Representation formulae for \((C_0)\) \(m\)-parameter operator semigroups

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Abstract. Some general representation formulae for \((C_0)\) \(m\)-parameter operator semigroups with rates of convergence are obtained by the probabilistic approach and multiplier enlargement method. These cover all known representation formulae for \((C_0)\) one- and \(m\)-parameter operator semigroups as special cases. When we consider special semigroups we recover well-known convergence theorems for multivariate approximation operators.

1. Introduction. Recently the study of representation formulae for \((C_0)\) operator semigroups has attracted much attention (Shaw [17, 18], Butzer–Hahn [2], Pfeifer [13–15] and Chen–Zhou [3]). They gave some general formulae that include earlier (Post–Widder, Hille–Phillips [8] and Chung [4]) concrete representation formulae. But most of the work done so far is confined to the one-parameter case, while Shaw’s method for the multi-parameter case is not an easy one to get new formulae and the results are without rates of convergence. In this article we try to give some general representation formulae for \((C_0)\) \(m\)-parameter operator semigroups. The main idea is the use of a probabilistic setting in the representation of operator semigroups, initiated by Chung [4] and developed by Butzer–Hahn [2] and Pfeifer [13], and the so-called multiplier enlargement method of Hsu–Wang [9, 19] and Shaw [17, 18]. At the same time, by introducing a modified second modulus of continuity of an operator semigroup and a Steklov-type element we establish quantitative estimates of the obtained formulae.

To the best of our knowledge, all existing representation formulae for \((C_0)\) one- and multi-parameter operator semigroups are special cases of our results. In particular, Shaw’s formulae [17, 18] for \(m\)-parameter operator

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semigroups are special cases of our results when specifying the random vectors considered. Also with our method it is easier to obtain new formulae.

At the end we give examples to show the application of our results in multivariate operator approximation theory when we consider particular operator semigroups.

2. Preliminaries. Let \( \mathcal{X} \) be a Banach space with norm \( \| \cdot \| \), and \( \mathcal{E}(\mathcal{X}) \) be the Banach algebra of endomorphisms of \( \mathcal{X} \). If \( T \in \mathcal{E}(\mathcal{X}) \), \( \| T \| \) also denotes the norm of \( T \). Let \( \mathbb{R}^m \) be the \( m \)-dimensional Euclidean space supplied with the usual definition of arithmetical operations and metric. We write \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \), \( t = t_1 + \ldots + t_m \), \( |t| = |t_1| + \ldots + |t_m| \) and denote the unit vectors by \( e_1, \ldots, e_m \), where \( e_k = (0, \ldots, 1, \ldots, 0) \) with 1 in the \( k \)th place and 0 elsewhere. Further, let \( \mathbb{R}_{+}^m = \{ t \in \mathbb{R}^m : t_k \geq 0, \ k = 1, \ldots, m \} \), the first closed \( 2^m \)-ant in \( \mathbb{R}^m \). \( Z_{+}^m \) denotes the set of all non-negative integers and \( \mathbb{N} \) is the set of all positive integers.

A family of bounded linear operators \( \{ T(t) : t \in \mathbb{R}_{+}^m \} \) on \( \mathcal{X} \) is called a \( (C_0) \) \( m \)-parameter operator semigroup in \( \mathcal{E}(\mathcal{X}) \) when the following three conditions are satisfied:

1. \( T(t+s) = T(t)T(s), \ t, s \in \mathbb{R}_{+}^m \);
2. \( T(0) = I \) (identity operator);
3. \( \lim_{t \to 0} \mathcal{E}(\mathcal{X}) \) \( s \)-converges \( T(t)f = f, \ f \in \mathcal{X} \).

It is known that \( \{ T(t) : t \in \mathbb{R}_{+}^m \} \) is then the direct product of \( m \) \( (C_0) \) one-parameter operator semigroups in \( \mathcal{E}(\mathcal{X}) \):

\[
T(t) = \prod_{k=1}^{m} T_k(t_k),
\]

where \( T_k(t_k) = T(t_ke_k) \). The operators \( \{ T_k(t_k) : 0 \leq t_k < \infty \} \) \( (k = 1, \ldots, m) \) commute with each other.

Let \( A_k \) be the infinitesimal generator of \( \{ T_k(t_k) : 0 \leq t_k < \infty \} \) with domain \( D(A_k) \), \( k = 1, \ldots, m \). Then if \( f \) is in \( D(A_k) \) so is \( T(t)f \) for each \( t \in \mathbb{R}_{+}^m \) and

\[
A_kT(t)f = T(t)A_kf.
\]

Further, if \( f \in D(A_j) \) and \( f \in D(A_jA_k) \) then \( f \in D(A_kA_j) \) and \( A_kA_jf = A_jA_kf \) \( (j, k = 1, \ldots, m) \). In the following we use the notation
$D^2 := \bigcap_{k,j=1}^{m} D(A_k A_j)$. 

$D^2$ is a linear subspace of $X$.

To each $k = 1, \ldots, m$, there correspond two numbers $M_k \geq 1$ and $\omega_k \geq 0$ such that 

$$\|T_k(t_k)\| \leq M_k e^{\omega_k t_k}, \quad 0 \leq t_k < \infty.$$  

Hence we have the inequality 

$$\|T(t)\| \leq M \exp(\omega(1 + \ldots + m)) = M e^{\omega}, \quad t \in \mathbb{R}_+^m, 
$$

where $M = M_1 \ldots M_m$ and $\omega = \max\{\omega_k : 1 \leq k \leq m\}$.

In the following it is always understood that $\{T(t) : t \in \mathbb{R}_+^m\}$ satisfies (5), unless otherwise specified.

For the above definitions and properties of operator semigroups we refer to Butzer–Berens [1], Hille–Phillips [8] or W. K"ohnen [11].

Let $(\Omega, A, P)$ be a probability space. For every real-valued random variable $X$ defined on $(\Omega, A, P)$, $E(X)$ denotes its expectation. If $\xi = E(X)$ exists then $\sigma^2 = \sigma^2(X) = E[(X - \xi)^2]$ is the variance of $X$. Let further $\Psi_X(u) = E(u^X)$, $u \geq 0$, and $\Psi^*_X(u) = E(e^{uX})$, $u \in \mathbb{R}$, denote the probability-generating function and the moment-generating function of $X$ respectively.

We need to consider $m$-dimensional random vectors, also denoted by $X, Y, \ldots$, on $(\Omega, A, P)$. For an $m$-dimensional random vector $X = (X_{01}, \ldots, X_{0m})$, we also use $E(X)$ to denote its expectation: 

$$E(X) := (E(X_{01}), \ldots, E(X_{0m}))$$

and define 

$$\sigma^2_i(X) := \sigma^2(X_{0i}).$$

It is not difficult to extend the theory of the extended Pettis integral developed in [13] to the multivariate case.

Let $\{T(t) : t \in \mathbb{R}_+^m\}$ be as above and $X$ be an $\mathbb{R}_+^m$-valued random vector such that 

$$\Psi^*_X(\omega) < \infty, \quad X = X_{01} + \ldots + X_{0m}.$$ 

Then for every $f \in \mathcal{X}$ define 

$$E[T(X)f] := \int_{\Omega} T(X)f \, dP,$$

which exists in the Bochner sense in $\mathcal{X}$ by the strong continuity of $\{T(t) : t \in \mathbb{R}_+^m\}$ and (5). Moreover, the map $E[T(X)] : f \rightarrow E[T(X)f]$ on $\mathcal{X}$ defines a bounded linear operator $E[T(X)] \in \mathcal{E}(\mathcal{X})$ with 

$$\|E[T(X)]\| \leq M \Psi^*_X(\omega).$$
$E[T(X)]$ is called the expectation of $T(X)$ and is understood as an extended Pettis integral following [13].

If $X$ and $Y$ are independent $\mathbb{R}^m_+$-valued random vectors such that $\Psi^*_X(\omega) < \infty$ and $\Psi^*_Y(\omega) < \infty$ then $E[T(X)], E[T(Y)]$ and $E[T(X + Y)]$ exist in $\mathcal{E}(X)$ and

$$E[T(X) \circ T(Y)] = E[T(X + Y)] = E[T(X)] \circ E[T(Y)],$$

where “$\circ$” denotes composition.

For the above see [13], [15], [16] and the references cited there.

