iA_jJ_2]\|$$
$$\leq \left(\frac{2}{h}\right)^{2m}\Bigg\{2 \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m-2} \prod_{k\neq i,j} M_k e^{\omega_k(\xi_k+\eta_k)} M_i e^{\omega_i\eta_i} M_j e^{\omega_j\eta_j}$$
$$\times \left\|\left(T_i\left(\frac{h}{2}\right)-I\right)\left(T_j\left(\frac{h}{2}\right)-I\right)f\right\| \prod_{k\neq i,j} d\xi_k\, d\eta_k\, d\eta_i\, d\eta_j$$
$$+ \left(\frac{1}{2}\right)^{2m} \underbrace{\int_0^{h} \cdots \int_0^{h}}_{2m-2} \prod_{k\neq i,j} M_k e^{\omega_k(\xi_k+\eta_k)} M_i e^{\omega_i\eta_i} M_j e^{\omega_j\eta_j}$$
$$\times \|(T_i(h)-I)(T_j(h)-I)f\| \prod_{k\neq i,j} d\xi_k d\eta_k d\eta_i d\eta_j\Bigg\}$$
$$\leq (2/h)^{2m} M\{2e^{(2m-2)\omega h/2}(h/2)^{2m-2} + (1/2)^{2m} e^{(2m-2)\omega h} h^{2m-2}\}$$
$$\times\, \omega_2(Tf,h)$$

$$\le Me^{(2m-2)\omega h}\{2(h/2)^{-2}+1/h^2\}\omega_2(Tf,h)$$
$$= 9Me^{2(m-1)\omega h}\omega_2(Tf,h)/h^2.$$

When $i=j$, the same estimate holds. ∎

LEMMA 3. *For any $\mathbb{R}^m_+$-valued random vector $Y=(Y_{01},\ldots,Y_{0m})$ with $E(Y)=x=(x_1,\ldots,x_m)$ and $f\in\mathcal{X}$,*

$$\begin{aligned}(9)\quad &\|E[T(Y)]f-T(x)f\|\\ &=\|E[T(Y)f]-T(x)f\|\\ &\le M\omega_2(Tf,h)\Big\{2E(e^{\omega\overline{Y}})\\ &\quad+\frac{9}{2}mMe^{2\omega\bar{x}}e^{2(m-1)h\omega}[E(e^{p\omega\overline{Y}})]^{1/p}\Big[\sum_{i=1}^{m}(E((Y_{0i}-x_i)^{2q}))^{1/q}\Big]/h^2\Big\},\end{aligned}$$

where $p>0$, $q>0$, $1/p+1/q=1$, $h>0$. If $\omega=0$, we have

$$(10)\qquad \|E[T(Y)]f-T(x)f\|\le 2M\omega_2(Tf,h)\Big[1+\frac{9mM}{4h^2}\sum_{i=1}^{m}\sigma^2(Y_{0i})\Big].$$

P r o o f. We have

$$\begin{aligned}(11)\quad \|E[T(Y)]f-T(x)f\|&=\|E[T(Y)f]-T(x)f\|\\ &\le\|E[T(Y)f]-E[T(Y)J_hf]\|+\|E[T(Y)J_hf]-T(x)J_hf\|\\ &\quad+\|T(x)J_hf-T(x)f\|\\ &=: I_1+I_2+I_3.\end{aligned}$$

Now,

$$\begin{aligned}(12)\qquad I_1&\le E[\|T(Y)(J_hf-f)\|]\\ &\le E[Me^{\omega\overline{Y}}\|J_hf-f\|]\le ME(e^{\omega\overline{Y}})\omega_2(Tf,h)\end{aligned}$$

by Lemma 2, and

$$(13)\qquad I_3\le Me^{\omega\bar{x}}\omega_2(Tf,h)\le ME(e^{\omega\overline{Y}})\omega_2(Tf,h)$$

by Jensen's inequality.

Note that $g:=J_h(f)\in D^2$, by Lemma 2. Apply Lemma 1 to get

$$\begin{aligned}(14)\quad I_2=\Big\|E\Big\{&T(x)[(Y_{01}-x_1)A_1+\ldots+(Y_{0m}-x_m)A_m]g\\ &+\int_0^1(1-u)T(x+u(Y-x))[(Y_{01}-x_1)A_1+\ldots\\ &\ldots+(Y_{0m}-x_m)A_m]^2g\,du\Big\}\Big\|\end{aligned}$$

$$= \Big\| E\Big\{ \int_0^1 (1-u)T(x+u(Y-x))[(Y_{01}-x_1)A_1 + \ldots$$
$$\ldots + (Y_{0m}-x_m)A_m]^2 g\,du\Big\}\Big\|$$
$$\le E\Big\{ \int_0^1 (1-u)\|T(x+u(Y-x))\|$$
$$\times \|[(Y_{01}-x_1)A_1 + \ldots + (Y_{0m}-x_m)A_m]^2 g\|\,du\Big\}$$
$$\le ME\Big\{ \int_0^1 (1-u)\exp(\omega(\overline{x+u(Y-x)})$$
$$\times \|[(Y_{01}-x_1)A_1 + \ldots + (Y_{0m}-x_m)A_m]^2 g\|\,du\Big\}$$
$$\le \frac{1}{2}Me^{2\omega\bar{x}}E\Big\{e^{\omega\overline{Y}}\sum_{i=1}^m\sum_{j=1}^m |Y_{0i}-x_i|\cdot|Y_{0j}-x_j|\cdot\|A_iA_jg\|\Big\}$$
$$\le \frac{1}{2}Me^{2\omega\bar{x}}E\Big\{e^{\omega\overline{Y}}\sum_{i=1}^m\sum_{j=1}^m \frac{1}{2}[(Y_{0i}-x_i)^2+(Y_{0j}-x_j)^2]\Big\}$$
$$\times 9Me^{2(m-1)h\omega}\omega_2(Tf,h)/h^2 \qquad \text{(by Lemma 2(iii))}$$
$$= \frac{9}{2}M^2e^{2\omega\bar{x}}e^{2(m-1)h\omega}m\sum_{i=1}^m E[e^{\omega\overline{Y}}(Y_{0i}-x_i)^2]\omega_2(Tf,h)/h^2$$
$$\le \frac{9}{2}mM^2e^{2\omega\bar{x}}e^{2(m-1)h\omega}[E(e^{p\omega\overline{Y}})]^{1/p}\Big[\sum_{i=1}^m (E((Y_{0i}-x_i)^{2q}))^{1/q}\Big]$$
$$\times \omega_2(Tf,h)/h^2$$

by Hölder's inequality.

Therefore by (11)–(14) we get (9).

If $\omega = 0$ we have $I_1 \le M\omega_2(Tf,h)$, $I_3 \le M\omega_2(Tf,h)$, and $I_2 \le \frac{9}{2}M^2m\times \sum_{i=1}^m \sigma^2(Y_{0i})\omega_2(Tf,h)/h^2$ so (10) follows. ∎

4. Main results. Here comes our first main result:

Theorem 1. *Let $X = (X_{01},\ldots,X_{0m})$ be an $\mathbb{R}_+^m$-valued random vector with $E(X) = x = (x_1,\ldots,x_m)$ and suppose that there exists a $\delta > 0$ such that $\Psi_{\overline{X}}^*(\delta) < \infty$. Then for any (C_0) m-parameter operator semigroup satisfying* (5), *and all $n > \max(p\omega/\delta, 1/\delta^2)$,*

$$\|\{E[T(X/n)]\}^n f - T(x)f\| \tag{15}$$
$$\le 2M\omega_2(Tf,1/\sqrt{n})\Big\{e^{\omega\bar{x}}\exp\Big[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi_{\overline{X}}^*(\delta)\Big]$$

$$+ 2^{1/q} 9M \frac{m^2 q^2}{e^2} e^{3\omega \bar{x}} e^{2(m-1)\omega/\sqrt{n}}$$
$$\times \exp\left[\left(\frac{2np\omega^2}{e^2(n\delta - p\omega)^2} + \frac{2ne^{\delta\bar{x}}}{e^2 q(\delta\sqrt{n}-1)^2}\right)\Psi^*_{\overline{X}}(\delta)\right]\Bigg\},$$

where $p, q > 0$, $1/p + 1/q = 1$, *is an arbitrary conjugate pair. If* $\omega = 0$, *then*

$$\|\{E[T(X/n)]\}^n f - T(x)f\| \le 2M\omega_2(Tf, 1/\sqrt{n})\left[1 + \frac{9}{4} mM \sum_{i=1}^{m} \sigma^2(X_{0i})\right]. \tag{16}$$

N o t e. All the right hand sides of (15) and (16) are finite.

