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On the global existence theorem for a free boundary problem for equations of a viscous compressible heat conducting fluid

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Abstract. We consider the motion of a viscous compressible heat conducting fluid in $\mathbb{R}^3$ bounded by a free surface which is under constant exterior pressure. Assuming that the initial velocity is sufficiently small, the initial density and the initial temperature are close to constants, the external force, the heat sources and the heat flow vanish, we prove the existence of global-in-time solutions which satisfy, at any moment of time, the properties prescribed at the initial moment.

1. Introduction. The main result of this paper is the global existence theorem for the following free boundary problem for a viscous compressible heat conducting fluid (see [4], Chs. 2 and 5):

$$
\begin{aligned}
&\varrho[v_t+(v\cdot\nabla)v]+\nabla p-\mu\Delta v-\nu\nabla\operatorname{div}v=\varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t+\operatorname{div}(\varrho v)=0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho c_v(\theta_t+v\cdot\nabla\theta)+\theta p_\theta\operatorname{div}v-\kappa\Delta\theta\\
&\qquad -\frac{\mu}{2}\sum_{i,j=1}^{3}(v_{i,x_j}+v_{j,x_i})^2-(\nu-\mu)(\operatorname{div}v)^2=\varrho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\cdot\overline{n}=-p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n}=-\frac{\varphi_t}{|\nabla\varphi|} && \text{on } \widetilde{S}^T,\\
&\frac{\partial\theta}{\partial n}=\theta_1 && \text{on } \widetilde{S}^T,\\
&\varrho|_{t=0}=\varrho_0,\quad v|_{t=0}=v_0,\quad \theta|_{t=0}=\theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\widetilde{\Omega}^T=\bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, $\Omega_t\subset\mathbb{R}^3$ is a bounded domain depending

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on t, $\Omega_0 = \Omega$, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t \times \{t\}$, $S_t = \partial\Omega_t$, $\varphi(x,t) = 0$ describes S_t, $\overline{n}$ is the unit outward vector normal to the boundary, i.e. $\overline{n} = \nabla\varphi/|\nabla\varphi|$. In (1.1), $v = v(x,t)$ is the velocity of fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature. Given functions are: $f = f(x,t)$, the external force field per unit mass; $r = r(x,t)$, the heat sources per unit mass; $\theta_1 = \theta_1(x,t)$, the heat flow per unit surface; $p = p(\varrho,\theta)$, the pressure; $c_v = c_v(\varrho,\theta)$, the specific heat at constant volume. Moreover, μ and ν denote the viscosity coefficients, κ the coefficient of the heat conductivity, and p_0 the external (constant) pressure. We assume that μ, ν, κ are constants and thermodynamic considerations imply that $c_v > 0$, $\kappa > 0$, $\nu \ge \frac{1}{3}\mu > 0$. Finally, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\begin{aligned}\mathbb{T} = \{T_{ij}\} &= \{-p\delta_{ij} + \mu(v_{i,x_j} + v_{j,x_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}\\ &\equiv \{-p\delta_{ij} + D_{ij}(v)\},\end{aligned}$$

where $i,j = 1,2,3$, and $\mathbb{D} = \mathbb{D}(v) = \{D_{ij}\}$ is the deformation tensor.

Let the domain Ω be given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi,t),\ \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1,\xi_2,\xi_3). \tag{1.2}$$

Therefore, we obtain the following relation between the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle:

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv X_u(\xi,t), \tag{1.3}$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface. Thus, if $\xi \in S = S_0$ then $X_u(\xi,t) \in S_t$ and $S_t = \{x : x = X_u(\xi,t),\ \xi \in S\}$.

Equation of continuity $(1.1)_2$ and $(1.1)_5$ give the conservation of the total mass, i.e.

$$\int_{\Omega_t} \varrho(x,t)\,dx = M. \tag{1.4}$$

In this paper we prove the existence of a global-in-time solution of problem (1.1) near a constant state.

To introduce the definition of the constant state consider the equation

$$p(\varrho_{\rm e},\theta_{\rm e}) = p_0, \tag{1.5}$$

where $\theta_{\rm e} = (1/|\Omega|)\int_\Omega \theta_0\,d\xi$. We assume that equation (1.5) is solvable with respect to $\varrho_{\rm e} > 0$.

Definition 1.1. Let $f = r = \theta_1 = 0$. Then by a *constant* (*equilibrium*) *state* we mean a solution $(v, \theta, \varrho, \Omega_t)$ of problem (1.1) such that $v = 0$, $\varrho = \varrho_{\mathrm{e}}$, $\theta = \theta_{\mathrm{e}}$, $\Omega_t = \Omega_{\mathrm{e}}$ for $t \ge 0$, where ϱ_{e} is a solution of equation (1.5) and $|\Omega_{\mathrm{e}}| = M/\varrho_{\mathrm{e}}$ ($|\Omega_{\mathrm{e}}| = \operatorname{vol}\Omega_{\mathrm{e}}$).

The paper is divided into five sections. In Section 2 we introduce some notation and auxiliary results. In Section 3 we present the local existence theorem (see Theorem 3.1) proved in [16], while in Section 4 we recall the differential inequality (see Theorem 4.1) obtained in [19]. Finally, Section 5 is devoted to the global existence theorem (see Theorem 5.5).

The analogous problem to (1.1) for a viscous compressible barotropic fluid was considered by W. M. Zajączkowski in [20]. Hence, in order to prove Theorem 5.5 we apply a method similar to the proof of the global existence theorem in the barotropic case (see [20], Theorem 6.5). We prove Theorem 5.5 under the appropriate choice of ϱ_0, v_0, θ_0, θ_1, p_0, κ and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\varrho, \theta)$ (see conditions (5.40)–(5.45)) and under the assumption that $\varphi(0) \le \varepsilon_1$ ($\varphi(t)$ is given in (4.5)), where ε_1 is sufficiently small. In Theorem 5.5 we obtain a global solution of (1.1) such that $(v, \vartheta_0, \vartheta, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{M}(t)$ for $t \in \mathbb{R}^1_+$ (where ϑ_0, ϑ, ϱ_σ, $\overline{\varrho}_{\Omega_t}$ are defined in (4.2) and $\mathfrak{M}(t)$ is defined at the beginning of Section 5) and $S_t \in W_2^{4-1/2}$.

The papers [21]–[23] of W. M. Zajączkowski and the paper [14] of V. A. Solonnikov and A. Tani are devoted to the motion of a compressible barotropic viscous capillary fluid bounded by a free surface.

The motion of a viscous compressible heat conducting fluid in a fixed domain was considered by A. Matsumura and T. Nishida in [5]–[9] and by A. Valli and W. M. Zajączkowski in [15], while the papers [11]–[13] of V. A. Solonnikov are concerned with free boundary problems for viscous incompressible fluids.

The papers [1], [2] of J. T. Beale are devoted to the global existence of solutions to free boundary problems, where the free boundary is unbounded and the gravitation is taken into account.

Problem (1.1) is considered also in the papers [16]–[19] of E. Zadrzyńska and W. M. Zajączkowski. In [16] the local existence of solutions to problem (1.1) is proved. In [18] conservation laws, and in [19] the differential inequality used in the proof of the global existence theorem are derived.

Finally, [17] is a survey of results concerning problem (1.1) and the free boundary problem with surface tension, analogous to problem (1.1).

2. Notation and auxiliary results. In Section 3 we use the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(Q_T)$, $l \in \mathbb{R}^1_+$ (see [3]), of functions defined in Q_T, where $Q_T = \Omega^T \equiv \Omega \times (0,T)$ ($\Omega \subset \mathbb{R}^3$ is a domain, $T < \infty$ or $T = \infty$) or $Q_T = S^T \equiv S \times (0,T)$, $S = \partial\Omega$.

We define $W_2^{l,l/2}(\Omega^T)$ as the space of functions u such that

$$\|u\|_{W_2^{l,l/2}(\Omega^T)} = \Big[\sum_{|\alpha|+2i\le[l]} \|D_\xi^\alpha \partial_t^i u\|^2_{L_2(\Omega^T)} + \sum_{|\alpha|+2i=[l]} \bigg(\int_0^T \int_\Omega \int_\Omega \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_{\xi'}^\alpha \partial_t^i u(\xi',t)|^2}{|\xi-\xi'|^{3+2(l-[l])}} \, d\xi\, d\xi'\, dt + \int_\Omega \int_0^T \int_0^T \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_\xi^\alpha \partial_{t'}^i u(\xi,t')|^2}{|t-t'|^{1+2(l/2-[l/2])}} \, dt\, dt'\, d\xi \bigg) \Big]^{1/2} < \infty,$$

where we use generalized (Sobolev) derivatives, $D_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3}$, $\partial_{\xi_j}^{\alpha_j} = \partial^{\alpha_j}/\partial \xi_j^{\alpha_j}$ $(j = 1, 2, 3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\partial_t^i = \partial^i/\partial t^i$ and $[l]$ is the integer part of l. In the case when l is an integer the second terms in the above formulae must be omitted, while in the case of $l/2$ being an integer the last terms in the above formulae must be omitted as well.