3. Auxiliary results. We need a Taylor expansion integral formula for $(C_0)$ $m$-parameter operator semigroups.

**Lemma 1.** Assume $\{T(t) : t \in \mathbb{R}^m_+\}$ is a $(C_0)$ $m$-parameter operator semigroup satisfying (5). Then for every $g \in D^2$ and $s,t \in \mathbb{R}^m_+$,

$$T(t)g - T(s)g = T(s)[(t_1 - s_1)A_1 g + \ldots + (t_m - s_m)A_m g]$$

$$+ \int_0^1 (1-u)T(s + u(t-s))((t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m)^2 g du.$$  

**Proof.** Let $G(u) = T(s + u(t-s))g \in \mathcal{X}, u \in [0,1]$. Then

$$G'(u) := \frac{dG(u)}{du}$$

$$= T(s + u(t-s))[(t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m]g$$

and

$$G''(u) = T(s + u(t-s))[(t_1 - s_1)A_1 + \ldots + (t_m - s_m)A_m]^2 g.$$  

Now (6) follows from the Taylor formula with integral remainder for Banach space valued functions (see, e.g., [5, Theorem 8.14.3]).

For our purpose we need a second modulus of continuity $\omega_2(Tf, \delta)$ and the Steklov operator $J_h(f)$ ($h > 0$) for a $(C_0)$ $m$-parameter operator semigroup $\{T(t) : t \in \mathbb{R}^m_+\}$ and $f \in \mathcal{X}$.

**Definition 1.**

$$\omega_2(Tf, \delta) = \sup_{t = (t_1, \ldots, t_m), 0 \leq t_i, t_j \leq \delta} \{ \| (T(t) - I)^2 f \|, \| (T_{t_i}(t_i) - I)(T_{t_j}(t_j) - I) f \| \}.$$  

We have $\omega_2(Tf, \delta) \to 0$ as $\delta \to 0$, by the strong continuity of $\{T(t) : t \in \mathbb{R}^m_+\}$.
Definition 2.

\[ J_h(f) = \left( \frac{2}{h} \right)^m \int_0^{h/2} \cdots \int_0^{h/2} \left[ 2T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) \right. \]

\[ - T(2\xi_1 + 2\eta_1, \ldots, 2\xi_m + 2\eta_m) \] \[ fd\xi_1 d\eta_1 \ldots d\xi_m d\eta_m. \]

The integral may be considered as a multi-\(X\)-valued Riemann integral.

We have the following lemma:

Lemma 2. (i) \( J_h(f) \in D^2 \) for all \( f \in X \);
(ii) \( \|f - J_h(f)\| \leq \omega_2(Tf, h) \);
(iii) \( \|A_i A_j J_h(f)\| \leq 9M e^{2(m-1)h_2} \omega_2(Tf, h)/h^2, 1 \leq i, j \leq m \).

Proof. (i) Let

\[ J_1 = \int_0^{h/2} \cdots \int_0^{h/2} T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) f d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m. \]

\[ J_2 = \int_0^{h/2} \cdots \int_0^{h/2} T(2\xi_1 + 2\eta_1, \ldots, 2\xi_m + 2\eta_m) f d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m. \]

\[ = \left( \frac{1}{2} \right)^m \int_0^{h/2} \cdots \int_0^{h/2} T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) f d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m. \]

It is not difficult to show that \( J_1 \in D^2, J_2 \in D^2 \) (cf. [1, p. 10]) and hence (i) holds.

(ii) We have

\[ \|f - J_h(f)\| = \left\| \left( \frac{2}{h} \right)^m \int_0^{h/2} \cdots \int_0^{h/2} \left[ f - 2T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) \right. \right. \]

\[ + T(2\xi_1 + 2\eta_1, \ldots, 2\xi_m + 2\eta_m) \] \[ f d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m \] \[ \left. \right\| \left. \right. \]

\[ = \left\| \left( \frac{2}{h} \right)^m \int_0^{h/2} \cdots \int_0^{h/2} \left[ T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) - I \right]^2 f \right. \]

\[ \times d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m \] \[ \right\| \]
\[ \leq \left( \frac{2}{h} \right)^{2m} \frac{h}{2} \int_0^h \ldots \int_0^h \left\| \left[ T(\xi_1 + \eta_1, \ldots, \xi_m + \eta_m) - I \right]^2 f \right\| \times d\xi_1 d\eta_1 \ldots d\xi_m d\eta_m \]

\[ \leq \omega_2(Tf, h). \]

(iii) When \( i \neq j \), similar to the one-parameter operator semigroup case (ibid.), we can show

\[ A_i A_j J_1 = \left( \frac{1}{2} \right)^{2m} \frac{h}{2} \int_0^h \ldots \int_0^h \prod_{k \neq i, j} T_k(\xi_k + \eta_k)T_i(\eta_i)T_j(\eta_j) \]

\[ \times (T_i(h/2) - I)(T_j(h/2) - I)f \prod_{k \neq i, j} d\xi_k d\eta_k d\eta_i d\eta_j \]

and

\[ A_i A_j J_2 = \left( \frac{1}{2} \right)^{2m} \frac{h}{2} \int_0^h \ldots \int_0^h \prod_{k \neq i, j} T_k(\xi_k + \eta_k)T_i(\eta_i)T_j(\eta_j) \]

\[ \times (T_i(h) - I)(T_j(h) - I)f \prod_{k \neq i, j} d\xi_k d\eta_k d\eta_i d\eta_j. \]

So

\[ \| A_i A_j J_1(f) \| = \|(2/h)^{2m} [2: A_i A_j J_1 - A_i A_j J_2] \| \]

\[ \leq \left( \frac{2}{h} \right)^{2m} \left\{ 2 \int_0^{h/2} \int_0^{h/2} \prod_{k \neq i, j} M_k e^{\omega_k(\xi_k + \eta_k)} M_i e^{\omega_i(\eta_i)} M_j e^{\omega_j(\eta_j)} \right\} \]

\[ \times \left\| (T_i(h/2) - I) \left( T_j(h/2) - I \right) f \prod_{k \neq i, j} d\xi_k d\eta_k d\eta_i d\eta_j \right\| \]

\[ + \left( \frac{1}{2} \right)^{2m} \frac{h}{2} \int_0^h \ldots \int_0^h \prod_{k \neq i, j} M_k e^{\omega_k(\xi_k + \eta_k)} M_i e^{\omega_i(\eta_i)} M_j e^{\omega_j(\eta_j)} \]

\[ \times \| (T_i(h) - I)(T_j(h) - I)f \| \prod_{k \neq i, j} d\xi_k d\eta_k d\eta_i d\eta_j \right\} \]

\[ \leq (2/h)^{2m} M \left\{ 2e^{(2m-2)\omega h/2}(h/2)^{2m-2} + (1/2)^{2m} e^{(2m-2)\omega h} h^{2m-2} \right\} \]

\[ \times \omega_2(Tf, h) \]
\[ \leq Me^{(2m-2)\omega h / 2} \{ 2(h/2)^{-2} + 1/h^2 \} \omega_2(Tf, h) \]
\[ = 9Me^{2(m-1)\omega h} \omega_2(Tf, h) / h^2. \]

When \( i = j \), the same estimate holds. \( \blacksquare \)

**Lemma 3.** For any \( \mathbb{R}^m_+ \)-valued random vector \( Y = (Y_0, \ldots, Y_m) \) with \( E(Y) = x = (x_1, \ldots, x_m) \) and \( f \in \mathcal{X} \),

\[ \|E[T(Y)]f - T(x)f\| \]
\[ = M\omega_2(Tf, h) \left\{ 2E(e^{\omega Y}) + \frac{9}{2} mMe^{2\omega_2} e^{2(m-1)\omega h} \left[ E(e^{p\omega Y}) \right]^{1/p} \left[ \sum_{i=1}^{m} (E((Y_i - x_i)^{2q}))^{1/q} \right] / h^2 \right\}, \]

where \( p > 0, q > 0, 1/p + 1/q = 1, h > 0 \). If \( \omega = 0 \), we have

\[ \|E[T(Y)]f - T(x)f\| \leq 2M\omega_2(Tf, h) \left[ 1 + \frac{9mM}{4h^2} \sum_{i=1}^{m} \sigma^2(Y_i) \right]. \]

**Proof.** We have

\[ \|E[T(Y)]f - T(x)f\| = \|E[T(Y)]f - E[T(Y)f] + E[T(Y)f] - T(x)f\| \]
\[ \leq \|E[T(Y)f] - E[T(Y)J_h f]\| + \|E[T(Y)J_h f] - T(x)J_h f\| \]
\[ + \|T(x)J_h f - T(x)f\| \]
\[ =: I_1 + I_2 + I_3. \]

Now,

\[ I_1 \leq E[\|T(Y)(J_h f - f)\|] \leq E[M e^{\omega Y} \|J_h f - f\|] \leq ME(e^{\omega Y}) \omega_2(Tf, h) \]

by Lemma 2, and

\[ I_2 \leq M e^{\omega_2} \omega_2(Tf, h) \leq ME(e^{\omega Y}) \omega_2(Tf, h) \]

by Jensen’s inequality.