P r o o f o f T h e o r e m 1. Let X_k be a sequence of independent random vectors identically distributed as X, and $Y = (1/n)\sum_{k=1}^{n} X_k$. Then

$$E(Y) = \frac{1}{n}\sum_{k=1}^{n} E(X_k) = x, \qquad E[T(Y)f] = \{E[T(X/n)]\}^n f.$$

For $u > 0$ we have

$$\begin{aligned}
\Psi^*_{\overline{Y}}(u) &= E(e^{(u/n)\Sigma_{k=1}^n \overline{X}_k}) = (E(e^{(u/n)\overline{X}}))^n \\
&\le \left(1 + \frac{u}{n}E(\overline{X}) + E\left(\frac{u^2\overline{X}^2}{2n^2} e^{(u/n)\overline{X}}\right)\right)^n \\
&\le \left(1 + \frac{u}{n}\bar{x} + \frac{u^2}{2n^2}\left(\frac{2}{\delta - u/n}\right)^2 e^{-2} E(e^{\delta\overline{X}})\right)^n \\
&\le e^{u\bar{x}} \exp\left[\frac{2nu^2}{e^2(n\delta - u)^2}\Psi^*_{\overline{X}}(\delta)\right]
\end{aligned}$$

whenever $u/n < \delta$.

Above we made use of the inequalities (see also Pfeifer [14, p. 275])

$$r^\alpha e^{\eta r} \le \left(\frac{\alpha}{\delta - \eta}\right)^\alpha e^{-\alpha} e^{\delta r} \qquad (\text{for } \eta < \delta,\ r > 0,\ \alpha > 0) \tag{17}$$

and

$$(1 + r)^n \le e^{nr}. \tag{18}$$

So for $n > p\omega/\delta \ge \omega/\delta$,

$$E(e^{\omega\overline{Y}}) \le e^{\omega\bar{x}} \exp\left\{\frac{2n\omega^2}{e^2(n\delta - \omega)^2}\Psi^*_{\overline{X}}(\delta)\right\}$$

and

$$[E(e^{p\omega\overline{Y}})]^{1/p} \le \left\{e^{p\omega\bar{x}} \exp\left[\frac{2np^2\omega^2}{e^2(n\delta - p\omega)^2}\Psi^*_{\overline{X}}(\delta)\right]\right\}^{1/p}$$
$$= e^{\omega\bar{x}} \exp\left[\frac{2np\omega^2}{e^2(n\delta - p\omega)^2}\Psi^*_{\overline{X}}(\delta)\right].$$

Observe that for $Y = (Y_{01}, \ldots, Y_{0m})$ we have

$$\begin{aligned}
&E((Y_{0i} - x_i)^{2q})\\
&\quad = E\left(\left(\frac{1}{n}\sum_{k=1}^{n} X_{ki} - x_i\right)^{2q}\right)\\
&\quad \le \left(\frac{2q}{\sqrt{n}}\right)^{2q} e^{-2q} E(e^{\sqrt{n}|(1/n)\Sigma_{k=1}^{n} X_{ki} - x_i|}) \qquad \text{(by (17))}\\
&\quad \le \left(\frac{2q}{\sqrt{n}}\right)^{2q} e^{-2q} [E(e^{(1/\sqrt{n})\Sigma_{k=1}^{n}(X_{ki} - x_i)}) + E(e^{(1/\sqrt{n})\Sigma_{k=1}^{n}(x_i - X_{ki})})]\\
&\quad \le \left(\frac{2q}{\sqrt{n}}\right)^{2q} e^{-2q} 2 \exp\left[E\left(\frac{1}{2}(X_{0i} - x_i)^2 e^{(1/\sqrt{n})|X_{0i} - x_i|}\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Taylor's expansion and (18))}\\
&\quad \le \left(\frac{2q}{\sqrt{n}}\right)^{2q} e^{-2q} 2 \exp\left[\left(\frac{2}{\delta - 1/\sqrt{n}}\right)^2 e^{-2} E\left(\frac{1}{2} e^{\delta|X_{0i} - x_i|}\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (17) for } 1/\sqrt{n} < \delta)\\
&\quad \le \left(\frac{2q}{\sqrt{n}}\right)^{2q} e^{-2q} 2 \exp\left[\frac{2n}{(\delta\sqrt{n} - 1)^2} e^{-2} e^{\delta\bar{x}} \Psi^*_{\overline{X}}(\delta)\right], \qquad 1 \le i \le m.
\end{aligned}$$

Hence we established that

$$[E((Y_{0i} - x_i)^{2q})]^{1/q} \le \frac{4q^2}{n} e^{-2} 2^{1/q} \exp\left[\frac{2n}{q(\delta\sqrt{n} - 1)^2} e^{-2} e^{\delta\bar{x}} \Psi^*_{\overline{X}}(\delta)\right].$$

Now apply Lemma 3 and take $h = 1/\sqrt{n}$:

$$\begin{aligned}
&\|\{E[T(X/n)]\}^n f - T(x)f\| = \|E[T(Y)]f - T(x)f\|\\
&\qquad \le M\omega_2(Tf, 1/\sqrt{n})\left\{2e^{\omega\bar{x}} \exp\left[\frac{2n\omega^2}{e^2(n\delta - \omega)^2}\Psi^*_{\overline{X}}(\delta)\right]\right.\\
&\qquad\quad + \frac{9}{2} m M e^{2\omega\bar{x}} e^{2(m-1)\omega/\sqrt{n}} e^{\omega\bar{x}} \exp\left[\frac{2np\omega^2}{e^2(n\delta - p\omega)^2}\Psi^*_{\overline{X}}(\delta)\right]\\
&\qquad\quad \left.\times\, m \frac{4q^2}{n} e^{-2} 2^{1/q} \exp\left[\frac{2n}{q(\delta\sqrt{n} - 1)^2} e^{-2} e^{\delta\bar{x}} \Psi^*_{\overline{X}}(\delta)\right] n\right\}
\end{aligned}$$

$$= 2M\omega_2(Tf, 1/\sqrt{n})\bigg\{e^{\omega\bar{x}} \exp\bigg[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi^*_{\overline{X}}(\delta)\bigg]$$

$$+ 2^{1/q}9M\frac{m^2q^2}{e^2}e^{3\omega\bar{x}}e^{2(m-1)\omega/\sqrt{n}}$$

$$\times \exp\bigg[\bigg(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} + \frac{2ne^{\delta\bar{x}}}{e^2q(\delta\sqrt{n}-1)^2}\bigg)\Psi^*_{\overline{X}}(\delta)\bigg]\bigg\}.$$

When $\omega = 0$, noting that $\sigma^2(Y_{0i}) = \sigma^2(X_{0i})/n$ by (10), we get (16). ■

A ramification of Theorem 1 follows:

THEOREM 2. *Let N be a $\mathbb{Z}_+$-valued random variable with $E(N) = \eta$, $\eta > 0$, and let $Y = (Y_{01}, \ldots, Y_{0m})$ be an $\mathbb{R}^m_+$-valued random vector independent of N with $E(Y) = \gamma = (\gamma_1, \ldots, \gamma_m)$. Assume that there exists a $\delta > 0$ such that*

$$\Psi_N(\Psi^*_{\overline{Y}}(\delta)) < \infty.$$

Then for $n > \max(p\omega/\delta, 1/\delta^2)$,

$$\|\{\Psi_N[E(T(Y/n))]\}^n f - T(\eta\gamma)f\| \tag{19}$$

$$\leq 2M\omega_2(Tf, 1/\sqrt{n})\bigg\{e^{\omega\eta\bar{\gamma}} \exp\bigg[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi_N(\Psi^*_{\overline{Y}}(\delta))\bigg]$$

$$+ 2^{1/q}9M\frac{m^2q^2}{e^2}e^{3\omega\eta\bar{\gamma}}e^{2(m-1)\omega/\sqrt{n}}$$

$$\times \exp\bigg[\bigg(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} + \frac{2ne^{\delta\eta\bar{\gamma}}}{e^2q(\delta\sqrt{n}-1)^2}\bigg)\Psi_N(\Psi^*_{\overline{X}}(\delta))\bigg]\bigg\},$$

where $p, q > 0$, $1/p + 1/q = 1$, is an arbitrary conjugate pair. If $\omega = 0$, then

$$\|\{\Psi_N[E(T(Y/n))]\}^n f - T(\eta\gamma)f\| \tag{20}$$

$$\leq 2M\omega_2(Tf, 1/\sqrt{n})\bigg\{1 + \frac{9}{4}mM\sum_{i=1}^m[\eta\sigma^2(Y_{0i}) + \sigma^2(N)\gamma_i^2]\bigg\}.$$

Proof. Consider $Y_k \overset{\text{i.i.d.}}{\sim} Y$, which are also independent of N. In Theorem 1, take $X = \sum_{k=1}^N Y_k$ (as usual, an empty sum equals 0). Then

$$E\bigg[T\bigg(\frac{1}{n}X\bigg)\bigg] = E\bigg[T\bigg(\frac{1}{n}\sum_{k=1}^N Y_k\bigg)\bigg] = \sum_{l=0}^\infty P(N=l)E\bigg[T\bigg(\frac{1}{n}\sum_{k=1}^l Y_k\bigg)\bigg]$$

$$= \sum_{l=0}^\infty P(N=l)\bigg[E\bigg(T\bigg(\frac{1}{n}Y\bigg)\bigg)\bigg]^l = \Psi_N\bigg(E\bigg(T\bigg(\frac{1}{n}Y\bigg)\bigg)\bigg),$$

$$E(X) = \sum_{l=0}^\infty P(N=l)E\bigg[\sum_{k=1}^l Y_k\bigg] = E(N)E(Y) = \eta\gamma.$$

Also

$$\Psi^*_{\overline{X}}(\delta) = E(e^{\delta \overline{X}}) = E(e^{\delta \Sigma_{k=1}^N \overline{Y}_k}) = \sum_{l=0}^{\infty} P(N=l) E(e^{\delta \Sigma_{k=1}^l \overline{Y}_k})$$
$$= \sum_{l=0}^{\infty} P(N=l)(E(e^{\delta \overline{Y}}))^l = \Psi_N(\Psi^*_{\overline{Y}}(\delta)).$$

If $X = (X_{01}, \ldots, X_{0m})$ we have

$$\sigma^2(X_{0i}) = \sigma^2\Big(\sum_{k=1}^N Y_{ki}\Big) = \sum_{l=0}^{\infty} P(N=l) E\Big(\Big(\sum_{k=1}^l Y_{ki}\Big)^2\Big) - \eta^2\gamma_i^2$$
$$= \sum_{l=0}^{\infty} P(N=l)(lE(Y_{0i}^2) + l(l-1)\gamma_i^2) - \eta^2\gamma_i^2$$
$$= \eta\sigma^2(Y_{0i}) + \sigma^2(N)\gamma_i^2.$$

Then (19), (20) follow by (15), (16). ∎

An application of Lemma 3 comes next:

THEOREM 3. *For each positive real number τ, let N_τ be a $\mathbb{Z}_+$-valued random variable with $E(N_\tau) = \tau\eta$, where $\eta \in \mathbb{R}_+$ is fixed. Let X be an $\mathbb{R}^m_+$-valued random vector with $E(X) = \gamma = (\gamma_1, \ldots, \gamma_m)$, independent of N_τ. Assume that there exists a $\delta > 0$ such that $\Psi^*_{\overline{X}}(\delta) < \infty$ and further there are $p, q > 0$ with $1/p + 1/q = 1$ such that*

$$\limsup_{\tau\to\infty} \Psi_{N_\tau}\bigg(\Psi^*_{\overline{X}}\Big(\frac{p\omega}{\tau}\Big)\bigg) = d_1 < \infty, \tag{21}$$

$$\limsup_{\tau\to\infty} \tau\bigg\{E\bigg[\Big(\frac{1}{\tau}N_\tau - \eta\Big)^{2q}\bigg]\bigg\}^{1/q} = d_2 < \infty \tag{22}$$

and

$$\limsup_{\tau\to\infty} \Psi^*_{N_\tau}\bigg(\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2} e^{\delta\bar{\gamma}} \Psi^*_{\overline{X}}(\delta)\bigg) = d_3 < \infty. \tag{23}$$

Then for $\tau > 1/\delta^2$,

$$\begin{aligned} &\|\Psi_{N_\tau}(E[T(X/\tau)])f - T(\eta\gamma)f\| \\ &\quad \le M\omega_2(Tf, 1/\sqrt{\tau})\bigg\{2d_1 \\ &\qquad + 9mMe^{2\omega\eta\bar{\gamma}} e^{2(m-1)\omega/\sqrt{\tau}} d_1^{1/p}\bigg[m2^{1/q}\Big(\frac{2q}{e}\Big)^2 d_3^{1/q} + d_2\sum_{i=1}^m \gamma_i^2\bigg]\bigg\}. \end{aligned} \tag{24}$$

If $\omega = 0$, *then*

(25)
$$\|\Psi_{N_\tau}(E[T(X/\tau)])f - T(\eta\gamma)f\| \le 2M\omega_2(Tf, 1/\sqrt{\tau})\Big\{1 + \frac{9}{4}mM\sum_{i=1}^{m}\Big[\eta\sigma^2(X_{0i}) + \gamma_i^2\frac{1}{\tau}\sigma^2(N_\tau)\Big]\Big\}.$$

Proof. Take random vectors $X_k \overset{\text{i.i.d.}}{\sim} X$, which are also independent of N_τ. Consider $Y_\tau = (1/\tau)\sum_{k=1}^{N_\tau} X_k$, where $Y_\tau = (Y_{01}, \ldots, Y_{0m})$, then apply Lemma 3 with $h = 1/\sqrt{\tau}$. We have

$$E[T(Y_\tau)]f = \sum_{l=0}^{\infty} P(N_\tau = l)E\Big[T\Big(\frac{1}{\tau}\sum_{k=1}^{l} X_k\Big)\Big]f = \Psi_{N_\tau}(E[T(X/\tau)]f),$$

$$E(Y_\tau) = \sum_{l=0}^{\infty} P(N_\tau = l)\frac{1}{\tau}E\Big(\sum_{k=1}^{l} X_k\Big) = \frac{1}{\tau}E(N_\tau)E(X) = \eta\gamma,$$

$$E(e^{\omega\overline{Y}_\tau}) \le E(e^{p\omega\overline{Y}_\tau}) = \Psi_{N_\tau}(\Psi^*_{\overline{X}}(p\omega/\tau)) \le d_1.$$

Furthermore,

$$\begin{aligned}
&(E((Y_{0i} - \eta\gamma_i)^{2q}))^{1/q} \\
&\quad = \Big\{E\Big(\Big[\frac{1}{\tau}\sum_{k=1}^{N_\tau} X_{ki} - \frac{1}{\tau}N_\tau\gamma_i + \frac{1}{\tau}N_\tau\gamma_i - \eta\gamma_i\Big]^{2q}\Big)\Big\}^{1/q} \\
&\quad \le 2\Big[E\Big(\Big(\frac{1}{\tau}\sum_{k=1}^{N_\tau}(X_{ki} - \gamma_i)\Big)^{2q}\Big)\Big]^{1/q} + 2\gamma_i^2\Big[E\Big(\Big(\frac{1}{\tau}N_\tau - \eta\Big)^{2q}\Big)\Big]^{1/q} \\
&\quad =: 2I_1 + 2I_2.
\end{aligned}$$