Similarly to $W_2^{l,l/2}(\Omega^T)$, using local mappings and a partition of unity we introduce the normed space $W_2^{l,l/2}(S^T)$ of functions defined on $S^T = S \times (0,T)$, where $S = \partial\Omega$. We also use the ordinary Sobolev spaces $W_2^l(Q)$, where $l \in \mathbb{R}^1_+$, $Q = \Omega$ ($\Omega \subset \mathbb{R}^3$ is a bounded domain) or $Q = S$. To simplify notation we write

$$\|u\|_{l,Q} = \|u\|_{W_2^{l,l/2}(Q)} \quad \text{if } Q = \Omega^T \text{ or } Q = S^T,$$
$$\|u\|_{l,Q} = \|u\|_{W_2^l(Q)} \quad \text{if } Q = \Omega \text{ or } Q = S.$$

Moreover, $\|u\|_{L_p(Q)} = |u|_{p,Q}$, $1 \le p \le \infty$.

Now, we introduce the spaces $\Gamma_k^l(\Omega)$ and $\Gamma_k^{l,l/2}(\Omega)$ of functions u defined on $\Omega \times (0,T)$ ($T < \infty$ or $T = \infty$) such that

$$|u|_{l,k,\Omega} \equiv \|u\|_{\Gamma_k^l(\Omega)} = \sum_{i\le l-k} \|\partial_t^i u\|_{l-i,\Omega} < \infty$$

and

$$\left| u \right|_{l,k,\Omega} \equiv \|u\|_{\Gamma_k^{l,l/2}(\Omega)} = \sum_{2i\le l-k} \|\partial_t^i u\|_{l-2i,\Omega} < \infty,$$

where $l \in \mathbb{R}^1_+$, $k \ge 0$.

Next, define the space $L_p(0,T;\Gamma_0^{l,l/2}(\Omega))$ (where $1 \le p \le \infty$) with the norm $\|u\|_{L_p(0,T;\Gamma_0^{l,l/2}(\Omega))} = \left| u \right|_{l,0,p,\Omega^T}$.

Moreover, let $C^{2,1}(Q)$ (resp. $C_B^{2,1}(Q)$) ($Q \subset \mathbb{R}^3 \times [0,\infty)$) denote the space of functions u such that $D_x^\alpha \partial_t^i u \in C^0(Q)$ (resp. $D_x^\alpha \partial_t^i u \in C_B^0(Q)$) for $|\alpha| + 2i \le 2$ ($C_B^0(Q)$ is the space of continuous bounded functions on Q).

Finally, the following seminorm is used:

$$\|u\|_{\kappa,Q^T} = \left(\int_0^T \frac{|u|^2_{2,Q}}{t^{2\kappa}}\, dt \right)^{1/2},$$

where $Q = \partial\Omega$.

Let X be whichever of the function spaces mentioned above. We say that a vector-valued function $u = (u_1, \ldots, u_\nu)$ belongs to X if $u_i \in X$ for any $1 \le i \le \nu$.

Moreover, we use the following lemmas.

LEMMA 2.1. *The following imbedding holds*:

$$W^l_r(\Omega) \subset L^\alpha_p(\Omega) \quad (\Omega \subset \mathbb{R}^3), \tag{2.1}$$

where $|\alpha| + 3/r - 3 \le l$, $l \in \mathbb{Z}$, $1 \le p, r \le \infty$; $L^\alpha_p(\Omega)$ *is the space of functions* u *such that* $|D^\alpha_x u|_{p,\Omega} < \infty$, *and* $W^l_r(\Omega)$ *is the Sobolev space.*

Moreover, *the following interpolation inequalities are true*:

$$|D^\alpha_x u|_{p,\Omega} \le c\varepsilon^{1-\kappa} |D^l_x u|_{r,\Omega} + c\varepsilon^{-\kappa}|u|_{r,\Omega}, \tag{2.2}$$

where $\kappa = |\alpha|/l + 3/(lr) - 3/(lp) < 1$, ε *is a parameter*, *and* $c > 0$ *is a constant independent of* u *and* ε. ■

Lemma 2.1 follows from Theorem 10.2 of [3].

LEMMA 2.2 (see [10]). *For a sufficiently regular* u *we have*

$$\|\partial^i_t u(t)\|_{2l-1-2i,\Omega} \le c(\|u\|_{2l,\Omega^T} + \|\partial^i_t u(0)\|_{2l-1-2i,\Omega}),$$

where $0 \le 2i \le 2l - 1$, $l \in \mathbb{N}$, *and* $c > 0$ *is a constant independent of* T. ■

Now, consider problem (1.1). For (1.1) the energy conservation law is satisfied (see [4], Ch. 5).

Assume that the internal energy per unit mass $\varepsilon = \varepsilon(\varrho, \theta)$ has the form

$$\varepsilon(\varrho, \theta) = a_0 \varrho^\alpha + h(\varrho, \theta), \tag{2.3}$$

where $a_0 > 0$, $\alpha > 0$, $h(\varrho, \theta) \ge h_* > 0$, a_0, α, h_* are constants and $h(\varrho, \theta)$ is a sufficiently regular function of its arguments. Moreover, we assume that $h(\varrho, \theta)$ has at $(\varrho_{\mathrm{e}}, \theta_{\mathrm{e}})$ (ϱ_{e} and θ_{e} are introduced in Definition 1.1) the only minimum point equal to h_*, i.e. $\min_{\varrho,\theta} h(\varrho, \theta) = h(\varrho_{\mathrm{e}}, \theta_{\mathrm{e}}) = h_*$.

In [19] it is shown that assumption (2.3) and the thermodynamical relation

$$d\varepsilon = \theta ds + \frac{p}{\varrho^2}\, d\varrho$$

(where s is the density of entropy per unit mass) imply the following relations between h, p and c_v:

$$\alpha a_0 \varrho^{\alpha+1} + \varrho^2 h_\varrho = p - \theta p_\theta \tag{2.4}$$

and

$$c_v = \frac{\partial \varepsilon}{\partial \theta} = h_\theta. \tag{2.5}$$

In [18] (Corollary 1) the following result is proved.

LEMMA 2.3. *Let conditions* (2.3)–(2.5) *be satisfied. Let*

$$f = 0, \quad \theta_1 \ge 0. \tag{2.6}$$

Assume that

$$\begin{aligned}\int_\Omega \varrho_0 \frac{v_0^2}{2}\, d\xi + \kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s,t')\, ds \\ + \int_\Omega \varrho_0 h(\varrho_0,\theta_0)\, d\xi - \inf_t \int_{\Omega_t} \varrho h(\varrho,\theta)\, dx \le \delta_0,\end{aligned} \tag{2.7}$$

$$\int_\Omega |\varrho_0 - \varrho_{\rm e}|\, d\xi \le \delta_0, \tag{2.8}$$

$$\frac{(\beta-1)^{\beta-1}}{\beta^\beta p_0^{\beta-1}} (a_0 \varrho_{\rm e}^\beta + p_0)^\beta - a_0 \varrho_{\rm e}^\beta \le \delta_0, \tag{2.9}$$

$$a_0 \Big(\int_\Omega \varrho_0^\beta\, d\xi - \inf_t \int_{\Omega_t} \varrho^\beta\, dx \Big) \le \delta_0, \tag{2.10}$$

where $\delta_0 > 0$ *is a sufficiently small constant and* $\beta = \alpha + 1$. *Then*

$$\begin{aligned}\frac{1}{2} \int_{\Omega_t} \varrho v^2\, dx + a_0 \Big(\int_{\Omega_t} \varrho^\beta\, dx - \inf_t \int_{\Omega_t} \varrho^\beta\, dx \Big) \\ + \int_{\Omega_t} \varrho h(\varrho,\theta)\, dx - \inf_t \int_{\Omega_t} \varrho h(\varrho,\theta)\, dx + p_0(|\Omega_t| - |\Omega_*|) \le c\widetilde{\delta},\end{aligned}$$

where $|\Omega_*| = \inf_t |\Omega_t|$, $c = \mathrm{const} > 0$ *is a constant and* $\widetilde{\delta} = \widetilde{\delta}(\delta_0)$, $\widetilde{\delta} \to 0$ *as* $\delta_0 \to 0$. ■

Remark 2.4 (see [18], Theorem 2.7). Assumptions (2.3)–(2.9) imply that $\mathrm{var}_t |\Omega_t| \le c\delta$, where $c > 0$ is a constant, $\mathrm{var}_t |\Omega_t| = \sup_t |\Omega_t| - \inf_t |\Omega_t|$, $\delta^2 = \widetilde{c}\delta_0$ and $\widetilde{c}$ is a constant. ■

Remark 2.5. Since

$$\int_{\Omega_t} \varrho h(\varrho,\theta)\, dx \ge h_* \int_{\Omega_t} \varrho\, dx = h_* M \tag{2.11}$$

assumption (2.7) is satisfied if

$$\begin{aligned}\int_\Omega \varrho_0 \frac{v_0^2}{2}\, d\xi + \kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s,t')\, ds \\ + \int_\Omega \varrho_0 (h(\varrho_0,\theta_0) - h_*)\, d\xi \le \delta_0. \ ■\end{aligned} \tag{2.12}$$

R e m a r k 2.6. By Remark 3 of [18] we have

$$\frac{(d-\inf_t\int_{\Omega_t}\varrho h(\varrho,\theta)\,dx)(\beta-1)}{\beta p_0}-c\delta \le |\Omega_t|\le c\delta+\frac{(d-\inf_t\int_{\Omega_t}\varrho h(\varrho,\theta)\,dx)(\beta-1)}{\beta p_0}$$

where $c>0$ is some constant, $\delta=\delta(\delta_0)$ with $\delta\to 0$ as $\delta_0\to 0$, and

$$d=\int_\Omega \varrho_0\bigg(\frac{v_0^2}{2}+a_0\varrho_0^2+h(\varrho_0,\theta_0)\bigg)\,d\xi+p_0|\Omega|+\kappa\sup_t\int_0^t dt'\int_{S_{t'}}\theta_1(s,t')\,ds.$$