Note that \( g := J_h(f) \in D^2 \), by Lemma 2. Apply Lemma 1 to get

\[ I_2 = \|E\left\{ T(x)[(Y_0 - x_1)A_1 + \ldots + (Y_m - x_m)A_m]g \right\} + \int_0^1 (1-u)T(x + u(Y - x))[(Y_0 - x_1)A_1 + \ldots \]
\[ \ldots + (Y_m - x_m)A_m]g du \right\| \]
such that
\( \Psi_1 \) satisfies (5), therefore by (11)–(14) we get (9).

If \( \omega = 0 \) we have
\[ I_1 \leq M \omega_2(Tf,h), \quad I_3 \leq M \omega_2(Tf,h), \quad \text{and} \quad I_2 \leq \frac{9}{2} M^2 m x \sum_{i=1}^m \sigma^2(Y_{0i}) \omega_2(Tf,h)/h^2 \]

so (10) follows.

4. Main results. Here comes our first main result:

**Theorem 1.** Let \( X = (X_{01}, \ldots, X_{0m}) \) be an \( \mathbb{R}_+^m \)-valued random vector with \( E(X) = x = (x_1, \ldots, x_m) \) and suppose that there exists a \( \delta > 0 \) such that \( \Psi_x^\delta(\delta) < \infty \). Then for any \( (C_0) \) m-parameter operator semigroup satisfying (5), and all \( n > \max(p\omega/\delta, 1/\delta^2) \),

\[
\| [E(T(X/n))]^n f - T(x)f \| \leq 2M \omega_2(Tf,1/\sqrt{n}) \left\{ e^{\omega_2} \exp \left[ \frac{2n \omega^2}{e^2(n\delta - \omega)^2} \Psi_x^\delta(\delta) \right] \right
\]

by Hölder’s inequality.

Therefore by (11)–(14) we get (9).

\[
\sum_{i=1}^m \sigma^2(Y_{0i}) \omega_2(Tf,h)/h^2
\]
\[ + 2^{1/2} q M \frac{m^2 q^2}{e^2} e^{2 \Delta \sqrt{\omega}} \exp \left( \frac{2 np \omega^2}{e^2 (n \delta - p \omega)} \right) \Psi_X^*(\delta) \]  

where \( p, q > 0, \frac{1}{p} + \frac{1}{q} = 1, \) is an arbitrary conjugate pair. If \( \omega = 0, \) then

\[
\| \{ E[T(X/n)] \}^n f - T(x) f \| \leq 2 M \omega_2 (Tf, 1/\sqrt{n}) \left[ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \sigma_i^2 (X_{0i}) \right].
\]

**Note.** All the right hand sides of (15) and (16) are finite.

**Proof of Theorem 1.** Let \( X_k \) be a sequence of independent random vectors identically distributed as \( X, \) and \( Y = (1/n) \sum_{k=1}^{n} X_k. \) Then

\[
E(Y) = \frac{1}{n} \sum_{k=1}^{n} E(X_k) = x, \quad E[T(Y) f] = \{ E[T(X/n)] \}^n f.
\]

For \( u > 0 \) we have

\[
\Psi_Y^*(u) = E(e^{u/n} \sum_{k=1}^{n} X_k) = (E(e^{u/n} X))^n
\]

\[
\leq \left( 1 + \frac{u}{n} E(X) + E \left( \frac{u^2 X^2}{2n^2} e^{u/n} X \right) \right)^n
\]

\[
\leq \left( 1 + \frac{u}{n} x + \frac{u^2}{2n^2} \left( \frac{2}{\delta - u/n} \right) e^{-2} E(e^{\delta X}) \right)^n
\]

\[
\leq e^{nu \bar{x}} \exp \left[ \frac{2nu^2}{e^2 (n \delta - u) \Psi_X^*(\delta)} \right]
\]

whenever \( u/n < \delta. \)

Above we made use of the inequalities (see also Pfeifer [14, p. 275])

\[
(\eta \leq \delta - \alpha \leq \eta e^{\alpha r/r^2} \quad \text{for } \eta < \delta, \ r > 0, \ \alpha > 0)
\]

and

\[
(1 + r)^n \leq e^{nr}.
\]

So for \( n > p \omega/\delta \geq \omega/\delta, \)

\[
E(e^{u X}) \leq e^{u \omega} \exp \left\{ \frac{2 n \omega^2}{e^2 (n \delta - \omega)^2} \Psi_X^*(\delta) \right\},
\]
and
\[
[E(e^{\rho \mathbf{Y}})]^{1/p} \leq \left\{ e^{\rho \mathbb{E} \left[ \prod_{k=1}^{n} X_{k_i} - x_i \right]} \right\}^{1/p}
\]
\[
eq e^{\rho \mathbb{E} \left[ 2n p \omega^2 e^{\frac{q}{\sqrt{n}}} \right] \psi^*_{X}(\delta)}.
\]

Observe that for \( Y = (Y_{01}, \ldots, Y_{0m}) \) we have
\[
E((Y_{0i} - x_i)^{2q})
\]
\[
= E \left( \left( \frac{1}{n} \sum_{k=1}^{n} X_{k_i} - x_i \right)^{2q} \right)
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} E(e^{\rho \mathbb{E} \left[ \prod_{k=1}^{n} X_{k_i} - x_i \right]}) \quad (\text{by (17)})
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} [E(e^{\rho \mathbb{E} \left[ \prod_{k=1}^{n} X_{k_i} - x_i \right]}) + E(e^{\rho \mathbb{E} \left[ \prod_{k=1}^{n} (X_{k_i} - x_i) \right]})]
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} \exp \left[ E \left( \frac{1}{2} \sum_{i=1}^{n} (X_{0i} - x_i)^{2} e^{\rho \mathbb{E} \left[ \prod_{k=1}^{n} X_{k_i} - x_i \right]} \right) \right]
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} \exp \left[ \frac{2n}{(\sqrt{n} - 1)^2} \right] e^{-2} \exp \left( \frac{1}{2} \mathbf{E} \left( \prod_{k=1}^{n} X_{k_i} - x_i \right) \right)
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} \exp \left( \frac{2n}{(\sqrt{n} - 1)^2} \right) \quad (\text{by Taylor’s expansion and (18)})
\]
\[
\leq \left( \frac{2q}{\sqrt{n}} \right)^{2q} e^{-2q} \exp \left( \frac{2n}{(\sqrt{n} - 1)^2} \right).
\]

Hence we established that
\[
[E((Y_{0i} - x_i)^{2q})]^{1/q} \leq \frac{4q^2}{n} e^{-2q^{1/q}} \exp \left( \frac{2n}{q(\sqrt{n} - 1)^2} \right) e^{-2} \mathbf{E} \left( \prod_{k=1}^{n} X_{k_i} - x_i \right).
\]

Now apply Lemma 3 and take \( h = 1/\sqrt{n} \):
\[
\| [E(T(X/n))]^{n} f - T(X)f \| = \| E(T(Y)) f - T(X)f \|
\]
\[
\leq M \omega_2(Tf, 1/\sqrt{n}) \left\{ 2 e^{\omega^2} \exp \left[ \frac{2n \omega^2}{e^2(n \delta - \omega)^2} \mathbf{E} \left( \prod_{k=1}^{n} X_{k_i} - x_i \right) \right] \right.
\]
\[
+ \frac{9}{2} m M e^{2q} e^{2(n-1)\omega/\sqrt{n}} e^{\omega^2} \exp \left[ \frac{2n \omega^2}{e^2(n \delta - \omega)^2} \mathbf{E} \left( \prod_{k=1}^{n} X_{k_i} - x_i \right) \right]
\]
\[
\times m \frac{4q^2}{n} e^{-2q^{1/q}} \exp \left[ \frac{2n}{q(\sqrt{n} - 1)^2} \right] e^{-2} \mathbf{E} \left( \prod_{k=1}^{n} X_{k_i} - x_i \right) \right\}.
\]
\[ = 2M \omega_2(Tf, 1/\sqrt{n}) \left\{ e^{\omega_2 \Psi_N(\sigma)} \exp \left[ \frac{2n \omega_2}{(n \delta - \omega)^2} \Psi_N^*(\delta) \right] \right\} + 2^{1/9} n M^{2/9} q^{2/9} e^{3 \omega_2 \exp(2m-1) \omega/\sqrt{n}} \times \exp \left[ \left( \frac{2n \omega_2}{(n \delta - \omega)^2} + \frac{2n e^{\delta \gamma}}{e^{(2(m-1) \omega/\sqrt{n})}} \right) \Psi_N^*(\delta) \right] \} \] 

When \( \omega = 0 \), noting that \( \sigma^2(Y_{0i}) = \sigma^2(X_{0i})/n \) by (10), we get (16). \( \blacksquare \)