We observe that

$$\begin{aligned}
I_1^q &= E\Big[\Big(\frac{1}{\tau}\sum_{k=1}^{N_\tau}(X_{ki} - \gamma_i)\Big)^{2q}\Big] \\
&\le \Big(\frac{2q}{\sqrt{\tau}}\Big)^{2q} e^{-2q}E[e^{(1/\sqrt{\tau})|\Sigma_{k=1}^{N_\tau}(X_{ki} - \gamma_i)|}] \qquad \text{(by (17))} \\
&\le \Big(\frac{2q}{e\sqrt{\tau}}\Big)^{2q}\{E[e^{(1/\sqrt{\tau})\Sigma_{k=1}^{N_\tau}(X_{ki} - \gamma_i)}] + E[e^{(1/\sqrt{\tau})\Sigma_{k=1}^{N_\tau}(\gamma_i - X_{ki})}]\} \\
&\le \Big(\frac{2q}{e\sqrt{\tau}}\Big)^{2q}\{E[(E(e^{(1/\sqrt{\tau})(X_{0i} - \gamma_i)}))^{N_\tau}] + E[(E(e^{(1/\sqrt{\tau})(\gamma_i - X_{0i})}))^{N_\tau}]\} \\
&\le \Big(\frac{2q}{e\sqrt{\tau}}\Big)^{2q}\Big\{E\Big[\Big(E\Big(1 + \frac{1}{\sqrt{\tau}}(X_{0i} - \gamma_i)
\end{aligned}$$

$$+ \frac{1}{2\tau}(X_{0i} - \gamma_i)^2 e^{(1/\sqrt{\tau})|X_{0i}-\gamma_i|}\bigg)\bigg)^{N_\tau}\bigg]$$

$$+ E\bigg[\bigg(E\bigg(1 + \frac{1}{\sqrt{\tau}}(\gamma_i - X_{0i}) + \frac{1}{2\tau}(X_{0i} - \gamma_i)^2 e^{(1/\sqrt{\tau})|X_{0i}-\gamma_i|}\bigg)\bigg)^{N_\tau}\bigg]\bigg\}$$

$$\le 2\bigg(\frac{2q}{e\sqrt{\tau}}\bigg)^{2q} E\{(\exp[E(X_{0i} - \gamma_i)^2 e^{(1/\sqrt{\tau})|X_{0i}-\gamma_i|}])^{N_\tau/(2\tau)}\} \quad (1/\sqrt{\tau} < \delta)$$

$$\le 2\bigg(\frac{2q}{e\sqrt{\tau}}\bigg)^{2q} E\bigg\{\bigg(\exp\bigg[\bigg(\frac{2}{\delta - 1/\sqrt{\tau}}\bigg)^2 e^{-2} E(e^{\delta|X_{0i}-\gamma_i|})\bigg]\bigg)^{N_\tau/(2\tau)}\bigg\}$$

$$\le 2\bigg(\frac{2q}{e\sqrt{\tau}}\bigg)^{2q} E\bigg\{\bigg(\exp\bigg[\bigg(\frac{2\sqrt{\tau}}{e(\sqrt{\tau}\,\delta - 1)}\bigg)^2 e^{\delta\bar{\gamma}} \Psi^*_{\overline{X}}(\delta)\bigg]\bigg)^{N_\tau/(2\tau)}\bigg\}$$

$$\le 2\bigg(\frac{2q}{e\sqrt{\tau}}\bigg)^{2q} d_3.$$

So

$$I_1 \le 2^{1/q} \frac{4q^2}{e^2\tau} d_3^{1/q}$$

and

$$I_2 = \gamma_i^2\bigg(E\bigg(\bigg(\frac{1}{\tau}N_\tau - \eta\bigg)^{2q}\bigg)\bigg)^{1/q} \le \gamma_i^2 d_2/\tau.$$

Therefore by inequality (9) of Lemma 3 for $Y = Y_\tau$ and $h = 1/\sqrt{\tau}$, we get

$$\begin{aligned}
&\|\Psi_{N_\tau}[E(T(X/\tau))]f - T(\eta\gamma)f\| \\
&\qquad \le M\omega_2(Tf, 1/\sqrt{\tau})\bigg\{2d_1 + \frac{9}{2}mMe^{2\omega\eta\bar{\gamma}} e^{2(m-1)\omega/\sqrt{\tau}} d_1^{1/p} \\
&\qquad\quad \times 2\sum_{i=1}^m \bigg[2^{1/q}\bigg(\frac{4q^2}{e^2\tau}d_3^{1/q}\bigg) + d_2\frac{\gamma_i^2}{\tau}\bigg]\tau\bigg\} \\
&\qquad = M\omega_2(Tf, 1/\sqrt{\tau})\bigg\{2d_1 + 9mMe^{2\omega\eta\bar{\gamma}} e^{2(m-1)\omega/\sqrt{\tau}} d_1^{1/p} \\
&\qquad\quad \times \bigg[m2^{1/q}\bigg(\frac{2q}{e}\bigg)^2 d_3^{1/q} + d_2\sum_{i=1}^m \gamma_i^2\bigg]\bigg\}.
\end{aligned}$$

If $\omega = 0$, we apply inequality (10) of Lemma 3. Note that

$$\begin{aligned}
\sigma^2(Y_{0i}) &= \sigma^2\bigg(\frac{1}{\tau}\sum_{k=1}^{N_\tau} X_{ki}\bigg) = \frac{1}{\tau^2}\sum_{l=0}^{\infty} P(N_\tau = l) E\bigg(\bigg(\sum_{k=1}^{l} X_{ki}\bigg)^2\bigg) - \eta^2\gamma_i^2 \\
&= \frac{1}{\tau^2}\sum_{l=0}^{\infty} P(N_\tau = l)(lE(X_{0i}^2) + l(l-1)\gamma_i^2) - \eta^2\gamma_i^2
\end{aligned}$$

$$= \frac{1}{\tau^2} E(N_\tau)E(X_{0i}^2) + \frac{1}{\tau^2}(E(N_\tau^2))\gamma_i^2 - \frac{1}{\tau^2}(E(N_\tau))\gamma_i^2 - \eta^2\gamma_i^2$$
$$= \frac{1}{\tau}\bigg[\eta\sigma^2(X_{0i}) + \gamma_i^2 \frac{1}{\tau}\sigma^2(N_\tau)\bigg].$$

By (10), when $h = 1/\sqrt{\tau}$, we obtain (25). ∎

Another generalization of Theorem 1 is presented next.

THEOREM 4. *Let $N = (N_1, \ldots, N_m)$ be a $\mathbb{Z}_+^m$-valued random vector with $E(N) = \eta = (\eta_1, \ldots, \eta_m)$. For each i $(1 \le i \le m)$, let $\{Y_{ki}\}_{k=1}^\infty$ be a sequence of i.i.d. real-valued random variables distributed as Y, a fixed random variable with $E(Y) = \gamma$. N and Y_{ki} are assumed to be independent. Also assume that there exists a $\delta > 0$ such that*

$$\Psi_{\overline{N}}(\Psi_Y^*(\delta)) < \infty.$$

Then for $n > \max(p\omega/\delta, 1/\delta^2)$,

$$\bigg\|\bigg\{E\bigg[T\bigg(\sum_{k_1=1}^{N_1} \frac{1}{n} Y_{k_1 1}, \ldots, \sum_{k_m=1}^{N_m} \frac{1}{n} Y_{k_m m}\bigg)\bigg]\bigg\}^n f - T(\gamma\eta) f\bigg\| \tag{26}$$
$$\le 2M\omega_2(Tf, 1/\sqrt{n})\bigg\{e^{\omega\gamma\bar\eta} \exp\bigg[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi_{\overline{N}}(\Psi_Y^*(\delta))\bigg]$$
$$+ 2^{1/q} 9M \frac{m^2q^2}{e^2} e^{3\omega\gamma\bar\eta} e^{2(m-1)\omega/\sqrt{n}}$$
$$\times \exp\bigg[\bigg(\frac{2np\omega^2}{e^2(n\delta - p\omega)^2} + \frac{2ne^{\delta\gamma\bar\eta}}{e^2 q(\delta\sqrt{n}-1)^2}\bigg)\Psi_{\overline{N}}(\Psi_Y^*(\delta))\bigg]\bigg\},$$

where $p, q > 0$, $1/p + 1/q = 1$, is an arbitrary conjugate pair. If $\omega = 0$, then

$$\bigg\|\bigg\{E\bigg[T\bigg(\sum_{k_1=1}^{N_1} \frac{1}{n} Y_{k_1 1}, \ldots, \sum_{k_m=1}^{N_m} \frac{1}{n} Y_{k_m m}\bigg)\bigg]\bigg\}^n f - T(\gamma\eta) f\bigg\| \tag{27}$$
$$\le 2M\omega_2(Tf, 1/\sqrt{n})\bigg\{1 + \frac{9}{4} mM \sum_{i=1}^m [\eta_i \sigma^2(Y) + \sigma^2(N_i)\gamma^2]\bigg\}.$$