Therefore, using (2.11) and the estimate

$$\int_{\Omega_t}\varrho^\beta\,dx\ge|\Omega_t|^{1-\beta}\Big(\int_{\Omega_t}\varrho\,dx\Big)^\beta=\frac{M^\beta}{|\Omega_t|^{\beta-1}}$$

we obtain

$$\inf_t\int_{\Omega_t}\varrho^\beta\,dx\ge\frac{M^\beta(\beta p_0)^{\beta-1}}{[c\delta\beta\varrho_0+(d-h_*M)(\beta-1)]^{\beta-1}}.$$

Hence, assumption (2.10) is satisfied if

$$a_0\bigg\{\int_\Omega\varrho_0^\beta\,dx-\frac{M^\beta(\beta p_0)^{\beta-1}}{[c\delta\beta\varrho_0+(d-h_*M)(\beta-1)]^{\beta-1}}\bigg\}\le\delta_0. \tag{2.13}$$

We see that the left-hand side of (2.13) tends to 0 as $\beta\to 1$, so for β sufficiently close to 1, it is as small as we wish. ∎

3. Local existence. To prove the local existence for (1.1) we rewrite it in the Lagrangian coordinates introduced by (1.2) and (1.3):

$$\begin{aligned}
&\eta u_t-\mu\nabla_u^2u-\nu\nabla_u\nabla_u\cdot u+\nabla_u p(\eta,\Gamma)=\eta g && \text{in } \Omega^T\equiv\Omega\times(0,T),\\
&\eta_t+\eta\nabla_u\cdot u=0 && \text{in } \Omega^T,\\
&\eta c_v(\eta,\Gamma)\Gamma_t-\kappa\nabla_u^2\Gamma=-\Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot u\\
&\qquad+\frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\nabla_\xi u_j+\xi_{x_j}\nabla_\xi u_i)^2\\
&\qquad+(\nu-\mu)(\nabla_u\cdot u)^2+\eta k && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,p)\cdot\overline{n}=-p_0\overline{n} && \text{on } S^T,\\
&\overline{n}\cdot\nabla_u\Gamma=\Gamma_1 && \text{on } S^T,\\
&u|_{t=0}=v_0 && \text{in } \Omega,\\
&\eta|_{t=0}=\varrho_0 && \text{in } \Omega,\\
&\Gamma|_{t=0}=\theta_0 && \text{in } \Omega,
\end{aligned} \tag{3.1}$$

where $u(\xi,t)=v(X_u(\xi,t),t)$, $\Gamma(\xi,t)=\theta(X_u(\xi,t),t)$, $\eta(\xi,t)=\varrho(X_u(\xi,t),t)$, $g(\xi,t)=f(X_u(\xi,t),t)$, $k(\xi,t)=r(X_u(\xi,t),t)$, $\nabla_u=\xi_x\nabla_\xi\equiv\{\xi_{ix}\partial_{\xi_i}\}$, $\mathbb{T}_u(u,p)=-pI+\mathbb{D}_u(u)$, $\mathbb{D}_u(u)=\{\mu(\xi_{kx_i}\partial_{\xi_k}u_j+\xi_{kx_j}\partial_{\xi_k}u_i)+(\nu-\mu)\delta_{ij}\nabla_u u\}$ (here the summation convention over the repeated indices is assumed), and $\Gamma_1(\xi,t)=\theta_1(X_u(\xi,t),t)$.

Let $A=\{a_{ij}\}$ be the Jacobi matrix of the transformation $x=X_u(\xi,t)$, where $a_{ij}=\delta_{ij}+\int_0^t\partial_{\xi_j}u_i(\xi,t')\,dt'$. Assuming that $|\nabla_\xi u|_{\infty,\Omega^T}\le M$ we obtain

$$0<c_1(1-Mt)^3\le\det\{x_\xi\}\le c_2(1+Mt)^3,\qquad t\le T,$$

where $c_1,c_2>0$ are constants and $T>0$ is sufficiently small. Moreover, $\det A=\exp(\int_0^t\nabla_u u\,dt')=\varrho_0/\eta$.

Let S_t be determined (at least locally) by the equation $\varphi(x,t)=0$. Then S is described by $\varphi(x(\xi,t),t)|_{t=0}=\widetilde{\varphi}(\xi)=0$. Thus, we have

$$\overline{n}(x(\xi,t),t)=-\frac{\nabla_x\varphi(x,t)}{|\nabla_x\varphi(x,t)|}\bigg|_{x=x(\xi,t)}\quad\text{and}\quad\overline{n}_0(\xi)=-\frac{\nabla_\xi\widetilde{\varphi}(\xi)}{|\nabla_\xi\widetilde{\varphi}(\xi)|}.$$

Now, we are able to formulate the local existence theorem.

Theorem 3.1 (see [16], Theorem 3.7). *Let* $S\in W_2^{4-1/2}$, $f\in C^{2,1}(\mathbb{R}^3\times[0,T])$, $r\in C^{2,1}(\mathbb{R}^3\times[0,T])$, $\theta_1\in C^{2,1}(\mathbb{R}^3\times[0,T])$, $v_0\in W_2^3(\Omega)$, $\theta_0\in W_2^3(\Omega)$, $1/\theta_0\in L_\infty(\Omega)$, $\theta_0>0$, $\varrho_0\in W_2^3(\Omega)$, $1/\varrho_0\in L_\infty(\Omega)$, $\varrho_0>0$, $c_v\in C^2(\mathbb{R}_+^2)$, $c_v>0$, $p\in C^3(\mathbb{R}_+^2)$. *Moreover, assume that the following compatibility conditions are satisfied*:

$$D_\xi^\alpha(\mathbb{D}_\xi(v_0)\cdot\overline{n}_0-p(\varrho_0,\theta_0)\overline{n}_0)=-D_\xi^\alpha(p_0\overline{n}_0),\quad|\alpha|\le1,\quad\text{on } S\tag{3.2}$$

and

$$D_\xi^\alpha(\overline{n}_0\cdot\nabla_\xi\theta_0)=D_\xi^\alpha(\theta_1(\xi,0)),\quad|\alpha|\le1,\quad\text{on } S.\tag{3.3}$$

Let $T^*>0$ *be so small that* $0<c_1(1-CK_0T^*)^3\le\det\{x_\xi\}\le c_2(1+CK_0T^*)^3$ (*where* $x(\xi,t)=\xi+\int_0^t u_0(\xi,t')\,dt'$ *for* $t\le T^*$, u_0 *is given by* (3.74) *of* [16], $K_0\le c(\|\varrho_0\|_{3,\Omega}+|\varrho_0|_{\infty,\Omega}+|1/\varrho_0|_{\infty,\Omega}+\|v_0\|_{3,\Omega}+\|\theta_0\|_{3,\Omega}+\|u_t(0)\|_{1,\Omega}+\|\Gamma_t(0)\|_{1,\Omega})$, $c>0$ *is a constant*, $C=C(K_0)$ *is a nondecreasing continuous function of* K_0 *satisfying* (3.94) *of* [16]). *Then there exists* T^{**} *with* $0<T^{**}\le T^*$ *such that for* $T\le T^{**}$ *there exists a unique solution* $(u,\Gamma,\eta)\in W_2^{4,2}(\Omega^T)\times W_2^{4,2}(\Omega^T)\times C^0(0,T;\Gamma_0^{3,3/2}(\Omega))$ *of problem* (3.1). *Moreover*, $\eta_t\in C^0(0,T;W_2^2(\Omega))\cap L_2(0,T;W_2^3(\Omega))$, $\eta_{tt}\in L_2(0,T;W_2^1(\Omega))$ *and*

$$\begin{aligned}&\|u\|_{4,\Omega^T}+\|\Gamma\|_{4,\Omega^T}\le CK_0,\\&\sup_t\|\eta\|_{3,\Omega}+\sup_t\|\eta_t\|_{2,\Omega}+\|\eta_t\|_{L_2(0,T;W_2^3(\Omega))}\\&\qquad+\|\eta_{tt}\|_{L_2(0,T;W_2^1(\Omega))}\le\Phi_1(T,T^aK_0)\|\varrho_0\|_{3,\Omega},\\&|1/\eta|_{\infty,\Omega^T}+|\eta|_{\infty,\Omega^T}\\&\qquad\le\Phi_2(T^{1/2}K_0)|1/\varrho_0|_{\infty,\Omega}+\Phi_3(T^{1/2}K_0)|\varrho_0|_{\infty,\Omega},\end{aligned}\tag{3.4}$$

where Φ_1, Φ_2 *and* Φ_3 *are increasing continuous functions*, *and* $a>0$. ∎

In order to consider the global existence we need

Remark 3.2. Assume that $g = 0$ and define

$$p_\sigma = p - p_0, \quad \gamma_0 = \Gamma - \theta_{\rm e}, \quad \eta_\sigma = \eta - \varrho_{\rm e}$$

(where $\theta_{\rm e}$ and $\varrho_{\rm e}$ are introduced in Definition 1.1). Then problem (3.1) can be written in the form

$$\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u p_\sigma = 0,\\
&\eta_{\sigma t} + \eta\nabla_u\cdot u = 0,\\
&\eta c_v(\eta,\Gamma)\gamma_{0t} - \kappa\nabla_u^2\gamma_0 + \Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot u\\
&\quad - \frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\nabla_\xi u_j + \xi_{x_j}\nabla_\xi u_i)^2 - (\nu-\mu)(\nabla_u\cdot u)^2 = \eta k,\\
&\mathbb{T}_u(u,p_\sigma)\cdot\overline{n} = 0,\\
&\overline{n}\cdot\nabla_u\gamma_0 = \Gamma_1,\\
&u|_{t=0} = v_0, \quad \eta_\sigma|_{t=0} = \varrho_{\sigma 0}, \quad \gamma_0|_{t=0} = \vartheta_{00},
\end{aligned} \tag{3.5}$$

where $\varrho_{\sigma 0} = \varrho_0 - \varrho_{\rm e}$ and $\vartheta_{00} = \theta_0 - \theta_{\rm e}$.