A ramification of Theorem 1 follows:

**Theorem 2.** Let \( N \) be a \( \mathbb{Z}_+ \)-valued random variable with \( E(N) = \eta, \eta > 0 \), and let \( Y = (Y_0, \ldots, Y_m) \) be an \( \mathbb{R}_+^m \)-valued random vector independent of \( N \) with \( E(Y) = \gamma = (\gamma_1, \ldots, \gamma_m) \). Assume that there exists a \( \delta > 0 \) such that

\[ \Psi_N(\Psi_N^*(\delta)) < \infty. \]

Then for \( n > \max(p \omega/\delta, 1/\delta^2) \),

\[ \| \{ \Psi_N[E(T(Y/n))] \}^n f - T(\eta \gamma) f \| \leq 2M \omega_2(Tf, 1/\sqrt{n}) \left\{ e^{\omega_2 \Psi_N(\sigma)} \exp \left[ \frac{2n \omega_2}{(n \delta - \omega)^2} \Psi_N(\Psi_N^*(\delta)) \right] \right\} + 2^{1/9} n M^{2/9} q^{2/9} e^{3 \omega_2 \exp(2m-1) \omega/\sqrt{n}} \times \exp \left[ \left( \frac{2n \omega_2}{(n \delta - \omega)^2} + \frac{2n e^{\delta \gamma}}{e^{(2(m-1) \omega/\sqrt{n})}} \right) \Psi_N(\Psi_N^*(\delta)) \right] \}, \]

where \( p, q > 0, 1/p + 1/q = 1, \) is an arbitrary conjugate pair. If \( \omega = 0 \), then

\[ \| \{ \Psi_N[E(T(Y/n))] \}^n f - T(\eta \gamma) f \| \leq 2M \omega_2(Tf, 1/\sqrt{n}) \left\{ 1 + \frac{9}{4} m M \sum_{i=1}^m \left[ \eta \sigma^2(Y_{0i}) + \sigma^2(N) \gamma_i^2 \right] \right\}. \]

**Proof.** Consider \( Y_k \overset{i.i.d.}{\sim} Y \), which are also independent of \( N \). In Theorem 1, take \( X = \sum_{k=1}^N Y_k \) (as usual, an empty sum equals 0). Then

\[ E \left[ T\left( \frac{1}{n} X \right) \right] = E \left[ T\left( \frac{1}{n} \sum_{k=1}^N Y_k \right) \right] = \sum_{l=0}^\infty P(N = l) E \left[ T\left( \frac{1}{n} \sum_{k=1}^l Y_k \right) \right] \]

\[ = \sum_{l=0}^\infty P(N = l) \left[ E \left( T\left( \frac{1}{n} Y \right) \right) \right]^l = \Psi_N \left( E \left( T\left( \frac{1}{n} Y \right) \right) \right), \]

\[ E(X) = \sum_{l=0}^\infty P(N = l) E \left[ \sum_{k=1}^l Y_k \right] = E(N) E(Y) = \eta \gamma. \]
Also
\[ \Psi_\delta^X = E(e^{\delta X}) = E(e^{\sum_{k=1}^N Y_k}) = \sum_{l=0}^{\infty} P(N = l)E(e^{\delta Y_l}) = \sum_{l=0}^{\infty} P(N = l)(E(e^{\delta Y_l}))^l = \Psi_N(\Psi_\delta^X(\delta)). \]

If \( X = (X_0, \ldots, X_m) \) we have
\[ \sigma^2(X_0) = \sigma^2\left(\sum_{k=1}^N Y_k\right) = \sum_{l=0}^{\infty} P(N = l)E\left(\left(\sum_{k=1}^l Y_k\right)^2\right) - \eta^2\gamma_i^2 \]
\[ = \sum_{l=0}^{\infty} P(N = l)(lE(Y_0^2) + l(l-1)\gamma_i^2) - \eta^2\gamma_i^2 \]
\[ = \eta\sigma^2(Y_0) + \sigma^2(N)\gamma_i^2. \]

Then (19), (20) follow by (15), (16).

An application of Lemma 3 comes next:

**Theorem 3.** For each positive real number \( \tau \), let \( N_\tau \) be a \( \mathbb{Z}_+ \)-valued random variable with \( E(N_\tau) = \tau \eta \), where \( \eta \in \mathbb{R}_+ \) is fixed. Let \( X \) be an \( \mathbb{R}_+^m \)-valued random vector with \( E(X) = \gamma = (\gamma_1, \ldots, \gamma_m) \), independent of \( N_\tau \). Assume that there exists a \( \delta > 0 \) such that \( \Psi_\delta^X(\delta) < \infty \) and further there are \( p, q > 0 \) with \( 1/p + 1/q = 1 \) such that

\[ \limsup_{\tau \to \infty} \tau \left\{ E\left(\left(1 - \frac{1}{\tau} N_\tau - \eta\right)^2\right)^{1/q}\right\} = d_2 < \infty \]

and

\[ \limsup_{\tau \to \infty} \frac{2}{e^2(\sqrt{\tau \delta} - 1)^2} e^{\delta \Psi_\delta^X(\delta)} = d_3 < \infty. \]

Then for \( \tau > 1/\delta^2 \),

\[ \|\Psi_N(T(X/\tau)) f - T(\eta \gamma) f\| \]
\[ \leq M\omega_2(Tf, 1/\sqrt{\tau})(2d_1 + 9mM e^{2\omega \gamma} e^{2(m-1)\omega/\sqrt{\tau}} d_1^{1/p}
\[ \left[m^{2l/q} \left(\frac{2q}{e}\right)^2 d_3^{1/q} + d_2 \sum_{i=1}^m \gamma_i^2 \right]. \]
If \( \omega = 0 \), then

\[
\| \Psi_{N_x} (E[T(X/\tau)])f - T(\eta \gamma) f \| \leq 2M \omega_2 (T_f, 1/\sqrt{T}) \left\{ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left[ \eta \sigma^2 (X_{0_i}) + \gamma_i^2 \frac{1}{\tau} \sigma^2 (N_x) \right] \right\}.
\]

**Proof.** Take random vectors \( X_k \) \( i.i.d. \) \( X \), which are also independent of \( N_x \). Consider \( Y_{\tau} = (1/\tau) \sum_{k=1}^{N_x} X_k \), where \( Y_x = (Y_{0_1}, \ldots, Y_{0_m}) \), then apply Lemma 3 with \( h = 1/\sqrt{T} \). We have

\[
E[T(Y_{\tau})] f = \sum_{l=0}^{\infty} P(N_x = l) E \left[ T \left( \frac{1}{\tau} \sum_{k=1}^{l} X_k \right) \right] f = \Psi_{N_x} (E[T(X/\tau)] f),
\]

\[
E(Y_{\tau}) = \sum_{l=0}^{\infty} P(N_x = l) \frac{1}{\tau} E \left( \sum_{k=1}^{l} X_k \right) = \frac{1}{\tau} E(N_x) E(X) = \eta \gamma,
\]

\[
E(e^{\omega T_{\tau}}) \leq E(e^{\rho T_{\tau}}) = \Psi_{N_x} (\Psi^{\ast}_{\tau} (\rho \omega/\tau)) \leq d_1.
\]

Furthermore,

\[
(E( (Y_{0_i} - \eta \gamma_i)^{2q} ))^{1/q} = \left\{ E \left( \left( \frac{1}{\tau} \sum_{k=1}^{N_x} X_{0_k} - \frac{1}{\tau} N_x \gamma_i + \frac{1}{\tau} N_x \gamma_i - \eta \gamma_i \right)^{2q} \right) \right\}^{1/q}
\]

\[
\leq \frac{2}{\gamma} \left[ E \left( \left( \frac{1}{\tau} \sum_{k=1}^{N_x} (X_{0_k} - \gamma_i) \right)^{2q} \right) \right]^{1/q} + 2 \gamma^2 \left[ E \left( \left( \frac{1}{\tau} N_x - \eta \right)^{2q} \right) \right]^{1/q}
\]

\[
= : 2 I_1 + 2 I_2.
\]