Proof. In Theorem 1, take $X := (\sum_{k_1=1}^{N_1} Y_{k_1 1}, \ldots, \sum_{k_m=1}^{N_m} Y_{k_m m})$ and let $X_k \overset{\text{i.i.d.}}{\sim} X$. Then

$$E(X) = \bigg(E\bigg[\sum_{k_1=1}^{N_1} Y_{k_1 1}\bigg], \ldots, E\bigg[\sum_{k_m=1}^{N_m} Y_{k_m m}\bigg]\bigg)$$
$$= (EN_1EY, \ldots, EN_mEY) = \gamma\eta.$$

We observe that

$$\begin{aligned}
\Psi^*_{\overline{X}}(\delta) &= E(e^{\delta \overline{X}}) \\
&= E(e^{\delta(\Sigma_{k_1=1}^{N_1} Y_{k_1 1}+\ldots+\Sigma_{k_m=1}^{N_m} Y_{k_m m})}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,\ldots,l_m))E(e^{\delta\Sigma_{k_1=1}^{l_1} Y_{k_1 1}+\ldots+\delta\Sigma_{k_m=1}^{l_m} Y_{k_m m}}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,\ldots,l_m))E(e^{\delta\Sigma_{k_1=1}^{l_1} Y_{k_1 1}})\ldots E(e^{\delta\Sigma_{k_m=1}^{l_m} Y_{k_m m}}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,\ldots,l_m))(E(e^{\delta Y}))^{l_1}\ldots(E(e^{\delta Y}))^{l_m} \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,\ldots,l_m))(E(e^{\delta Y}))^{l_1+\ldots+l_m} \\
&= E((E(e^{\delta Y}))^{N_1+\ldots+N_m}) = \Psi_{\overline{N}}(\Psi^*_Y(\delta)).
\end{aligned}$$

Thus (26) follows from (15).

If $\omega = 0$ we see that

$$\sigma^2(X_{0i}) = \sigma^2\Big(\sum_{k_i=1}^{N_i} Y_{k_i i}\Big) = \eta_i\sigma^2(Y) + \sigma^2(N_i)\gamma^2,$$

established similarly to the fact at the end of the proof of Theorem 2. Now (27) follows from (16). ■

5. Further results: multiplier enlargement formulae. In this section we modify the formulae obtained in the previous section by the so-called multiplier enlargement method (see [6]) initiated by Hsu–Wang [9, 19] in the 60's and also used by Shaw [17, 18] in the representation of operator semigroups. The modified representation formulae have a larger range of applications and when we specify the random vectors (variables) considered, we recover the representation formulae for m-parameter operator semigroups of Shaw [18]. For simplicity we only consider equibounded operator semigroups, i.e.,

$$\|T(t)\| \le M \qquad \text{for all } t \in \mathbb{R}_+^m.$$

Here we only need to give two versions related to Theorems 1 and 4. Others can be similarly obtained.

Theorem 5. *Suppose* $\|T(t)\| \le M$ *for all* $t \in \mathbb{R}_+^m$, *and* α_n *is a sequence of positive real numbers with* $\liminf_{n\to\infty}\alpha_n > 0$. *For each* $n \in \mathbb{N}$ *let* $X(n)$ *be an* $\mathbb{R}_+^m$*-valued random vector with* $E[X(n)] = x/\alpha_n$. *Assume*

$\limsup_{n\to\infty} \alpha_n \sigma_i^2(X(n)) < \infty$, $i = 1, \ldots, m$. *Then*

$$\left\| \left\{ E\left[T\left(\frac{\alpha_n}{n} X(n) \right) \right] \right\}^n f - T(x) f \right\| \le 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left[1 + \frac{9}{4} mM\alpha_n \sum_{i=1}^m \sigma_i^2(X(n)) \right]. \tag{28}$$

P r o o f. For each fixed n, let $X_k \overset{\text{i.i.d.}}{\sim} X(n)$, $k = 1, \ldots, n$, and consider

$$Y := \frac{1}{n} \sum_{k=1}^n \alpha_n X_k.$$

Then

$$E(Y) = E\left(\frac{1}{n} \sum_{k=1}^n \alpha_n X_k \right) = \alpha_n E(X(n)) = \alpha_n x / \alpha_n = x$$

and

$$E[T(Y)]f = E\left[T\left(\frac{1}{n} \sum_{k=1}^n \alpha_n X_k \right) \right] f = \left\{ E\left[T\left(\frac{\alpha_n}{n} X(n) \right) \right] \right\}^n f.$$

Furthermore,

$$\sigma_i^2(Y) = \sigma_i^2\left(\frac{1}{n} \sum_{k=1}^n \alpha_n X_k \right) = \frac{\alpha_n^2}{n^2} n \sigma_i^2(X(n)) = \frac{\alpha_n^2}{n} \sigma_i^2(X(n)).$$

Now take $h = (\alpha_n/n)^{1/2}$; then by (10), we get (28). ∎

THEOREM 6. *Let α_n be a sequence of positive real numbers satisfying*

$$\lim_{n\to\infty} \alpha_n/n = 0 \quad \textit{and} \quad \liminf_{n\to\infty} \alpha_n > 0.$$

For each $n \in \mathbb{N}$, let $N(n) := (N_1(n), \ldots, N_m(n))$ be a $\mathbb{Z}_+^m$-valued random vector with $E(N(n)) = (1/\alpha_n)\eta = (1/\alpha_n)(\eta_1, \ldots, \eta_m)$ and

$$\limsup_{n\to\infty} \alpha_n \sigma_i^2(N(n)) < \infty \qquad \textit{for all } i = 1, \ldots, m.$$

For each i $(1 \le i \le m)$, let $\{Y_{ki}(n)\}_{k=1}^\infty$ be a sequence of i.i.d. random variables, distributed as $Y(n)$, where $Y(n)$ is a fixed real-valued random variable for each $n \in \mathbb{N}$ and $E(Y(n)) = \alpha_n \gamma$. Suppose that $Y_{0i}(n)$ $(i = 1, \ldots, m)$ and $N(n)$ are all independent. Assume also that

$$\limsup_{n\to\infty} \sigma^2(Y(n))/\alpha_n^2 < \infty.$$

Consider the equibounded operator semigroup $\{T(t) : t \in \mathbb{R}_+^m\}$ with

$$\|T(t)\| \le M \qquad \textit{for all } t \in \mathbb{R}_+^m.$$

Then

$$\left\|\left\{E\left[T\Big(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_1 1}(n),\dots,\sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_m m}(n)\Big)\right]\right\}^n f-T(\gamma\eta)f\right\| \tag{29}$$

$$\le 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\left\{1+\frac{9}{4}mM\sum_{i=1}^{m}\left[\frac{\eta_i}{\alpha_n^2}\sigma^2(Y(n))+\gamma^2\alpha_n\sigma_i^2(N(n))\right]\right\}.$$