Let the assumptions of Theorem 3.1 be satisfied and let (u,Γ,η) be the corresponding local solution of problem (3.1). Then by Theorems 3.5, 3.6 and Lemma 3.3 of [16] for a solution $(u,\,\gamma_0,\,\eta_\sigma)$ of (3.5) such that

$$T^a(\|u\|_{4,\Omega^T} + \|v_0\|_{3,\Omega} + \|\vartheta_{00}\|_{3,\Omega} + \|\varrho_{\sigma 0}\|_{3,\Omega})\varphi_1(T,K_0) \le \delta$$

(where $a > 0$ is a constant, φ_1 is an increasing continuous function of its arguments, $\delta > 0$ is sufficiently small) the following estimate holds:

$$\begin{aligned}
&\|u\|_{4,\Omega^T} + \|\eta_\sigma\|_{3,\Omega^T} + |\eta_\sigma|_{3,0,\infty,\Omega^T} + \|\gamma_0\|_{4,\Omega^T}\\
&\quad\le \varphi_2(T,K_0)(\|v_0\|_{3,\Omega} + \|\varrho_{\sigma 0}\|_{3,\Omega} + \|\vartheta_{00}\|_{3,\Omega}\\
&\qquad + \|k\|_{2,\Omega^T} + \|k(0)\|_{1,\Omega} + \|\Gamma_1\|_{3-1/2,S^T} + |D^2_{\xi,t}\Gamma_1|_{1/4,S^T}),
\end{aligned} \tag{3.6}$$

where φ_2 is an increasing continuous function of its arguments. ■

4. Differential inequality. In order to prove the global existence of solutions we need the differential inequality derived in [19] (Theorem 3.13). Assume that the existence of a sufficiently smooth local solution of problem (1.1) has been proved and consider the motion near the constant state (see Definition 1.1) $v_{\rm e} = 0$, $p_{\rm e} = p_0$, $\theta_{\rm e} = (1/|\Omega|)\int_\Omega \theta_0\,d\xi$ and $\varrho_{\rm e}$, where $\varrho_{\rm e}$ is a solution of the equation

$$p(\varrho_{\rm e},\theta_{\rm e}) = p_0. \tag{4.1}$$

Let

$$
\begin{aligned}
&p_\sigma = p - p_0, \quad \varrho_\sigma = \varrho - \varrho_{\mathrm{e}}, \quad \vartheta_0 = \theta - \theta_{\mathrm{e}},\\
&\vartheta = \theta - \theta_{\Omega_t}, \quad \overline{\varrho}_{\Omega_t} = \varrho - \varrho_{\Omega_t},
\end{aligned} \tag{4.2}
$$

where $\theta_{\Omega_t} = (1/|\Omega_t|)\int_{\Omega_t}\theta\,dx$, and $\varrho_{\Omega_t} = \varrho_{\Omega_t}(t)$ is a solution of the problem

$$
p(\varrho_{\Omega_t}, \theta_{\Omega_t}) = p_0, \quad \varrho_{\Omega_t}|_{t=0} = \varrho_{\mathrm{e}}. \tag{4.3}
$$

Then problem (1.1) takes the form

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p_\sigma) = \varrho f && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega_t,\ t\in[0,T],\\
&\varrho c_v(\varrho,\theta)(\vartheta_{0t} + v\cdot\nabla\vartheta_0) + \theta p_\theta(\varrho,\theta)\operatorname{div} v\\
&\quad - \kappa\Delta\vartheta_0 - \frac{\mu}{2}\sum_{i,j}(\partial_{x_i}v_j + \partial_{x_j}v_i)^2\\
&\quad - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in} \Omega_t,\ \ t\in[0,T],\\
&\mathbb{T}(v,p_\sigma)\cdot\overline{n} = 0 && \text{on } S_t,\ t\in[0,T],\\
&\partial\vartheta_0/\partial n = \theta_1 && \text{on } S_t,\ t\in[0,T],
\end{aligned} \tag{4.4}
$$

where $\mathbb{T}(v,p_\sigma) = \{\mu(\partial_{x_i}v_j + \partial_{x_j}v_i) + (\nu-\mu)\delta_{ij}\operatorname{div} v - p_\sigma\delta_{ij}\}$ and T is the time of the local existence.

Define

$$
\begin{aligned}
\overline{\varphi}(t) = &\int_{\Omega_t}\varrho\sum_{1\le|\alpha|+i\le 3}|D_x^\alpha\partial_t^i v|^2\,dx\\
&+\int_{\Omega_t}\left(\frac{p_1}{\varrho}\varrho_\sigma^2 + \overline{\varrho}_{\Omega_t}^2 + \frac{p_2\varrho c_v}{p_\theta\theta}\vartheta_0^2\right)dx\\
&+\int_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\sum_{1\le|\alpha|+i\le 3}|D_x^\alpha\partial_t^i\varrho_\sigma|^2\,dx\\
&+\int_{\Omega_t}\frac{\varrho c_v}{\theta}\sum_{1\le|\alpha|+i\le 3}|D_x^\alpha\partial_t^i\vartheta_0|^2\,dx,\\
\varphi(t) = &|v|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t} + |\varrho_\sigma|^2_{3,0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},\\
\Phi(t) = &|v|^2_{4,1,\Omega_t} + |\vartheta_0|^2_{4,1,\Omega_t} - \|\vartheta_0\|^2_{0,\Omega_t} + \|\vartheta\|^2_{0,\Omega_t}\\
&+ |\varrho_\sigma|^2_{3,0,\Omega_t} - \|\varrho_\sigma\|^2_{0,\Omega_t} + \|\overline{\varrho}_{\Omega_t}\|^2_{0,\Omega_t},\\
F(t) = &\|f_{ttt}\|^2_{0,\Omega_t} + |f|^2_{2,0,\Omega_t} + \|r_{ttt}\|^2_{0,\Omega_t} + |r|^2_{2,0,\Omega_t}\\
&+ \|r\|_{0,\Omega_t} + |\theta_1|^2_{4,1,\Omega_t} + \|\theta_1\|_{1,\Omega_t},\\
\psi(t) = &\|v\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t}.
\end{aligned} \tag{4.5}
$$

The following theorem is proved in [19] (see Theorem 3.13).

THEOREM 4.1. *Let $\nu > \frac{1}{3}\mu$. Then for a sufficiently smooth solution* $(v, \vartheta_0, \varrho)$ *of* (4.4) *we have*

$$\frac{d\overline{\varphi}}{dt} + c_0\Phi \le c_1 P(\varphi)\Big(\varphi + \int_0^t \|v\|^2_{4,\Omega_{t'}}\,dt'\Big)(1+\varphi^3)(\varphi+\Phi) + c_2F + c_3\psi, \tag{4.6}$$

where P is an increasing continuous function; $0 < c_0 < 1$ *is a constant depending on* ϱ^*, ϱ_*, θ^*, θ_*, μ, ν, κ; *and* c_i $(i = 1,2,3)$ *are positive constants depending on* ϱ_*, ϱ^*, θ_*, θ^*, $\int_0^t \|v\|_{3,\Omega_{t'}}dt'$, $\|S\|_{4-1/2}$, T *and the constants from the imbedding Lemma* 2.1 *and the Korn inequalities from* [20] (*Section* 5). ■

Remark 4.2. Theorem 3.13 of [20] was proved under the assumption that $\nu \ge \mu$. This assumption implies that

$$\frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t} \ge 0, \tag{4.7}$$

where $E_{\Omega_t}(v) = \int_{\Omega_t}\sum_{i,j=1}^3 (v_{i,x_j} + v_{j,x_i})^2$.

It turns out that the condition $\nu \ge \mu$ is too restrictive and we can now show that (4.7) is satisfied for $\nu > \frac{1}{3}\mu$, which is assumed in Theorem 4.1. In fact, we have

$$\begin{aligned}
&\frac{\mu}{2}E_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{0,\Omega_t}\\
&\quad= \frac{\mu}{2}\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx + (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx\\
&\quad= \frac{\mu}{2}\sum_{i\ne j}\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx + \frac{\mu}{2}\sum_{i=j}\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx\\
&\qquad+ (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx\\
&\quad= \frac{\mu}{2}\sum_{i\ne j}\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx\\
&\qquad+ \frac{\mu}{2}\varepsilon_1\sum_{i=j}\int_{\Omega_t}(v_{i,x_j}+v_{j,x_i})^2\,dx\\
&\qquad+ \frac{\mu}{2}(1-\varepsilon_1)\cdot 4\sum_{i}\int_{\Omega_t}(v_{i,x_j})^2\,dx\\
&\qquad+ (\nu-\mu)\int_{\Omega_t}(\operatorname{div} v)^2\,dx \equiv I,
\end{aligned}$$

where $\varepsilon_1 \in (0,1)$.