We observe that

\[
I_1^q = E \left[ \left( \frac{1}{\tau} \sum_{k=1}^{N_x} (X_{0_k} - \gamma_i) \right)^{2q} \right]
\]

\[
\leq \frac{2 q}{e^{\sqrt{T}}} \left\{ E \left( e^{(1/\sqrt{T})|X_{0_k} - \gamma_i|} \right) \right\} (by (17))
\]

\[
\leq \frac{2 q}{e^{\sqrt{T}}} \left\{ E \left( e^{(1/\sqrt{T})|X_{0_k} - \gamma_i|} \right) + E \left[ e^{(1/\sqrt{T})|X_{0_k} - \gamma_i|} \right] \right\}
\]

\[
\leq \frac{2 q}{e^{2\sqrt{T}}} \left\{ E \left[ e^{(1/\sqrt{T})|X_{0_k} - \gamma_i|} \right] \right\} + E \left[ e^{(1/\sqrt{T})|X_{0_k} - \gamma_i|} \right] \right\}
\]

\[
\leq \frac{2 q}{e^{2\sqrt{T}}} \left\{ E \left[ e^{(1/\sqrt{T})(X_{0_k} - \gamma_i)} \right] \right\} + E \left[ e^{(1/\sqrt{T})(\gamma_i - X_{0_k})} \right] \right\}
\]

\[
\leq \left( \frac{2 q}{e^{2\sqrt{T}}} \right)^q E \left[ e^{\left( 1 + \frac{1}{\sqrt{T}} (X_{0_k} - \gamma_i) \right)} \right]
\]
\[
\frac{1}{2\tau}(X_{0i} - \gamma_i)^2 e^{(1/\sqrt{\tau})|X_{0i} - \gamma_i|})^{N_\tau} + E\left([E\left(1 + \frac{1}{\sqrt{\tau}}(\gamma_i - X_{0i}) + \frac{1}{2\tau}(X_{0i} - \gamma_i)^2 e^{(1/\sqrt{\tau})|X_{0i} - \gamma_i|})^{N_\tau} \right)]\right) \\
\leq 2\left(\frac{2q}{e\sqrt{\tau}}\right)^{2q} E\left\{\left[\exp\left[\left(\frac{2}{\delta - 1/\sqrt{\tau}}\right)^2 e^{-2}\left(E\left|X_{0i} - \gamma_i\right|\right)\right]\right]^{N_\tau/(2\tau)}\right\} \\
\leq 2\left(\frac{2q}{e\sqrt{\tau}}\right)^{2q} E\left\{\left[\exp\left[\left(\frac{2\sqrt{\tau}}{e(\sqrt{\tau}\delta - 1)}\right)^2 e^{i\gamma}\Psi_i(\delta)\right]\right]^{N_\tau/(2\tau)}\right\} \\
\leq 2\left(\frac{2q}{e\sqrt{\tau}}\right)^{2q} d_3.
\]

So
\[
I_1 \leq 2^{1/q} \frac{4q^2}{\tilde{e}^2 \tau} d_3^{1/q}
\]

and
\[
I_2 = \gamma_i^2 \left(E\left[\left(\frac{1}{\tau} N_\tau - \eta\right)^{2q}\right]\right)^{1/q} \leq \gamma_i^2 d_2^{\tau}.
\]

Therefore by inequality (9) of Lemma 3 for \( Y = Y_\tau \) and \( h = 1/\sqrt{\tau} \), we get
\[
\|\Psi_{N_\tau}|E(T(X/\tau))|f - T(\eta_\tau)\| f
\leq M\omega_2(T f, 1/\sqrt{\tau}) \left\{2d_1 + 9mM e^{2\omega_\tau\gamma_i} \sum_{l=0}^{2(2^{(m-1)}\omega_\tau\gamma_i)} d_1^{1/p} \right\} \\
\times \left[2^{1/q} \left(e^{2\omega_\tau\gamma_i}\right)^{1/q} + d_2 \right] \right\} \\
= M\omega_2(T f, 1/\sqrt{\tau}) \left\{2d_1 + 9mM e^{2\omega_\tau\gamma_i} \sum_{l=0}^{2(2^{(m-1)}\omega_\tau\gamma_i)} d_1^{1/p} \right\} \\
\times \left[m2^{1/q} \left(e^{2\omega_\tau\gamma_i}\right)^{1/q} + d_2 \right] \right\}.
\]

If \( \omega = 0 \), we apply inequality (10) of Lemma 3. Note that
\[
\sigma^2(Y_{0i}) = \sigma^2\left(\frac{1}{\tau} \sum_{k=1}^{N_\tau} X_{0i}\right) = \frac{1}{\tau^2} \sum_{l=0}^{\infty} P(N_\tau = l) E\left[\left(\sum_{k=0}^{l} X_{0i}\right)^2\right] - \eta^2 \gamma_i^2 \\
= \frac{1}{\tau^2} \sum_{l=0}^{\infty} P(N_\tau = l) (l E(X_{0i}^2) + l(l - 1)\gamma_i^2) - \eta^2 \gamma_i^2
\]
\[ \frac{1}{\tau^2} E(N_r) E(X_{11}^2) + \frac{1}{\tau^2} (E(N_r^2) - E(N_r)) \gamma^2_i - \eta^2 \gamma^2_i \]
\[ = \frac{1}{\tau} \left[ \eta \sigma^2(X_{11}) + \gamma^2_i \frac{1}{\tau} \sigma^2(N_r) \right]. \]

By (10), when \( h = 1/\sqrt{\tau} \), we obtain (25). \( \blacksquare \)

Another generalization of Theorem 1 is presented next.

**Theorem 4.** Let \( N = (N_1, \ldots, N_m) \) be a \( \mathbb{Z}_+^m \)-valued random vector with \( E(N) = \eta = (\eta_1, \ldots, \eta_m) \). For each \( i \) (\( 1 \leq i \leq m \)), let \( \{ Y_{k_i} \}_{k_i=1}^{\infty} \) be a sequence of i.i.d. real-valued random variables distributed as \( Y \), a fixed random variable with \( E(Y) = \gamma \). \( N \) and \( Y_{k_i} \) are assumed to be independent. Also assume that there exists a \( \delta > 0 \) such that \( \Psi_N(\Psi^*_Y(\delta)) < \infty \).

Then for \( n > \max(p\omega/\delta, 1/\delta^2) \),

\[ \left\| \left\{ E \left[ T \left( \sum_{k_i=1}^{N_i} \frac{1}{n} Y_{k_i1}, \ldots, \sum_{k_m=1}^{N_m} \frac{1}{n} Y_{k_m m} \right) \right] \right\}^n f - T(\gamma \eta) f \right\| \]
\[ \leq 2M \omega_2(T f, 1/\sqrt{n}) \left\{ e^{\omega \gamma \eta} \exp \left[ \frac{2m \omega^2}{e^2 (n \delta - \omega)^2} \Psi_N(\Psi^*_Y(\delta)) \right] \right. \]
\[ + \left. 2^{1/q} \frac{9M}{m^2} \left( \eta_i \upsilon^2 + \upsilon^2 \gamma^2 \right) \left[ 1 + \frac{2}{4} m M \sum_{i=1}^{m} (\eta_i \sigma^2 + \sigma^2 (N_i) \gamma^2) \right] \right\} \],

where \( p, q > 0, 1/p + 1/q = 1, \) is an arbitrary conjugate pair. If \( \omega = 0 \), then

\[ \left\| \left\{ E \left[ T \left( \sum_{k_i=1}^{N_i} \frac{1}{n} Y_{k_i1}, \ldots, \sum_{k_m=1}^{N_m} \frac{1}{n} Y_{k_m m} \right) \right] \right\}^n f - T(\gamma \eta) f \right\| \]
\[ \leq 2M \omega_2(T f, 1/\sqrt{n}) \left\{ 1 + \frac{9}{4} M M \sum_{i=1}^{m} (\eta_i \sigma^2 + \sigma^2 (N_i) \gamma^2) \right\} \].

**Proof.** In Theorem 1, take \( X := (\sum_{k_1=1}^{N_1} Y_{k_11}, \ldots, \sum_{k_m=1}^{N_m} Y_{k_m m}) \) and let \( X_k^{1.1.\delta} \) \( X \). Then

\[ E(X) = \left( E \left[ \sum_{k_1=1}^{N_1} Y_{k_11} \right], \ldots, E \left[ \sum_{k_m=1}^{N_m} Y_{k_m m} \right] \right) \]
\[ = (EN_1 Y, \ldots, EN_m Y) = \gamma \eta. \]
We observe that
\[ \Psi_X^*(\delta) = E(e^{\delta X}) \]
\[ = \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} P(N = (l_1, \ldots, l_m))E(e^{\delta(Y_{k_1} + \ldots + Y_{k_m})}) \]
\[ = \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} P(N = (l_1, \ldots, l_m))E(e^{\delta(Y_{k_1})}) \cdots E(e^{\delta(Y_{k_m})}) \]
\[ = E((E(e^{\delta Y}))^{N_1 + \ldots + N_m}) = \psi_{\Psi_X^*}(\delta). \]

Thus (26) follows from (15).