P r o o f. We want to apply Lemma 3. Let

$$X_k\overset{\text{i.i.d.}}{\sim}X:=\Big(\sum_{k_1=1}^{N_1(n)}Y_{k_1 1}(n),\dots,\sum_{k_m=1}^{N_m(n)}Y_{k_m m}(n)\Big)$$

and

$$Y=\frac{1}{n}\sum_{k=1}^{n}X_k.$$

Then

$$\begin{aligned}E(Y)=E(X)&=\Big(E\Big(\sum_{k_1=1}^{N_1(n)}Y_{k_1 1}(n)\Big),\dots,E\Big(\sum_{k_m=1}^{N_m(n)}Y_{k_m m}(n)\Big)\Big)\\&=\left(\frac{1}{\alpha_n}\eta_1\alpha_n\gamma,\dots,\frac{1}{\alpha_n}\eta_m\alpha_n\gamma\right)=(\eta_1\gamma,\dots,\eta_m\gamma)=\gamma\eta.\end{aligned}$$

Moreover,

$$E[T(Y)]f=\left\{E\left[T\Big(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_1 1}(n),\dots,\sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_m m}(n)\Big)\right]\right\}^n f.$$

Furthermore,

$$\begin{aligned}\sigma_i^2(Y)=\sigma_i^2\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)&=\frac{1}{n}\sigma^2\Big(\sum_{k_i=1}^{N_i(n)}Y_{k_i i}(n)\Big)\\&=\frac{1}{n}[E(N_i(n))\sigma^2(Y_{0i}(n))+\sigma^2(N_i(n))(E(Y_{0i}(n)))^2]\\&=\frac{1}{n}\left[\frac{1}{\alpha_n}\eta_i\sigma^2(Y(n))+\sigma^2(N_i(n))\alpha_n^2\gamma^2\right].\end{aligned}$$

Pick $h:=(\alpha_n/n)^{1/2}$; then by (10) of Lemma 3 we have

$$\left\|\left\{E\left[T\Big(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_1 1}(n),\dots,\sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_m m}(n)\Big)\right]\right\}^n f-T(\gamma\eta)f\right\|$$

$$\leq 2M\omega_2(Tf,(\alpha_n/n)^{1/2})$$
$$\times\left\{1+\frac{9mM}{4}\frac{n}{\alpha_n}\sum_{i=1}^{m}\frac{1}{n}\left[\frac{1}{\alpha_n}\eta_i\sigma^2(Y_{0i}(n))+\sigma^2(N_i(n))\alpha_n^2\gamma^2\right]\right\}$$
$$\leq 2M\omega_2(Tf,(\alpha_n/n)^{1/2})$$
$$\times\left\{1+\frac{9}{4}mM\sum_{i=1}^{m}\left[\frac{\eta_i}{\alpha_n^2}\sigma^2(Y(n))+\gamma^2\alpha_n\sigma_i^2(N(n))\right]\right\}. \ \blacksquare$$

6. Applications. In this section we specify the random vectors (variables) and α_n of Theorems 1–6 to derive some concrete representation formulae for (C_0) m-parameter operator semigroups. We also illustrate how to get the results on multivariate approximation operators from the corresponding ones on operator semigroups. Unless otherwise mentioned all (C_0) m-parameter operator semigroups considered satisfy (5).

EXAMPLE 1. Take $X=(X_{01},\ldots,X_{0m})$ that follows the multi-point distribution with $EX=x=(x_1,\ldots,x_m)$:

$$P(X=e_i)=x_i \qquad (e_i=(0,\ldots,1,\ldots,0))$$

and

$$P(X=0)=1-\overline{x}, \qquad \text{where } 0<\overline{x}<1\ (\overline{x}=x_1+\ldots+x_m).$$

Then

$$\Psi^*_{\overline{X}}(\delta)=E(e^{\delta\overline{X}})=P(\overline{X}=0)+P(\overline{X}=1)e^{\delta}=1-\overline{x}+\overline{x}e^{\delta}<\infty.$$

Furthermore, we have

$$E[T(X/n)]=I+\sum_{i=1}^{m}x_i(T_i(1/n)-I).$$

Hence by Theorem 1 there is a constant $K=K(\omega,M,x,\delta,m)$ such that

$$\|(I+\sum_{i=1}^{m}x_i(T_i(1/n)-I))^n f-T(x)f\| \leq K\omega_2(Tf,\ 1/\sqrt{n})\to 0 \qquad (n\to\infty). \tag{30}$$

From the above result on operator semigroups we are able to recover the approximation theorem for multivariate Bernstein operators as follows. Choose

$\mathcal{X}:=\mathrm{BUC}(\mathbb{R}^m)$

$:=\{f: f$ is a bounded uniformly continuous function from $\mathbb{R}^m$ into $\mathbb{R}\}$

and define

$$T(t)f(x):=f(x+t)=f(x_1+t_1,\ldots,x_m+t_m)$$

for each $f \in \mathcal{X}$ and $x \in \mathbb{R}^m$. Then $\{T(t) : t \in \mathbb{R}_+^m\}$ is a (C_0) m-parameter operator semigroup in $\mathcal{E}(\mathcal{X})$.

Now let $x = 0$, $t = (t_1, \ldots, t_m)$, $0 < \overline{t} < 1$, $0 < t_i < 1$, $i = 1, \ldots, m$. Then

$$\begin{aligned}\Big\{I + \sum_{i=1}^{m} t_i[T_i(1/n) - I]\Big\}^n f(0) &\\ = \sum_{k\in\mathbb{Z}_+^m,\, \bar{k}\le n} f(k_1/n, \ldots, k_m/n) \frac{n!}{k_1! \ldots k_m!(n - k_1 - \ldots - k_m)!} &\\ \times t_1^{k_1} \ldots t_m^{k_m} (1 - \overline{t})^{n-\bar{k}} &\\ = B_n^f(t_1, \ldots, t_m), &\end{aligned}$$

where $B_n^f(t_1, \ldots, t_m)$ is the m-variate Bernstein operator over a simplex (cf. [12]). So by (30), we obtain

$$\lim_{n\to\infty} B_n^f(t_1, \ldots, t_m) = T(t)f(0) = f(t_1, \ldots, t_m), \qquad \text{uniformly.}$$

Remark. The fact that the approximation theorem for the Bernstein operator can be derived from simple operator semigroup considerations has been observed by many authors (see, e.g., [1, p. 28], [10] and [14]). When considering other representation formulae for m-parameter operator semigroups in the following examples we may derive other known convergence theorems for multivariate approximation operators, but we avoid to go into details here.

Example 2. Let α_n be a sequence of positive real numbers with $\liminf_{n\to\infty} \alpha_n > 0$ and $\lim_{n\to\infty} \alpha_n/n = 0$. For each $n \in \mathbb{N}$ take $X(n) = (X_{01}(n), \ldots, X_{0m}(n))$ to have a modified multi-point distribution:

$$P(X(n) = e_i) = x_i/\alpha_n, \qquad 1 \le i \le m,$$
$$P(X(n) = 0) = 1 - \overline{x}/\alpha_n \qquad (0 < \overline{x}/\alpha_n < 1 \text{ and } x_i > 0).$$

Then

$$E[X(n)] = \frac{x}{\alpha_n} \qquad (x = (x_1, \ldots, x_m))$$

and

$$\sigma_i^2(X(n)) = E(X_i^2(n)) - (E(X_i(n)))^2 = \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}.$$

For an equibounded (C_0) m-parameter operator semigroup $\{T(t) : t \in \mathbb{R}_+^m\}$ with $\|T(t)\| \le M$ for all $t \in \mathbb{R}_+^m$, we have

$$E\left[T\left(\frac{\alpha_n}{n}X(n)\right)\right] = T(0)P(X(n)=0) + \sum_{i=1}^{m} T\left(\frac{\alpha_n}{n}e_i\right)P(X(n)=e_i)$$
$$= I + \sum_{i=1}^{m}\frac{x_i}{\alpha_n}\left(T_i\left(\frac{\alpha_n}{n}\right) - I\right).$$

Thus by Theorem 5, we obtain

$$\left\|\left\{I + \sum_{i=1}^{m}\frac{x_i}{\alpha_n}\left(T_i\left(\frac{\alpha_n}{n}\right) - I\right)\right\}^n f - T(x)f\right\| \tag{31}$$
$$\le 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\left[1 + \frac{9}{4}mM\alpha_n\sum_{i=1}^{m}\left(\frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}\right)\right]$$
$$= 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\left[1 + \frac{9}{4}mM\sum_{i=1}^{m}\left(x_i - \frac{x_i^2}{\alpha_n}\right)\right] \to 0 \qquad (n\to\infty).$$

Remark. (30) is the special case of (31) when $\alpha_n \equiv 1$, but (30) is true for an arbitrary (C_0) m-parameter operator semigroup.