Since $(\xi_1+\xi_2+\xi_3)^2 \le 3(\xi_1^2+\xi_2^2+\xi_3^2)$ the last two terms in I are estimated from below by

$$\left[\nu - (1+2\varepsilon_1)\frac{\mu}{2}\right] \int\limits_{\Omega_t} (\operatorname{div} v)^2\, dx.$$

Assuming that $\nu = (1+2\varepsilon_1)\mu/3$ we obtain $\varepsilon_1 = \frac{3}{2\mu}(\nu - \mu/3)$, so

$$I \ge \frac{\mu}{2}\varepsilon_1 \int\limits_{\Omega_t} (v_{i,x_j} + v_{j,x_i})^2\, dx = \frac{3}{4}\left(\nu - \frac{\mu}{3}\right) \int\limits_{\Omega_t} (v_{i,x_j} + v_{j,x_i})^2\, dx$$
$$> 0 \quad \text{for } \nu > \tfrac{1}{3}\mu. \ \blacksquare$$

5. Global existence. We assume that

$$f = 0, \quad \theta_1 \ge 0, \tag{5.1}$$

and

$$\|r\|^2_{0,\Omega_t} + |r|^2_{2,0,\Omega_t} + \|r\|_{0,\Omega_t} + |\theta_1|^2_{4,1,\Omega_t} + \|\theta_1\|_{1,\Omega_t} \le \eta_1 e^{-\eta_2 t}, \tag{5.2}$$

where $\eta_1 > 0$ is sufficiently small and $\eta_2 > 1$.

Let $\varphi(t)$ and $\Phi(t)$ be defined by (4.5). We introduce the spaces

$$\mathfrak{N}(t) = \{(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) : \varphi(t) < \infty\},$$
$$\mathfrak{M}(t) = \Big\{(v, \vartheta_0, \vartheta, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) : \varphi(t) + \int\limits_0^t \Phi(t')\, dt' < \infty\Big\}.$$

Notice that $(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{N}(t)$ iff $\overline{\varphi}(t) < \infty$ and $(v, \vartheta_0, \vartheta, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{M}(t)$ iff $\overline{\varphi}(t) + \int_0^t \Phi(t')\, dt' < \infty$. Moreover, $c'\varphi(t) \le \overline{\varphi}(t) \le c''\varphi(t)$, where $c', c'' > 0$ are constants.

LEMMA 5.1. *Let the assumptions of Theorem* 3.1 *be satisfied. Let the initial data* v_0, ϱ_0, θ_0, S *of problem* (1.1) *be such that* $(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{N}(0)$ *and* $S \in W_2^{4-1/2}$. *Let*

$$\int\limits_{\Omega} \varrho_0 v_0\, d\xi = 0, \qquad \int\limits_{\Omega} \varrho_0 \xi\, d\xi = 0.$$

Moreover, assume

$$\overline{\varphi}(0) \le \varepsilon_1, \tag{5.3}$$

where ε_1 *is sufficiently small. Then the local solution* (v, θ, ϱ) *of problem* (1.1) *is such that* $(v, \vartheta_0, \vartheta, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{M}(t)$ *for* $t \le T$, *where* T *is the time of local existence and*

$$\varphi(t) + \int\limits_0^t \Phi(t')\, dt' \le c\varepsilon_1. \tag{5.4}$$

P r o o f. Take $(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{N}(0)$, $S \in W_2^{4-1/2}$. Then $(v_0, \vartheta_{00}, \varrho_{\sigma 0}) \in W_2^3(\Omega)$ $(\varrho_{\sigma 0} = \varrho_0 - \varrho_{\rm e}$, $\vartheta_{00} = \theta_0 - \theta_{\rm e})$ and by Theorem 3.1, Remark 3.2 and (5.2) there exists a solution of problem (1.1) such that

$$u \in W_2^{4,2}(\Omega^T), \quad \vartheta_0 \in W_2^{4,2}(\Omega^T), \quad \eta_\sigma \in W_2^{3,3/2}(\Omega^T) \cap C^0(0,T;\Gamma_0^{3,3/2}(\Omega)) \tag{5.5}$$

and

$$\begin{aligned} \|u\|^2_{4,\Omega^T} + \|\eta_\sigma\|^2_{3,\Omega^T} + |\!|\!|\eta_\sigma|\!|\!|^2_{3,0,\infty,\Omega^T} + \|\gamma_0\|^2_{4,\Omega^T} \\ \le c(\|v_0\|^2_{3,\Omega} + \|\varrho_{\sigma 0}\|^2_{3,\Omega} + \|\vartheta_{00}\|^2_{3,\Omega}) \le c\overline{\varphi}(0) \le c\varepsilon_1, \end{aligned} \tag{5.6}$$

where $u = v(x(\xi,t),t)$, $\eta_\sigma = \varrho_\sigma(x(\xi,t),t)$, $\gamma_0 = \vartheta_0(x(\xi,t),t)$.

Using estimate (5.6) for the local solution and the imbeddings (see Lemmas 2.2 and 2.1)

$$\sup_t(\|u\|^2_{3,\Omega} + \|u_t\|^2_{1,\Omega}) \le c(\|u\|^2_{4,\Omega^T} + \|u(0)\|^2_{3,\Omega} + |u(0)|^2_{1,0,\Omega}) \le c\overline{\varphi}(0) \le c\varepsilon_1$$

and

$$\int_0^t |u_\xi|_{\infty,\Omega}\, dt' \le cT^{1/2}\|u\|_{4,\Omega^T} \le cT^{1/2}\overline{\varphi}(0)$$

we have the following estimate for the solution η_σ of $(3.5)_2$ (see [22], Lemma 6.1):

$$\begin{aligned} N_1 \equiv \sup_t(\|\eta_{\sigma tt}\|^2_{0,\Omega} + \|\eta_{\sigma t}\|^2_{2,\Omega} + \|\eta_\sigma\|^2_{3,\Omega}) \\ + \|\eta_{\sigma tt}\|^2_{L_2(0,T;W_2^1(\Omega))} + \|\eta_{\sigma t}\|^2_{L_2(0,T;W_2^3(\Omega))} \\ \le \varphi_1(T,\varphi(0)) \le c\varepsilon_1, \end{aligned} \tag{5.7}$$

where φ_1 is an increasing continuous function of its arguments.

Repeating the proof of Lemma 3.10 of [19] we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v^2_{xxt} + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xxt} + \frac{\varrho c_v}{\theta}\vartheta^2_{0xxt}\right)dx \\ &\quad + C(\|v_{xxt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma xxt}\|^2_{0,\Omega_t} + \|\vartheta_{0xxt}\|^2_{1,\Omega_t}) \\ &\le (\varepsilon_1' + cN)(\|v_{xttt}\|^2_{0,\Omega_t} + \|v_{xxxt}\|^2_{0,\Omega_t} + \|v_{xxtt}\|^2_{0,\Omega_t} + \|\vartheta_{0xttt}\|^2_{0,\Omega_t} \\ &\quad + \|\vartheta_{0xxtt}\|^2_{0,\Omega_t} + \|\vartheta_{0xxxt}\|^2_{0,\Omega_t}) + cM(1+N)^2 + cF(t), \end{aligned} \tag{5.8}$$

where $C, c > 0$ are constants, $N = N_1 + N_2$, $N_2 = \sup_t(\|u\|^2_{3,\Omega} + \|u_t\|^2_{1,\Omega} + \|\gamma_0\|^2_{3,\Omega} + \|\gamma_{0t}\|^2_{1,\Omega})$ and M is such that $\int_0^T M\, dt' \le c\overline{\varphi}(0)$ holds in view of the estimates for the local solution.

Similarly, using Lemma 3.11 of [19] yields

$$\begin{aligned}(5.9)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_{xxt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma xxt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0xxt}^2\right)dx\\ &\quad+C(\|v_{xtt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma xtt}\|_{0,\Omega_t}^2+\|\vartheta_{0xtt}\|_{1,\Omega_t}^2)\\ &\le(\varepsilon_2'+cN)(\|v_{xttt}\|_{0,\Omega_t}^2+\|v_{xxtt}\|_{0,\Omega_t}^2+\|v_{xxxt}\|_{0,\Omega_t}^2\\ &\quad+\|\vartheta_{0xttt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxtt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2)+c(1+N)^2\\ &\quad\times(\|v_{xxt}\|_{1,\Omega_t}^2+\|\vartheta_{0xxt}\|_{1,\Omega_t}^2)+cM(1+N)^2+cF(t).\end{aligned}$$