If \( \omega = 0 \) we see that
\[ \sigma^2(X_0) = \sigma^2 \left( \sum_{k=1}^{N_1} Y_{k_1} \right) = \eta_1 \sigma^2(Y) + \sigma^2(N_1) \gamma^2, \]
established similarly to the fact at the end of the proof of Theorem 2. Now (27) follows from (16). □

5. Further results: multiplier enlargement formulae. In this section we modify the formulae obtained in the previous section by the so-called multiplier enlargement method (see [6]) initiated by Hsu–Wang [9, 19] in the 60’s and also used by Shaw [17, 18] in the representation of operator semigroups. The modified representation formulae have a larger range of applications and when we specify the random vectors (variables) considered, we recover the representation formulae for \( m \)-parameter operator semigroups of Shaw [18]. For simplicity we only consider equibounded operator semigroups, i.e.,
\[ \|T(t)\| \leq M \quad \text{for all } t \in \mathbb{R}^m. \]
Here we only need to give two versions related to Theorems 1 and 4. Others can be similarly obtained.

**Theorem 5.** Suppose \( \|T(t)\| \leq M \) for all \( t \in \mathbb{R}^m_+ \), and \( \alpha_n \) is a sequence of positive real numbers with \( \liminf_{n \to \infty} \alpha_n > 0 \). For each \( n \in \mathbb{N} \) let \( X(n) \) be an \( \mathbb{R}^m_+ \)-valued random vector with \( E[X(n)] = x/\alpha_n \). Assume
\[
\limsup_{n \to \infty} \alpha_n \sigma^2_i(X(n)) < \infty, \quad i = 1, \ldots, m. \quad \text{Then}
\]
\[
(28) \quad \left\| \left\{ E \left[ T \left( \frac{\alpha_n}{n} X(n) \right) \right] \right\} f - T(x)f \right\|
\leq 2M \omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} m M \alpha_n \sum_{i=1}^{m} \sigma^2_i(X(n)) \right].
\]

Proof. For each fixed \( n \), let \( X_k \overset{\text{i.i.d.}}{\sim} X(n), \; k = 1, \ldots, n \), and consider
\[
Y := \frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k.
\]
Then
\[
E(Y) = E \left( \frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k \right) = \alpha_n E(X(n)) = \alpha_n x/\alpha_n = x
\]
and
\[
E[T(Y)]f = E \left[ T \left( \frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k \right) \right] f = \left\{ E \left[ T \left( \frac{\alpha_n}{n} X(n) \right) \right] \right\} f.
\]
Furthermore,
\[
\sigma^2_i(Y) = \sigma^2_i \left( \frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k \right) = \frac{\alpha^2}{n^2} \sigma^2_i(X(n)) = \frac{\alpha^2}{n} \sigma^2_i(X(n)).
\]
Now take \( h = (\alpha_n/n)^{1/2} \); then by (10), we get (28). \( \blacksquare \)

**Theorem 6.** Let \( \alpha_n \) be a sequence of positive real numbers satisfying
\[
\lim_{n \to \infty} \alpha_n/n = 0 \quad \text{and} \quad \liminf_{n \to \infty} \alpha_n > 0.
\]
For each \( n \in \mathbb{N} \), let \( N(n) := (N_1(n), \ldots, N_m(n)) \) be a \( \mathbb{Z}_+^m \)-valued random vector with \( E(N(n)) = (1/\alpha_n) (\eta_1, \ldots, \eta_m) \) and
\[
\limsup_{n \to \infty} \alpha_n \sigma^2_i(N(n)) < \infty \quad \text{for all} \; i = 1, \ldots, m.
\]
For each \( i \) \( (1 \leq i \leq m) \), let \( \{ Y_{0i}(n) \}_{k=1}^{\infty} \) be a sequence of i.i.d. random variables, distributed as \( Y(n) \), where \( Y(n) \) is a fixed real-valued random variable for each \( n \in \mathbb{N} \) and \( E(Y(n)) = \alpha_n \gamma \). Suppose that \( Y_{0i}(n) \) \( (i = 1, \ldots, m) \) and \( N(n) \) are all independent. Assume also that
\[
\limsup_{n \to \infty} \sigma^2(Y(n))/\alpha_n^2 < \infty.
\]
Consider the equibounded operator semigroup \( \{ T(t) : t \in \mathbb{R}_+^m \} \) with
\[
\| T(t) \| \leq M \quad \text{for all} \; t \in \mathbb{R}_+^m.
\]
Then
\[
\left\| \left\{ E \left[ T \left( \sum_{k=1}^{N_1(n)} \frac{1}{n} Y_{k_1}(n), \ldots, \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m}(n) \right) \right] \right\} \right\|^n f - T(\gamma \eta) f \leq 2M\omega(T, (\alpha_n/n)^{1/2}) \left\{ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left[ \frac{\eta_i}{\alpha_n \eta} \sigma^2(Y(n)) + \gamma^2 \alpha_n \sigma^2(Y(n)) \right] \right\}. 
\]

**Proof.** We want to apply Lemma 3. Let

\[
X \sim \left( \sum_{k=1}^{N_1(n)} Y_{k_1}(n), \ldots, \sum_{k_m=1}^{N_m(n)} Y_{k_m}(n) \right)
\]

and

\[
Y = \frac{1}{n} \sum_{k=1}^{n} X_k.
\]

Then
\[
E(Y) = E(X) = \left( E \left( \sum_{k_1=1}^{N_1(n)} Y_{k_1}(n) \right), \ldots, E \left( \sum_{k_m=1}^{N_m(n)} Y_{k_m}(n) \right) \right) = \left( \frac{1}{\alpha_n} \eta_1 \alpha_n \gamma, \ldots, \frac{1}{\alpha_n} \eta_m \alpha_n \gamma \right) = (\eta_1 \gamma, \ldots, \eta_m \gamma) = \gamma \eta.
\]

Moreover,
\[
E[T(Y)] f = \left\{ E \left[ T \left( \sum_{k=1}^{N_1(n)} \frac{1}{n} Y_{k_1}(n), \ldots, \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m}(n) \right) \right] \right\}^n f.
\]

Furthermore,
\[
\sigma^2(Y) = \sigma^2 \left( \frac{1}{n} \sum_{k=1}^{n} X_k \right) = \frac{1}{n} \sigma^2 \left( \sum_{k_1=1}^{N_1(n)} Y_{k_1}(n) \right)
\]
\[
= \frac{1}{n} \left[ E(N_1(n)) \sigma^2(Y_1(1)) + \sigma^2(N_1(n)) E(Y_1(1))^2 \right]
\]
\[
= \frac{1}{n} \left[ \frac{1}{\alpha_n} \eta_1 \sigma^2(Y(1)) + \sigma^2(N_1(n)) \alpha_n^2 \gamma^2 \right].
\]

Pick \( h := (\alpha_n/n)^{1/2} \); then by (10) of Lemma 3 we have
\[
\left\| \left\{ E \left[ T \left( \sum_{k=1}^{N_1(n)} \frac{1}{n} Y_{k_1}(n), \ldots, \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m}(n) \right) \right] \right\} \right\|^n f - T(\gamma \eta) f \]
\[\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \times \left\{ 1 + \frac{9mM}{4} \frac{n}{\alpha_n} \sum_{i=1}^{m} \frac{1}{n} \left[ \frac{1}{\alpha_n} \sigma_2^2(Y_{0i}(n)) + \sigma_2^2(N_i(n)) \alpha_n^2 \gamma^2 \right] \right\} \]

\[\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \times \left\{ 1 + \frac{9}{4} mM \sum_{i=1}^{m} \left[ \frac{\eta_i}{\alpha_n^2} \sigma_2^2(Y(n)) + \gamma^2 \alpha_n\sigma_i^2(N(n)) \right] \right\}. \]

6. Applications. In this section we specify the random vectors (variables) and \(\alpha_n\) of Theorems 1–6 to derive some concrete representation formulae for \((C_0)\) \(m\)-parameter operator semigroups. We also illustrate how to get the results on multivariate approximation operators from the corresponding ones on operator semigroups. Unless otherwise mentioned all \((C_0)\) \(m\)-parameter operator semigroups considered satisfy (5).