Inequalities (31) and the following (32)–(34) are Shaw's formulae [17, 18] supplied with rates of convergence.

Example 3. Assume α_n is as in Example 2. For each n, let $X(n) := (X_{01}(n),\ldots,X_{0m}(n))$ follow the negative multi-point distribution:

$$P(X(n)=(k_1,\ldots,k_m)) = \binom{\overline{k}}{k}\left(1+\frac{1}{\alpha_n}\overline{x}\right)^{-1}\prod_{i=1}^{m}\left(\frac{x_i}{\alpha_n+\overline{x}}\right)^{k_i}$$

for all $k=(k_1,\ldots,k_m)\in\mathbb{Z}_+^m = \{(n_1,\ldots,n_m) : n_i\in\mathbb{Z}_+, 1\le i\le m\}$, where $x=(x_1,\ldots,x_m)\in\mathbb{R}_+^m$ is fixed and

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-\overline{k}+1)}{k_1!\ldots k_m!}.$$

Then

$$P(X_{0i}(n)=k_i) = \left(1+\frac{1}{\alpha_n}x_i\right)^{-1}\left(\frac{x_i}{\alpha_n+x_i}\right)^{k_i}$$

(see [7, p. 165, (8.4)]),

$$E(X_{0i}(n)) = \sum_{k_i=0}^{\infty} k_i\left(1+\frac{1}{\alpha_n}x_i\right)^{-1}\left(\frac{x_i}{\alpha_n+x_i}\right)^{k_i} = \frac{x_i}{\alpha_n}$$

and

$$\sigma_i^2(X(n)) = \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n}.$$

For an equibounded (C_0) m-parameter operator semigroup $\{T(t) : t \in \mathbb{R}_+^m\}$ with $\|T(t)\| \le M$ for all $t \in \mathbb{R}_+^m$, we have

$$\begin{aligned}
&E\Big[T\Big(\frac{\alpha_n}{n}X(n)\Big)\Big] \\
&\quad = \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty} T\Big(\frac{\alpha_n}{n}(k_1,\ldots,k_m)\Big)\binom{\overline{k}}{k}\Big(1+\frac{1}{\alpha_n}\overline{x}\Big)^{-1}\prod_{i=1}^{m}\Big(\frac{x_i}{\alpha_n+\overline{x}}\Big)^{k_i} \\
&\quad = \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\binom{\overline{k}}{k}\Big(1+\frac{1}{\alpha_n}\overline{x}\Big)^{-1}\prod_{i=1}^{m}\Big(\frac{x_iT_i(\alpha_n/n)}{\alpha_n+\overline{x}}\Big)^{k_i} \\
&\quad = \Big(1+\frac{1}{\alpha_n}\overline{x}\Big)^{-1}\Big[I-\frac{x_1T_1(\alpha_n/n)+\ldots+x_mT_m(\alpha_n/n)}{\alpha_n+\overline{x}}\Big]^{-1} \\
&\quad = \Big\{I+\frac{I}{\alpha_n}\overline{x}-\frac{x_1}{\alpha_n}T_1(\alpha_n/n)-\ldots-\frac{x_m}{\alpha_n}T_m(\alpha_n/n)\Big\}^{-1} \\
&\quad = \Big\{I-\sum_{i=1}^{m}\frac{x_i}{\alpha_n}(T_i(\alpha_n/n)-I)\Big\}^{-1}.
\end{aligned}$$

By Theorem 5 we obtain

$$\begin{aligned}
(32)\qquad &\Big\|\Big\{I-\sum_{i=1}^{m}\frac{x_i}{\alpha_n}(T_i(\alpha_n/n)-I)\Big\}^{-n}f-T(x)f\Big\| \\
&\le 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\Big[1+\frac{9}{4}mM\alpha_n\sum_{i=1}^{m}\Big(\frac{x_i}{\alpha_n}+\frac{x_i^2}{\alpha_n^2}\Big)\Big] \\
&= 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\Big[1+\frac{9}{4}mM\sum_{i=1}^{m}\Big(x_i+\frac{x_i^2}{\alpha_n}\Big)\Big] \to 0 \qquad (n\to\infty).
\end{aligned}$$

Example 4. In Theorem 6 take $N(n)$ that follows the multi-point distribution:

$$P(N(n)=e_i)=x_i/\alpha_n, \qquad 1\le i\le m,$$
$$P(N(n)=0)=1-\overline{x}/\alpha_n, \qquad \text{where } x=(x_1,\ldots,x_m)\in\mathbb{R}_+^m, \text{ is fixed.}$$

Here α_n is as in Example 2. Let $Y_{0i}(n)$, $1\le i\le m$, be exponentially distributed with density $(1/\alpha_n)e^{-v/\alpha_n}$, $v\in\mathbb{R}_+$. Then

$$E(N(n))=\frac{1}{\alpha_n}x=\Big(\frac{1}{\alpha_n}x_1,\ldots,\frac{1}{\alpha_n}x_m\Big), \qquad E(Y_{0i}(n))=\alpha_n,$$
$$\sigma^2(N_i(n))=\frac{x_i}{\alpha_n}-\frac{x_i^2}{\alpha_n^2} \quad\text{and}\quad \sigma^2(Y_{0i}(n))=\alpha_n^2.$$

Also

$$E\left[T_i\left(\frac{1}{n}Y_{0i}(n)\right)\right] = \int\limits_0^\infty T_i(v/n)\frac{1}{\alpha_n}e^{-v/\alpha_n}\,dv = \left(I - \frac{\alpha_n}{n}A_i\right)^{-1}$$

(cf. [8, p. 360] and [18, p. 226]).

Furthermore, we have

$$\begin{aligned} E\Bigg[T\Bigg(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_1 1}(n),\ldots,\sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_m m}(n)\Bigg)\Bigg] \\ = T(0)P(N(n)=0) + \sum_{i=1}^m E\left[T_i\left(\frac{1}{n}Y_{k_i}(n)\right)\right]P(N(n)=e_i) \\ = I + \sum_{i=1}^m \frac{x_i}{\alpha_n}\left[\left(I - \frac{\alpha_n}{n}A_i\right)^{-1} - I\right]. \end{aligned}$$

By (29) of Theorem 6 for an equibounded (C_0) m-parameter operator semigroup $\{T(t) : t \in \mathbb{R}_+^m\}$ with $\|T(t)\| \le M$ for all $t \in \mathbb{R}_+^m$, we get

$$\left\|\left\{I + \sum_{i=1}^m (x_i/\alpha_n)\left[\left(I - \frac{\alpha_n}{n}A_i\right)^{-1} - I\right]\right\}^n f - T(x)f\right\| \tag{33}$$

$$\begin{aligned} &\le 2M\omega_2(Tf, (\alpha_n/n)^{1/2})\left\{1 + \frac{9}{4}mM\sum_{i=1}^m\left[\frac{x_i}{\alpha_n^2}\alpha_n^2 + \alpha_n\left(\frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}\right)\right]\right\} \\ &= 2M\omega_2(Tf, (\alpha_n/n)^{1/2})\left[1 + \frac{9}{4}mM\sum_{i=1}^m(2x_i - x_i^2/\alpha_n)\right] \to 0 \qquad (n\to\infty). \end{aligned}$$