Next, Lemma 3.12 of [19] implies

$$\begin{aligned}(5.10)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_{ttt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma ttt}^2+\frac{\varrho c_v}{\theta}\vartheta_{0ttt}^2\right)dx\\ &\quad+C(\|v_{ttt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma ttt}\|_{0,\Omega_t}^2+\|\vartheta_{0ttt}\|_{1,\Omega_t}^2)\\ &\le(\varepsilon_3'+N+M)(\|v_{ttt}\|_{0,\Omega_t}^2+\|\vartheta_{0ttt}\|_{0,\Omega_t}^2)\\ &\quad+cN(\|v_{xtt}\|_{1,\Omega_t}^2+\|v_{xttt}\|_{0,\Omega_t}^2+\|v_{xxt}\|_{1,\Omega_t}^2+\|\vartheta_{0xtt}\|_{1,\Omega_t}^2\\ &\quad+\|\vartheta_{0xttt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma ttt}\|_{0,\Omega_t}^2)+c\|v_{xtt}\|_{1,\Omega_t}^2\\ &\quad+c\|\vartheta_{0xtt}\|_{1,\Omega_t}^2+cM(1+N)^2+cF(t),\end{aligned}$$

where in virtue of the continuity equation $(4.4)_2$ we have

$$(5.11)\qquad \|\varrho_{\sigma ttt}\|_{0,\Omega_t}^2\le c(1+N)\|v_{xtt}\|_{0,\Omega_t}^2+cM(1+N)^2.$$

Finally, to estimate $\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2$ rewrite the equation

$$p(\varrho_{\Omega_t},\theta_{\Omega_t})-p(\varrho_{\rm e},\theta_{\rm e})=0$$

using the Taylor formula as

$$p_\varrho(\varrho_{\Omega_t}-\varrho_{\rm e})+p_\theta(\theta_{\Omega_t}-\theta_{\rm e})=0.$$

Hence

$$\begin{aligned}(5.12)\quad &\|\overline{\varrho}_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta\|_{0,\Omega_t}^2\\ &\le\|\varrho_\sigma\|_{0,\Omega_t}^2+\|\varrho_{\rm e}-\varrho_{\Omega_t}\|_{0,\Omega_t}^2+\|\vartheta_0\|_{0,\Omega_t}^2+\|\theta_{\rm e}-\theta_{\Omega_t}\|_{0,\Omega_t}^2\\ &\le c(\|\varrho_\sigma\|_{0,\Omega_t}^2+\|\vartheta_0\|_{0,\Omega_t}^2+\|\theta_{\rm e}-\theta_{\Omega_t}\|_{0,\Omega_t}^2)\\ &\le c\left(\|\varrho_\sigma\|_{0,\Omega_t}^2+\|\vartheta_0\|_{0,\Omega_t}^2+\left\|\frac{1}{|\Omega_t|}\int\limits_{\Omega_t}\vartheta_0\,dx\right\|_{0,\Omega_t}^2\right)\le c\overline{\varphi}(0)\le c\varepsilon_1,\end{aligned}$$

where to estimate $\|\varrho_\sigma\|_{0,\Omega_t}^2$ and $\|\vartheta_0\|_{0,\Omega_t}^2$ we have used (5.6).

From (5.8)–(5.12) and (5.1)–(5.2), for sufficiently small ε_1', ε_2', ε_3', N, $\int_0^T M\,dt'$ and η_1 from (5.2), we deduce that $(v(t), \varrho_\sigma(t), \overline{\varrho}_{\Omega_t}(t), \vartheta_0(t), \vartheta(t)) \in \mathfrak{M}(t)$ for $t \le T$ and (5.4) is satisfied. Of course to prove the last statement the standard technique of mollifiers or differences should be used. This concludes the proof. ∎

LEMMA 5.2. *Assume that there exists a local solution to problem* (1.1) *which belongs to* $\mathfrak{M}(t)$ *for* $t \le T$, *i.e. let the assumptions of Lemma* 5.1 *be satisfied. Let the assumptions of Lemma* 2.3 *be satisfied. Then there exist* $\delta_1, \delta_2 \in (0,1)$ *sufficiently small such that*

$$\|p_\sigma\|_{0,\Omega_t}^2 \le \delta_1, \tag{5.13}$$

$$\|\vartheta_0\|_{0,\Omega_t}^2 + \|\varrho_\sigma\|_{0,\Omega_t}^2 \le \delta_2, \tag{5.14}$$

where $\delta_1 = c\varepsilon_1\delta' + c(\delta')\widetilde{\delta}$, $\delta_2 = c\varepsilon_1\delta' + c(\delta')(\delta_0 + \widetilde{\delta})$, $\delta' \in (0,1)$ *is as small as needed*, $c(\delta')$ *is a decreasing function of* δ', *and* δ_0 *and* $\widetilde{\delta}$ *are taken from Lemma* 2.3.

Proof. Estimate (5.13) can be proved in exactly the same way as estimate (6.13) in [20]. In order to prove (5.14) we use the relation

$$|\Omega| - |\Omega_{\rm e}| = \frac{1}{\varrho_{\rm e}} \int_\Omega (\varrho_{\rm e} - \varrho_0)\,d\xi.$$

Hence, by assumption (2.8) of Lemma 2.3 we have

$$||\Omega| - |\Omega_{\rm e}|| \le c\delta_0. \tag{5.15}$$

Using (5.15) and Remark 2.4 we obtain

$$||\Omega_t| - |\Omega_{\rm e}|| \le c\delta, \tag{5.16}$$

where $\delta = \delta(\delta_0) \to 0$ as $\delta_0 \to 0$.

If $\varrho > \varrho_{\rm e}$ then $(1/\varrho_{\rm e})\int_{\Omega_t} |\varrho - \varrho_{\rm e}|\,dx = |\Omega_{\rm e}| - |\Omega_t|$.

If $\varrho < \varrho_{\rm e}$ then $(1/\varrho_{\rm e})\int_{\Omega_t} |\varrho - \varrho_{\rm e}|\,dx = |\Omega_t| - |\Omega_{\rm e}|$.

Therefore, from (5.16) it follows that

$$\int_{\Omega_t} (\varrho - \varrho_{\rm e})^2\,dx \le c\int_{\Omega_t} |\varrho - \varrho_{\rm e}|\,dx \le c\delta. \tag{5.17}$$

Hence

$$\|\varrho_\sigma\|_{0,\Omega_t} \le \delta_2'. \tag{5.18}$$

Using the Taylor formula we have

$$p_\sigma = p_1\varrho_\sigma + p_2\vartheta_0, \tag{5.19}$$

where $p_1 = p_1(\varrho, \theta)$ and $p_2 = p_2(\varrho_{\rm e}, \theta)$ (see (3.4) of [19]). Now estimates (5.13), (5.18) and formula (5.19) yield

$$\|\vartheta_0\|_{0,\Omega_t} \le \delta_2''. \tag{5.20}$$

By (5.19) and (5.20) we get (5.14). ∎

LEMMA 5.3. *Assume that there exists a local solution of* (1.1) *in* $\mathfrak{M}(t)$ *for* $0 \le t \le T$. *Let the assumptions of Lemma* 2.3 *be satisfied. Assume that the initial data are in* $\mathfrak{N}(0)$ *and*

$$\overline{\varphi}(0) \le \gamma, \quad \gamma \in (0, 1/2], \tag{5.21}$$

where γ *is sufficiently small. Then the solution at* $t \in [0, T]$ *belongs to* $\mathfrak{N}(t)$ *and*

$$\overline{\varphi}(t) \le \gamma. \tag{5.22}$$

Proof. Assumption (5.21) and Lemma 5.1 imply that the estimate (4.6) can be written as

$$\frac{d\overline{\varphi}}{dt} + c_0\Phi \le c_1'\Big(\varphi + \int_0^t \Phi(t')\,dt'\Big)(1+\varphi^3)(\varphi + \Phi) + c_2 F + c_3\psi. \tag{5.23}$$

Since $\Phi + \|\vartheta_0\|_{0,\Omega_t}^2 + \|\varrho_\sigma\|_{0,\Omega_t}^2 \ge \varphi$, using Lemma 5.2 we obtain

$$\Phi + \delta_2 \ge \varphi, \tag{5.24}$$

where δ_2 is independent of γ.

Next, by Lemmas 5.2 and 2.3 and assumptions (5.1)–(5.2) we have

$$F + \psi \le \eta + \delta_1 + c\widetilde{\delta}, \tag{5.25}$$

where $\eta = \eta_1 e^{-\eta_2 t}$ and δ_1, $\widetilde{\delta}$ are sufficiently small. Using (5.24) in (5.23) gives

$$\frac{d\overline{\varphi}}{dt} + c_0\Phi \le c_1'\Big(\varphi + \int_0^t \Phi(t')\,dt'\Big)(1+\varphi^3)(2\Phi + \delta_2) + c_2 F + c_3\psi. \tag{5.26}$$

Assuming that the initial data are so small that

$$2c_1'\Big(\varphi + \int_0^t \Phi(t')\,dt'\Big)(1+\varphi^3)\Phi \le \frac{c_0\Phi}{2},$$

instead of (5.26) we get

$$\frac{d\overline{\varphi}}{dt} + \frac{c_0\Phi}{2} \le c_1'\Big(\varphi + \int_0^t \Phi(t')\,dt'\Big)(1+\varphi^3)\delta_2 + c_2 F + c_3\psi. \tag{5.27}$$

Now using (5.24), (5.25), (5.21) and Lemma 5.1 in (5.27) yields

$$\frac{d\overline{\varphi}}{dt} + \frac{c_0\Phi}{2} \le c_4\bigg(\gamma\delta_2 + \frac{\delta_2}{2} + \eta + \delta_1 + c\widetilde{\delta}\bigg). \tag{5.28}$$

By (5.21), $\overline{\varphi}(0) \le \gamma$, $\gamma \in (0, 1/2]$. Assume that $t_* = \inf\{t \in [0, T] : \overline{\varphi}(t) > \gamma\}$. Consider (5.28) in the interval $[0, t_*]$. From the definition of t_* we have

$\overline{\varphi}(t_*) = \gamma$. Therefore (5.28) implies

$$\overline{\varphi}_t(t_*) \le -\frac{\gamma}{2} + c_4\left(\gamma\delta_2 + \frac{\delta_2}{2} + \eta + \delta_1 + c\widetilde{\delta}\right). \tag{5.29}$$

Assume that δ_2, η, δ_1 and $\widetilde{\delta}$ are so small that

$$c_4\left(\gamma\delta_2 + \frac{\delta_2}{2} + \eta + \delta_1 + c\widetilde{\delta}\right) < \frac{\gamma}{4}.$$

Hence (5.29) yields $\overline{\varphi}_t(t_*) < 0$, a contradiction. Therefore (5.22) holds. ∎

Lemma 5.3 suggests that the solution can be continued to the interval $[T, 2T]$, but to do this we need the following facts:

(5.30) (a) The existence of the transformation $x = x(\xi, t)$ and its inverse for $t \in [T, 2T]$.
(b) The validity of the Korn inequality with the same constant for the whole interval $[0, 2T]$.
(c) The variations of the shape of Ω_t for $t \in [0, 2T]$ are so small that the constants in Lemma 2.1 (imbedding (2.1)) can be chosen independently of t.