**Example 1.** Take \(X = (X_{01}, \ldots, X_{0m})\) that follows the multi-point distribution with \(EX = x = (x_1, \ldots, x_m)\):

\[P(X = e_i) = x_i (e_i = (0, \ldots, 1, \ldots, 0))\]

and

\[P(X = 0) = 1 - \pi, \quad \text{where } 0 < \pi < 1 \ (\pi = x_1 + \ldots + x_m).\]

Then

\[\Psi^\star X(\delta) = E(e^{\delta X}) = P(X = 0) + P(X = 1) e^{\delta} = 1 - \pi + \pi e^{\delta} < \infty.\]

Furthermore, we have

\[E[T(X/n)] = I + \sum_{i=1}^{m} x_i(T_i(1/n) - I).\]

Hence by Theorem 1 there is a constant \(K = K(\omega, M, x, \delta, m)\) such that

\[\|(I + \sum_{i=1}^{m} x_i(T_i(1/n) - I))^n f - T(x)f\| \leq K\omega_2(Tf, 1/\sqrt{n}) \to 0 \quad (n \to \infty).\]

From the above result on operator semigroups we are able to recover the approximation theorem for multivariate Bernstein operators as follows. Choose

\[\mathcal{X} := \text{BUC}(\mathbb{R}^m)\]

\[:= \{ f : f \text{ is a bounded uniformly continuous function from } \mathbb{R}^m \text{ into } \mathbb{R} \}\]

and define

\[T(t)f(x) := f(x + t) = f(x_1 + t_1, \ldots, x_m + t_m)\]
for each \( f \in \mathcal{X} \) and \( x \in \mathbb{R}^m \). Then \( \{T(t) : t \in \mathbb{R}_+^m \} \) is a \((C_0)\) \(m\)-parameter operator semigroup in \( \mathcal{E}(\mathcal{X}) \).

Now let \( x = 0, t = (t_1, \ldots, t_m), 0 < t < 1, 0 < t_i < 1, i = 1, \ldots, m \). Then

\[
\left\{ I + \sum_{i=1}^m t_i [T_i(1/n) - I] \right\}^n f(0)
\]

\[
= \sum_{k \in \mathbb{N}_0^n, k \leq n} f(k_1/n, \ldots, k_m/n) \frac{n!}{k_1! \ldots k_m! (n - k_1 - \ldots - k_m)!} \times t_1^{k_1} \ldots t_m^{k_m} (1 - t)^{n-k}
\]

\[
= B_n^f(t_1, \ldots, t_m),
\]

where \( B_n^f(t_1, \ldots, t_m) \) is the \( m \)-variate Bernstein operator over a simplex (cf. [12]). So by (30), we obtain

\[
\lim_{n \to \infty} B_n^f(t_1, \ldots, t_m) = T(t)f(0) = f(t_1, \ldots, t_m), \quad \text{uniformly.}
\]

Remark. The fact that the approximation theorem for the Bernstein operator can be derived from simple operator semigroup considerations has been observed by many authors (see, e.g., [1, p. 28], [10] and [14]). When considering other representation formulae for \( m \)-parameter operator semigroups in the following examples we may derive other known convergence theorems for multivariate approximation operators, but we avoid to go into details here.

Example 2. Let \( \alpha_n \) be a sequence of positive real numbers with

\[
\liminf_{n \to \infty} \alpha_n > 0 \quad \text{and} \quad \lim_{n \to \infty} \alpha_n/n = 0.
\]

For each \( n \in \mathbb{N} \) take \( X(n) = (X_{01}(n), \ldots, X_{0m}(n)) \) to have a modified multi-point distribution:

\[
P(X(n) = e_i) = x_i/\alpha_n, \quad 1 \leq i \leq m,
\]

\[
P(X(n) = 0) = 1 - \pi/\alpha_n \quad (0 < \pi/\alpha_n < 1 \text{ and } x_i > 0).
\]

Then

\[
E[X(n)] = \frac{x}{\alpha_n} \quad (x = (x_1, \ldots, x_m))
\]

and

\[
\sigma_i^2(X(n)) = E(X_i^2(n)) - (E(X_i(n)))^2 = \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}.
\]

For an equibounded \((C_0)\) \(m\)-parameter operator semigroup \( \{T(t) : t \in \mathbb{R}_+^m \} \) with \( \|T(t)\| \leq M \) for all \( t \in \mathbb{R}_+^m \), we have
$E\left[T\left(\frac{\alpha_n}{n} X(n)\right)\right] = T(0)P(X(n) = 0) + \sum_{i=1}^{m} T\left(\frac{\alpha_n}{n} e_i\right) P(X(n) = e_i)$

$= I + \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left(T_i\left(\frac{\alpha_n}{n}\right) - I\right)$.

Thus by Theorem 5, we obtain

$$\left\| \left\{ I + \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left(T_i\left(\frac{\alpha_n}{n}\right) - I\right) \right\} f - T(x)f \right\|_n$$

$$\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} mM \sum_{i=1}^{m} \left( \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2} \right) \right]$$

$$= 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} mM \sum_{i=1}^{m} \left( x_i - \frac{x_i^2}{\alpha_n} \right) \right] \to 0 \quad (n \to \infty).$$

Remark. (30) is the special case of (31) when $\alpha_n \equiv 1$, but (30) is true for an arbitrary $(C_0)$ $m$-parameter operator semigroup.

Inequalities (31) and the following (32)–(34) are Shaw’s formulae [17, 18] supplied with rates of convergence.

**Example 3.** Assume $\alpha_n$ is as in Example 2. For each $n$, let $X(n) := (X_{01}(n), \ldots, X_{0m}(n))$ follow the negative multi-point distribution:

$$P(X(n) = (k_1, \ldots, k_m)) = \binom{k}{k} \left( 1 + \frac{1}{\alpha_n x} \right)^{-1} \prod_{i=1}^{m} \left( \frac{x_i}{\alpha_n + x_i} \right)^{k_i}$$

for all $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m_+ = \{(n_1, \ldots, n_m) : n_i \in \mathbb{Z}_+, 1 \leq i \leq m\}$, where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$ is fixed and

$$\binom{n}{k} = \frac{n(n-1) \ldots (n-k+1)}{k_1! \ldots k_m!}.$$

Then

$$P(X_{0i}(n) = k_i) = \left( 1 + \frac{1}{\alpha_n x_i} \right)^{-1} \left( \frac{x_i}{\alpha_n + x_i} \right)^{k_i}$$

(see [7, p. 165, (8.4)]),

$$E(X_{0i}(n)) = \sum_{k_i=0}^{\infty} k_i \left( 1 + \frac{1}{\alpha_n x_i} \right)^{-1} \left( \frac{x_i}{\alpha_n + x_i} \right)^{k_i} = \frac{x_i}{\alpha_n}$$

and

$$\sigma^2_i(X(n)) = \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n}.$$
For an equibounded \((C_0)\) \(m\)-parameter operator semigroup \(\{T(t) : t \in \mathbb{R}_+^m\}\) with \(\|T(t)\| \leq M\) for all \(t \in \mathbb{R}_+^m\), we have

\[
E \left[T \left( \frac{\alpha_n}{n} X(n) \right) \right]
\]

\[
= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} T \left( \frac{\alpha_n}{n} (k_1, \ldots, k_m) \right) \left( \frac{k}{k} \right) \left( 1 + \frac{1}{\alpha_n} \right)^{-1} \prod_{i=1}^{m} \left( \frac{x_i}{\alpha_n + x} \right)^{k_i}
\]

\[
= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left( \frac{k}{k} \right) \left( 1 + \frac{1}{\alpha_n} \right)^{-1} \prod_{i=1}^{m} \left( \frac{x_i T_i(\alpha_n/n)}{\alpha_n + x} \right)^{k_i}
\]

\[
= \left( 1 + \frac{1}{\alpha_n} \right)^{-1} \left[ I - \frac{x_1 T_1(\alpha_n/n) + \ldots + x_m T_m(\alpha_n/n)}{\alpha_n + x} \right]^{-1}
\]

\[
= \left\{ I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n}(T_i(\alpha_n/n) - I) \right\}^{-1}
\]

By Theorem 5 we obtain

\[
\left\| I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n}(T_i(\alpha_n/n) - I) \right\|^{-n} f - T(x)f \leq 2M \omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} m M \alpha_n \sum_{i=1}^{m} \left( \frac{x_i}{\alpha_n} + \frac{x_i^2}{\alpha_n^2} \right) \right]
\]

\[
= 2M \omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left( x_i + \frac{x_i^2}{\alpha_n} \right) \right] \rightarrow 0 \quad (n \rightarrow \infty).
\]

**Example 4.** In Theorem 6 take \(N(n)\) that follows the multi-point distribution:

\[
P(N(n) = e_i) = x_i/\alpha_n, \quad 1 \leq i \leq m,
\]

\[
P(N(n) = 0) = 1 - \frac{x}{\alpha_n}, \quad \text{where} \quad x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m, \quad \text{is fixed}.
\]

Here \(\alpha_n\) is as in Example 2. Let \(Y_i(n), 1 \leq i \leq m\), be exponentially distributed with density \((1/\alpha_n)e^{-v/\alpha_n}, v \in \mathbb{R}_+\). Then

\[
E(N(n)) = \frac{1}{\alpha_n} x = \left( \frac{1}{\alpha_n} x_1, \ldots, \frac{1}{\alpha_n} x_m \right), \quad E(Y_{0i}(n)) = \alpha_n,
\]

\[
\sigma^2(N_i(n)) = \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2} \quad \text{and} \quad \sigma^2(Y_{0i}(n)) = \alpha_n^2.
\]
Also
\[
E \left[ T_i \left( \frac{1}{n} Y_{0i}(n) \right) \right] = \int_0^\infty T_i(v/n) \frac{1}{\alpha_n} e^{-v/\alpha_n} \, dv = \left( I - \frac{\alpha_n}{n} A_i \right)^{-1}
\]
(cf. [8, p. 360] and [18, p. 226]).