EXAMPLE 5. Take α_n and $Y_{0i}(n)$ as in Example 4. Let $N(n)$ have the negative multi-point distribution:

$$P(N(n) = (l_1,\ldots,l_m)) = \binom{\overline{l}}{l}\left(1 + \frac{1}{\alpha_n}\overline{x}\right)^{-1}\prod_{i=1}^m\left(\frac{x_i}{\alpha_n + \overline{x}}\right)^{l_i}$$

for all $l = (l_1,\ldots,l_m) \in \mathbb{Z}_+^m$, where $x = (x_1,\ldots,x_m) \in \mathbb{R}_+^m$ is fixed. Then

$$E(N(n)) = \frac{1}{\alpha_n}x = \left(\frac{1}{\alpha_n}x_1,\ldots,\frac{1}{\alpha_n}x_m\right), \qquad E(Y_{0i}(n)) = \alpha_n,$$

$$\sigma^2(N_i(n)) = \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n} \quad\text{and}\quad \sigma^2(Y_{0i}(n)) = \alpha_n^2.$$

Furthermore, we observe that

$$E\Big[T\Big(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_11}(n),\dots,\sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_mm}(n)\Big)\Big]$$
$$=\sum_{l\in\mathbb{Z}_+^m}P(N(n)=l)E\Big[T\Big(\sum_{k_1=1}^{l_1}\frac{1}{n}Y_{k_11}(n)\Big),\dots,T\Big(\sum_{k_m=1}^{l_m}\frac{1}{n}Y_{k_mm}(n)\Big)\Big]$$
$$=\sum_{l\in\mathbb{Z}_+^m}\binom{\overline{l}}{l}\Big(1+\frac{1}{\alpha_n}\overline{x}\Big)^{-1}\prod_{i=1}^m\Big(\frac{x_iE[T_i(Y/n)]}{\alpha_n+\overline{x}}\Big)^{l_i}$$
$$=\Big\{I+\frac{1}{\alpha_n}\overline{x}I-\sum_{i=1}^m\frac{x_i}{\alpha_n}E[T_i(Y/n)]\Big\}^{-1}$$
$$=\Big\{I+\frac{1}{\alpha_n}\overline{x}I-\sum_{i=1}^m\frac{x_i}{\alpha_n}\Big(I-\frac{\alpha_n}{n}A_i\Big)^{-1}\Big\}^{-1}$$
$$=\Big\{I-\sum_{i=1}^m\frac{x_i}{\alpha_n}\Big[\Big(I-\frac{\alpha_n}{n}A_i\Big)^{-1}-I\Big]\Big\}^{-1}.$$

Thus by (29) of Theorem 6, for an equibounded (C_0) m-parameter operator semigroup $\{T(t):t\in\mathbb{R}_+^m\}$ with $\|T(t)\|\le M$ for all $t\in\mathbb{R}_+^m$, we have

$$\Big\|\Big\{I-\sum_{i=1}^m\frac{x_i}{\alpha_n}\Big[\Big(I-\frac{\alpha_n}{n}A_i\Big)^{-1}-I\Big]\Big\}^{-n}f-T(x)f\Big\| \tag{34}$$
$$\le 2M\omega_2(Tf,(\alpha_n/n)^{1/2})\Big\{1+\frac{9}{4}mM\sum_{i=1}^m\Big[\frac{x_i}{\alpha_n^2}\alpha_n^2+\alpha_n\Big(\frac{x_i^2}{\alpha_n^2}+\frac{x_i}{\alpha_n}\Big)\Big]\Big\}$$
$$=2M\omega_2(Tf,(\alpha_n/n)^{1/2})\Big[1+\frac{9}{4}mM\sum_{i=1}^m\Big(2x_i+\frac{x_i^2}{\alpha_n}\Big)\Big]\to 0\qquad(n\to\infty).$$

EXAMPLE 6. Take N to be a non-negative integer-valued random variable that follows the geometric distribution over $\mathbb{Z}_+$:

$$P(N=k)=\frac{1}{1+\eta}\Big(\frac{\eta}{1+\eta}\Big)^k\qquad\text{for all } k\in\mathbb{Z}_+,\ \text{where } \eta>0 \text{ is a parameter.}$$

Let also $Y\equiv(x_1,\dots,x_m)\in\mathbb{R}_+^m$. Then

$$E(N)=\eta>0,\qquad EY\equiv(x_1,\dots,x_m)=x.$$

Furthermore,

$$\Psi_N(\Psi_{\overline{X}}^*(\delta))=E(e^{\delta(x_1+\ldots+x_m)N})=\frac{1}{1+\eta-e^{\delta(x_1+\ldots+x_m)}\eta}<\infty$$

for $\delta<(1/\overline{x})\ln(1+1/\eta)$. So by Theorem 2, there is a constant $K=$

$K(M,\omega,\delta,\eta,x)$ such that for sufficiently large n,

$$\begin{aligned}(35)\qquad &\|\{\Psi_N(E[T(Y/n)])\}^n f - T(\eta x)f\| \\ &= \|\{I+\eta[I-T(x_1/n,\dots,x_m/n)]\}^{-n}f - T(\eta x)f\| \\ &\le K\omega_2(Tf,1/\sqrt{n}) \to 0 \qquad (n\to\infty).\end{aligned}$$

EXAMPLE 7. In Theorem 3, take $p=q=2$, and N_τ to be the Poisson process ($\tau\in\mathbb{R}_+$):

$$P(N_\tau = k) = e^{-\eta\tau}\frac{(\eta\tau)^k}{k!} \qquad \text{for all } k\in\mathbb{Z}_+^m, \text{ where } \eta>0 \text{ is a parameter.}$$

Consider $X\equiv(x_1,\dots,x_m)=x\in\mathbb{R}_+^m$. Then

$$E(N_\eta)=\eta\tau,\quad \Psi_{N_\tau}(s)=e^{(s-1)\eta\tau},\quad \Psi_{\overline{X}}^*(\delta)=e^{\delta\overline{x}}<\infty.$$

Furthermore, note that

$$\begin{aligned}d_1 &= \limsup_{\tau\to\infty}\Psi_{N_\tau}\left(\Psi_{\overline{X}}^*\left(\frac{2\omega}{\tau}\right)\right) = \limsup_{\tau\to\infty}\exp[(e^{\frac{2\omega}{\tau}\overline{x}}-1)\eta\tau] \\ &\le \limsup_{\tau\to\infty}\exp\left[\eta\tau\frac{2\omega}{\tau}\overline{x}e^{(2\omega/\tau)\overline{x}}\right]<\infty, \\ d_2 &= \limsup_{\tau\to\infty}\tau\left\{E\left[\left(\frac{1}{\tau}N_\tau-\eta\right)^4\right]\right\}^{1/2} = \limsup_{\tau\to\infty}\tau\left(3\frac{\eta^2}{\tau^2}+\frac{\eta}{\tau^3}\right)^{1/2}<\infty\end{aligned}$$

and

$$\begin{aligned}d_3 &= \limsup_{\tau\to\infty}\Psi_{N_\tau}^*\left(\frac{2}{e^2(\sqrt{\tau}\delta-1)^2}e^{\delta\overline{x}}\Psi_{\overline{X}}^*(\delta)\right) \\ &= \limsup_{\tau\to\infty}\exp\left\{\left(\exp\left[\frac{2}{e^2(\sqrt{\tau}\delta-1)^2}e^{2\delta\overline{x}}\right]-1\right)\eta\tau\right\} \\ &\le \limsup_{\tau\to\infty}\exp\left\{\frac{2}{e^2(\sqrt{\tau}\delta-1)^2}e^{2\delta\overline{x}}\exp\left[\frac{2}{e^2(\sqrt{\tau}\delta-1)^2}e^{2\delta\overline{x}}\right]\eta\tau\right\}<\infty.\end{aligned}$$

So by (24) of Theorem 3 there exists a constant $K=K(\delta,M,\omega,d_1,d_2,d_3)$ such that for sufficiently large n we have

$$\begin{aligned}(36)\qquad &\|\Psi_{N_\tau}(E(T(X/\tau)))f - T(\eta x)f\| \\ &= \|\exp[\eta\tau(T(x_1/\tau,\dots,x_m/\tau)-I)]f - T(\eta x)f\| \\ &\le K\omega_2(Tf,1/\sqrt{\tau}) \to 0 \qquad (\tau\to\infty).\end{aligned}$$

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Reçu par la Rédaction le 26.9.1994
Révisé le 28.5.1995