Generally, to prove the global existence we need these facts for all t. Theorem 2.7 of [18] implies that the volume of Ω_t does not change much but we have not shown yet any restriction on the variations of its shape.

It is sufficient to show (c), because (a) and (b) follow.

LEMMA 5.4. *Assume that there exists a local solution of* (1.1) *in* $\mathfrak{M}(t)$ *for* $0 \le t \le T$ *with initial data in* $\mathfrak{N}(0)$ *sufficiently small* (see (5.3)). *Then there exist constants* $\mu_1 > 0$ *and* $\mu_2 > 0$ (μ_2 *is sufficiently small*) *such that*

$$\overline{\varphi}(t) \le c e^{-\mu_1 t}(\overline{\varphi}(0) + \mu_2), \qquad t \le T, \tag{5.31}$$

where $c > 0$ *is a constant and* T *is the time of local existence. Moreover*, *if we assume* (5.2) *with* $\eta_1 = 0$, *then* (5.31) *holds with* $\mu_2 = 0$.

Proof. Inequalities (3.20) and (3.28) of [19] (see the proof of Lemma 3.1 and the assertion of Lemma 3.2 of [19]) imply

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \left[\varrho(v^2 + v_t^2) + \frac{1}{\varrho}(p_1\varrho_\sigma^2 + p_\varrho \varrho_{\sigma t}^2) + \frac{\varrho c_v}{\theta}\left(\frac{p_2}{p_\theta}\vartheta_0^2 + \vartheta_{0t}^2\right)\right] dx \\ + c_0(\|v\|_{1,\Omega_t}^2 + \|v_t\|_{1,\Omega_t}^2 + \|\operatorname{div} v\|_{0,\Omega_t}^2 + \|\operatorname{div} v_t\|_{0,\Omega_t}^2 \\ + \|\vartheta_{0x}\|_{0,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{0,\Omega_t}^2) \\ \le C_2\varphi^2(t)(1 + \varphi(t)) + C_1 F(t).\end{aligned} \tag{5.32}$$

Multiplying both sides of (5.32) by $e^{-\alpha t}$ (where $0<\alpha<1$) we obtain

$$
\begin{aligned}
(5.33)\quad &\frac{d}{dt}\bigg\{e^{-\alpha t}\int\limits_{\Omega_t}\bigg[\varrho(v^2+v_t^2)+\frac{1}{\varrho}(p_1\varrho_\sigma^2+p_\varrho\varrho_{\sigma t}^2)\\
&\quad+\frac{\varrho c_v}{\theta}\Big(\frac{p_2}{p_\theta}\vartheta_0^2+\vartheta_{0t}^2\Big)\bigg]dx\bigg\}+\alpha e^{-\alpha t}\int\limits_{\Omega_t}\bigg[\varrho(v^2+v_t^2)\\
&\quad+\frac{1}{\varrho}(p_1\varrho_\sigma^2+p_\varrho\varrho_{\sigma t}^2)+\frac{\varrho c_v}{\theta}\Big(\frac{p_2}{p_\theta}\vartheta_0^2+\vartheta_{0t}^2\Big)\bigg]dx\\
&\quad+c_0e^{-\alpha t}(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\operatorname{div}v\|_{0,\Omega_t}^2+\|\operatorname{div}v_t\|_{0,\Omega_t}^2\\
&\quad+\|\vartheta_{0x}\|_{0,\Omega_t}^2+\|\vartheta_{0t}\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2)\\
&\le C_2e^{-\alpha t}\varphi^2(t)(1+\varphi(t))+C_1e^{-\alpha t}F(t).
\end{aligned}
$$

Next multiplying inequality (4.6) by $e^{-\alpha t}$ we have

$$
\begin{aligned}
(5.34)\quad &\frac{d\overline{\varphi}_1}{dt}+\alpha\varphi_1+c_0\Phi_1\\
&\le c_1P(\varphi)\Big(\varphi+\int\limits_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'\Big)(1+\varphi^3)(\varphi_1+\Phi_1)\\
&\quad+c_2F_1+c_3\psi_1,
\end{aligned}
$$

where $\varphi_1=\varphi e^{-\alpha t}$, $\overline{\varphi}_1=\overline{\varphi}e^{-\alpha t}$, $\Phi_1=\Phi e^{-\alpha t}$, $F_1=Fe^{-\alpha t}$, and $\psi_1=\psi e^{-\alpha t}$.

From the assumption that the initial data are sufficiently small we deduce that $\varphi+\int_0^t\|v\|_{4,\Omega_{t'}}^2\,dt'$ is also small (see Lemma 5.1). Therefore, for sufficiently small data from (5.34) we get

$$
(5.35)\qquad \frac{d\overline{\varphi}_1}{dt}+c_4(\varphi_1+\Phi_1)\le c_5F_1+c_6(\|p_\sigma\|_{0,\Omega_t}^2+\|v\|_{0,\Omega_t}^2)e^{-\alpha t}.
$$

Now, applying the same argument as in the proof of Lemma 6.2 of [20] we obtain

$$
(5.36)\qquad \|p_\sigma\|_{0,\Omega_t}^2\le\varepsilon(\|p_{\sigma x}\|_{0,\Omega_t}^2+\|v_{xx}\|_{0,\Omega_t}^2)+c(\varepsilon)(\|v\|_{0,\Omega_t}^2+\|v_t\|_{0,\Omega_t}^2).
$$

Moreover,

$$
(5.37)\qquad \|p_{\sigma x}\|_{0,\Omega_t}^2\le c(\|\varrho_{\sigma x}\|_{0,\Omega_t}^2+\|\vartheta_{0x}\|_{0,\Omega_t}^2).
$$

Using (5.36) and (5.37) in (5.35) we have

$$
(5.38)\qquad \frac{d\overline{\varphi}_1}{dt}+c_4(\varphi_1+\Phi_1)\le c_5F_1+c_7(\|v\|_{0,\Omega_t}^2+\|v_t\|_{0,\Omega_t}^2)e^{-\alpha t}.
$$

Multiplying (5.33) by a sufficiently large constant c_8, adding the result to

(5.38) and using the fact that $\overline{\varphi}(0)$ is sufficiently small we obtain

$$\frac{d\widetilde{\overline{\varphi}}}{dt} + c_9(\widetilde{\varphi} + \widetilde{\Phi}) \le c_{10}F_1, \tag{5.39}$$

where

$$\begin{aligned}
\widetilde{\varphi} &= \varphi_1 + c_8 e^{-\alpha t} \int_{\Omega_t} \bigg[\varrho(v^2 + v_t^2) + \frac{1}{\varrho}(p_1\varrho_\sigma^2 + p_\sigma \varrho_{\sigma t}^2) \\
&\quad + \frac{\varrho c_v}{\theta}\bigg(\frac{p_2}{p_\theta}\vartheta_0^2 + \vartheta_{0t}^2\bigg)\bigg]\, dx, \\
\widetilde{\overline{\varphi}} &= \overline{\varphi}_1 + c_8 e^{-\alpha t} \int_{\Omega_t} \bigg[\varrho(v^2 + v_t^2) + \frac{1}{\varrho}(p_1\varrho_\sigma^2 + p_\sigma \varrho_{\sigma t}^2) \\
&\quad + \frac{\varrho c_v}{\theta}\bigg(\frac{p_2}{p_\theta}\vartheta_0^2 + \vartheta_{0t}^2\bigg)\bigg]\, dx, \\
\widetilde{\Phi} &= \Phi_1 + c_8 e^{-\alpha t}(\|v\|_{1,\Omega_t}^2 + \|v_t\|_{1,\Omega_t}^2 + \|\operatorname{div} v\|_{0,\Omega_t}^2 \\
&\quad + \|\operatorname{div} v_t\|_{0,\Omega_t}^2 + \|\vartheta_{0x}\|_{0,\Omega_t}^2 + \|\vartheta_{0t}\|_{1,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{0,\Omega_t}^2).
\end{aligned}$$

There exist constants c_0', $c_0'' > 0$ such that

$$c_0'\overline{\varphi}_1 \le \widetilde{\overline{\varphi}} \le c_0''\overline{\varphi}_1 \quad \text{and} \quad c_0'\Phi_1 \le \widetilde{\Phi} \le c_0''\Phi_1.$$

Hence by assumptions (5.1) and (5.2) inequality (5.39) implies

$$\overline{\varphi}_1 \le c_{12} e^{-c_{10}t}(\overline{\varphi}(0) + c_{11}), \tag{5.40}$$

where $c_{11} > 0$ is sufficiently small.