Furthermore, we have
\[
E \left[ T \left( \sum_{k_1=1}^{N_{i1}(n)} \frac{1}{n} Y_{k_11}(n), \ldots, \sum_{k_m=1}^{N_{im}(n)} \frac{1}{n} Y_{k_m m}(n) \right) \right] = T(0) P(N(n) = 0) + \sum_{i=1}^{m} E \left[ T_i \left( \frac{1}{n} Y_{k_i}(n) \right) \right] P(N(n) = e_i)
\]
\[
= I + \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left[ \left( I - \frac{\alpha_n}{n} A_i \right)^{-1} - I \right].
\]

By (29) of Theorem 6 for an equibounded \((C_0)\) \(m\)-parameter operator semigroup \(\{T(t) : t \in \mathbb{R}_+^m\}\) with \(\|T(t)\| \leq M\) for all \(t \in \mathbb{R}_+^m\), we get
\[
\| I + \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left[ \left( I - \frac{\alpha_n}{n} A_i \right)^{-1} - I \right] \| f - T(x) f \|
\]
\[
\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left( \frac{x_i}{\alpha_n} \alpha_i^2 + \alpha_n \left( \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2} \right) \right) \right]
\]
\[
= 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left[ 1 + \frac{9}{4} m M \sum_{i=1}^{m} (2x_i - x_i^2/\alpha_n) \right] \to 0 \quad (n \to \infty).
\]

**Example 5.** Take \(\alpha_n\) and \(Y_{0i}(n)\) as in Example 4. Let \(N(n)\) have the negative multi-point distribution:
\[
P(N(n) = (l_1, \ldots, l_m)) = \binom{l}{l} \left( I + \frac{1}{\alpha_n} \right)^{-1} \prod_{i=1}^{m} \left( \frac{x_i}{\alpha_n + x_i} \right)^{l_i}
\]
for all \(l = (l_1, \ldots, l_m) \in \mathbb{Z}_+^m\), where \(x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m\) is fixed. Then
\[
E(N(n)) = \frac{1}{\alpha_n} x = \left( \frac{1}{\alpha_n} x_1, \ldots, \frac{1}{\alpha_n} x_m \right), \quad E(Y_{0i}(n)) = \alpha_n,
\]
\[
\sigma^2(N_i(n)) = \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n} \quad \text{and} \quad \sigma^2(Y_{0i}(n)) = \alpha_n^2.
\]
Furthermore, we observe that
\[
E \left[ \sum_{k_1 = 1}^{N_1(n)} \frac{1}{n} Y_{k_1}(n), \ldots, \sum_{k_m = 1}^{N_m(n)} \frac{1}{n} Y_{k_m}(n) \right]
\]
\[
= \sum_{l \in \mathbb{Z}_m^+} \left( \prod_{i=1}^{m} \left( x_i \frac{E[T_i(Y/n)]}{\alpha_n + \tau} \right) \right)^{l_i},
\]
\[
= \left\{ I + \frac{1}{\alpha_n} \mathbb{I} - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} E[T_i(Y/n)] \right\}^{-1},
\]
\[
= \left\{ I + \frac{1}{\alpha_n} \mathbb{I} - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left( I - \frac{\alpha_n}{n} A_i \right) \right\}^{-1}
= \left\{ I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left[ \left( I - \frac{\alpha_n}{n} A_i \right)^{-1} - I \right] \right\}^{-1}.
\]

Thus by (29) of Theorem 6, for an equibounded \((C_0)\) \(m\)-parameter operator semigroup \(\{T(t) : t \in \mathbb{R}_+^m\}\) with \(\|T(t)\| \leq M\) for all \(t \in \mathbb{R}_+^m\), we have

\[
\left\| \left\{ I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} \left[ \left( I - \frac{\alpha_n}{n} A_i \right)^{-1} - I \right] \right\}^{-n} f - T(x)f \right\| \leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left\{ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left( \frac{x_i}{\alpha_n^2} + \frac{\alpha_n}{\alpha_i} \left( \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n} \right) \right) \right\}
= 2M\omega_2(Tf, (\alpha_n/n)^{1/2}) \left\{ 1 + \frac{9}{4} m M \sum_{i=1}^{m} \left( 2x_i + \frac{x_i^2}{\alpha_n} \right) \right\} \to 0 \quad (n \to \infty).
\]

**Example 6.** Take \(N\) to be a non-negative integer-valued random variable that follows the geometric distribution over \(\mathbb{Z}_+\):

\[
P(N = k) = \frac{1}{1 + \eta} \left( \frac{\eta}{1 + \eta} \right)^k \quad \text{for all } k \in \mathbb{Z}_+,
\]

where \(\eta > 0\) is a parameter. Let also \(Y \equiv (x_1, \ldots, x_m) \in \mathbb{R}_+^m\). Then

\[
E(N) = \eta > 0, \quad E Y \equiv (x_1, \ldots, x_m) = x.
\]

Furthermore,

\[
\Psi_N(\Psi^*_X(\delta)) = E(e^{\delta(x_1 + \ldots + x_m)N}) = \frac{1}{1 + \eta - e^{\delta(x_1 + \ldots + x_m)\eta} < \infty}
\]

for \(\delta < (1/\tau) \ln(1 + 1/\eta)\). So by Theorem 2, there is a constant \(K = \)
$K(M, \omega, \delta, \eta, x)$ such that for sufficiently large $n$,
\begin{equation}
\|\{\Psi_N(E[T(Y/n)])\}^n f - T(\eta x)f\|
\leq \|\{I + \eta[I - T(x_1/n, \ldots, x_m/n)]\}^{-n} f - T(\eta x)f\|
\leq K\omega_2(Tf, 1/\sqrt{n}) \to 0 \quad (n \to \infty).
\end{equation}

**Example 7.** In Theorem 3, take $p = q = 2$, and $N_\tau$ to be the Poisson process ($\tau \in \mathbb{R}_+$):

\[ P(N_\tau = k) = e^{-\eta \tau} \frac{(\eta \tau)^k}{k!} \] for all $k \in \mathbb{Z}^m_+$, where $\eta > 0$ is a parameter.

Consider $X \equiv (x_1, \ldots, x_m) = x \in \mathbb{R}^m_+$. Then

\[ E(N_\eta) = \eta \tau, \quad \Psi_{N_\tau}(s) = e^{(s-1)\eta \tau}, \quad \Psi_X^\tau(\delta) = e^{\delta \tau} < \infty. \]

Furthermore, note that

\[ d_1 = \lim \sup_{\tau \to \infty} \Psi_{N_\tau} \left( \frac{2\omega}{\tau} \right) = \lim \sup_{\tau \to \infty} \exp[(e^{\frac{2\omega}{\tau}} - 1)\eta \tau] \]

\[ d_2 = \lim \sup_{\tau \to \infty} \tau \left\{ E \left[ \left( \frac{1}{\tau} N_\tau - \eta \right)^4 \right] \right\}^{1/2} = \lim \sup_{\tau \to \infty} \tau \left( \frac{3\eta^2}{\tau^2} + \frac{\eta}{\tau^3} \right)^{1/2} < \infty \]

and

\[ d_3 = \lim \sup_{\tau \to \infty} \Psi_{N_\tau} \left( \frac{2}{e^2(\sqrt{\tau} \delta - 1)^2} e^{2\delta \tau} \right) \]

\[ = \lim \sup_{\tau \to \infty} \exp \left\{ \frac{2}{e^2(\sqrt{\tau} \delta - 1)^2} - 1 \right\} \eta \tau \]

\[ \leq \lim \sup_{\tau \to \infty} \exp \left\{ \frac{2}{e^2(\sqrt{\tau} \delta - 1)^2} e^{2\delta \tau} \right\} < \infty. \]

So by (24) of Theorem 3 there exists a constant $K = K(\delta, M, \omega, d_1, d_2, d_3)$ such that for sufficiently large $n$ we have
\begin{equation}
\|\{\Psi_N(E[T(X/\tau)])\} f - T(\eta x)f\|
\leq \|\exp[\eta \tau(T(x_1/\tau, \ldots, x_m/\tau) - I)] f - T(\eta x)f\|
\leq K\omega_2(Tf, 1/\sqrt{\tau}) \to 0 \quad (\tau \to \infty).
\end{equation}

**References**


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