For α sufficiently small, from (5.40) we obtain (5.31). ■

Finally, we prove the main result of this paper.

THEOREM 5.5. *Let $\nu > \frac{1}{3}\mu$. Let* (5.1), (5.2) *with $\eta_1 = 0$ and the assumptions of Theorem* 3.1 *with $r \in C_B^{2,1}(\mathbb{R}^3 \times [0,+\infty))$ and $\theta_1 \in C_B^{2,1}(\mathbb{R}^3 \times [0,+\infty))$ be satisfied. Furthermore, let $(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{N}(0)$ and*

$$\varphi(0) \le \varepsilon_1, \tag{5.41}$$

where $\varepsilon_1 \in (0,1)$ is sufficiently small. Let the following compatibility conditions be satisfied:

$$\begin{aligned}
&D^\alpha \partial_t^i(\mathbb{T}\cdot\overline{n} + p_0\overline{n})|_{t=0,S} = 0, \qquad |\alpha| + i \le 2, \\
&D^\alpha \partial_t^i(\overline{n}\cdot\nabla\theta - \theta_1)|_{t=0,S} = 0, \qquad |\alpha| + i \le 2.
\end{aligned} \tag{5.42}$$

Assume also that the internal energy per unit mass $\varepsilon = \varepsilon(\varrho,\theta)$ has the form (2.3) *and conditions* (2.4)–(2.5) *hold. Moreover, assume that*

$$\int\limits_{\Omega} \varrho_0 \frac{v_0^2}{2}\, d\xi + \kappa \sup_t \int\limits_0^t dt' \int\limits_{S_{t'}} \theta_1(s,t')\, ds + \int\limits_{\Omega} \varrho_0(h(\varrho_0,\theta_0) - h_*)\, d\xi \le \varepsilon_2, \tag{5.43}$$

$$\int\limits_{\Omega} |\varrho_0 - \varrho_{\mathrm{e}}|\, d\xi \le \varepsilon_2, \tag{5.44}$$

$$\frac{(\beta-1)^{\beta-1}}{\beta^\beta p_0^{\beta-1}} (a_0 \varrho_{\mathrm{e}}^\beta + p_0)^\beta - a_0 \varrho_{\mathrm{e}}^\beta \le \varepsilon_2, \tag{5.45}$$

$$a_0 \bigg\{ \int\limits_{\Omega} \varrho_0^\beta\, dx - \frac{M^\beta (\beta p_0)^{\beta-1}}{[c\delta\beta\varrho_0 + (d - h_* M)(\beta-1)]^{\beta-1}} \bigg\} \le \varepsilon_2, \tag{5.46}$$

where $\varepsilon_2 > 0$ *is a sufficiently small constant*, $\beta = \alpha + 1$, *and* $c > 0$ *and* $\delta > 0$ *are the constants from Remark* 2.6. *Assume*, *finally*, *that*

$$\int\limits_{\Omega} \varrho_0 v_0 (a + b \times \xi)\, d\xi = 0, \quad \int\limits_{\Omega} \varrho_0 \xi\, d\xi = 0, \quad \int\limits_{\Omega} \varrho_0\, d\xi = M, \tag{5.47}$$

where a, b *are arbitrary constant vectors. Then there exists a global solution of* (1.1) *such that* $(v, \vartheta_0, \vartheta, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{M}(t)$ *for* $t \in \mathbb{R}_+^1$ *and* $S_t \in W_2^{4-1/2}$.

P r o o f. The theorem is proved step by step using the local existence in a fixed time interval. Under the assumption that

$$(v, \vartheta_0, \varrho_\sigma, \overline{\varrho}_{\Omega_t}) \in \mathfrak{N}(0), \tag{5.48}$$

Theorem 3.1 and Remark 3.2 yield the local existence of solutions of (1.1) such that

$$\begin{aligned} & u \in W_2^{4,2}(\Omega^T), \quad \vartheta_0 \in W_2^{4,2}(\Omega^T), \\ & \eta_\sigma \in W_2^{3,3/2}(\Omega^T) \cap C^0(0,T; \Gamma_0^{3,3/2}(\Omega)), \end{aligned} \tag{5.49}$$

where T is the time of the existence. By (5.48) and (5.49), Lemma 5.1 implies that the local solution belongs to $\mathfrak{M}(t)$ for $t \le T$. For small ε_1 the existence time T is correspondingly large, so we can assume it is a fixed positive number.

To prove the last result we needed the Korn inequalities (see [20]) and Lemma 2.1 (imbedding (2.2)).The constants in those theorems depend on Ω_t and the shape of S_t, so generally they are functions of t. In view of (5.41), Lemma 5.1 gives

$$\varphi(t) + \int\limits_0^t \Phi(t')\, dt' \le c\varepsilon_1.$$

Hence we obtain

$$\Big|\int_0^t v\,dt'\Big| \le c\varepsilon_1, \qquad t\in[0,T]. \tag{5.50}$$

Therefore from the relation

$$x=\xi+\int_0^t u(\xi,t')\,dt', \qquad \xi\in S,\ t\in T, \tag{5.51}$$

it follows that for sufficiently small ε_1 and fixed T, the shape of Ω_t, $t\le T$, does not change too much, so the constants from the imbedding Lemma 2.1 can be chosen independent of time.

Since $\overline{\varphi}(t)\le c''\varphi(t)$, (5.41), Lemma 5.3 and Remarks 2.5–2.6 imply

$$\overline{\varphi}(T)\le c''\varepsilon_1, \tag{5.52}$$

for sufficiently small ε_1 and ε_2.

Now we wish to extend the solution to the interval $[T,2T]$. Using (5.52) we can prove the existence of local solutions in $\mathfrak{M}(t)$ for $T\le t\le 2T$. To prove

$$\overline{\varphi}(2T)\le c''\varepsilon_1 \tag{5.53}$$

we need inequality (4.6), where the constants depend on the constants from the imbedding theorems and the Korn inequalities for $t\in[T,2T]$. Therefore, we have to show that the shape of S_t, $t\le 2T$, does not change more that for $t\le T$. For this we need the following (see (5.30)). Assume that there exists a local solution in $[0,kT]$. Then in view of Lemma 5.4, for $t\in[0,kT]$ we have

$$\begin{aligned}
&\Big|\int_0^t v\,dx\Big|+\Big|\int_0^t v_x\,dx\Big| \\
&\quad\le c_2\int_0^t \|v\|_{3,\Omega_{t'}}\,dt' \le c_2\sum_{i=0}^{k-1}\int_{iT}^{(i+1)T}\|v\|_{3,\Omega_{t'}}\,dt' \\
&\quad\le c_2T^{1/2}\sum_{i=0}^{k-1}\Big(\int_{iT}^{(i+1)T}\|v\|^2_{3,\Omega_{t'}}\,dt'\Big)^{1/2} \le c_3T^{1/2}\sum_{i=0}^{k-1}\Big(\int_{iT}^{(i+1)T}\overline{\varphi}(t')\,dt'\Big)^{1/2} \\
&\quad\le c_3T^{1/2}\sum_{i=0}^{k-1}\Big[\overline{\varphi}(iT)\int_{iT}^{(i+1)T}e^{-\mu_1(t-iT)}\,dt\Big]^{1/2} \\
&\quad\le c_3[T(1-e^{-\mu_1T})/\mu_1]^{1/2}\sum_{i=0}^{k-1}(\overline{\varphi}(iT))^{1/2}
\end{aligned}$$

$$\leq c_3[T(1-e^{-\mu_1 T})/\mu_1]^{1/2}\sum_{i=0}^{k-1}(\overline{\varphi}(iT))^{1/2}$$
$$\leq c_3\{T[(1-e^{-\mu_1 T})/\mu_1]\overline{\varphi}(0)(1+e^{-\mu_1 T}+e^{-2\mu_1 T}+\ldots)\}^{1/2}$$
$$= c_3[T(1/\mu_1)\overline{\varphi}(0)(1-e^{-\mu_1 T})(1-e^{-\mu_1 T})^{-1}]^{1/2}$$
$$= c_3[T(1/\mu_1)\overline{\varphi}(0)]^{1/2} \leq c_4 T^{1/2}\varepsilon_1^{1/2}.$$

Taking $k=2$ and ε_1 sufficiently small we see that $|\int_0^t v(x,t')\,dt'|$ is small for any $t\in[T,2T]$, so (5.51) implies that the shape of S_t changes no more than in $[0,T]$ and then the differential inequality (4.6) can be shown for this interval with the same constants, too. Hence in view of Lemma 5.1 the solution of (1.1) belongs to $\mathfrak{M}(t)$ for $t\in[T,2T]$. Next Lemmas 5.1–5.3 and Remarks 2.5–2.6 imply (5.53).

Repeating the above considerations for the intervals $[kT,(k+1)T]$, $k\geq 2$, we prove the existence for all $t\in\mathbb{R}_+^1$. This concludes the proof of the theorem. ■

Remark 5.6. Lemma 5.4 implies that $\varphi(t)\to 0$ as $t\to\infty$. Hence the considered motion converges to the constant state. ■

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