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Logarithmic structure of the generalized bifurcation set

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Abstract. Let $G : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}$ be a holomorphic family of functions. If $\Lambda \subset \mathbb{C}^n \times \mathbb{C}^r$, $\pi_r : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^r$ is an analytic variety then

$$Q_\Lambda(G) = \{(x, u) \in \mathbb{C}^n \times \mathbb{C}^r : G(\cdot, u) \text{ has a critical point in } \Lambda \cap \pi_r^{-1}(u)\}$$

is a natural generalization of the bifurcation variety of G. We investigate the local structure of $Q_\Lambda(G)$ for locally trivial deformations of $\Lambda_0 = \pi_r^{-1}(0)$. In particular, we construct an algorithm for determining logarithmic stratifications provided G is versal.

1. Introduction. Motivation of this paper lies in theoretical questions in optics where a central role is played by isotropic, Lagrangian and coisotropic varieties in a symplectic space. The geometrical framework convenient for investigations of these varieties is based mainly on the action of symplectic relations (cf. [4]).

Let $\Omega = (T^*\mathbb{R}^k \times T^*\mathbb{R}^n, \pi_2^*\omega_{\mathbb{R}^n} - \pi_1^*\omega_{\mathbb{R}^k})$ be a product symplectic space. Lagrangian submanifolds of Ω (symplectic relations) act on subsets of $(T^*\mathbb{R}^k, \omega_{\mathbb{R}^k})$ preserving their symplectic properties. In this way one can investigate the symplectic projections $\pi_{\mathbb{R}^n}|_S : S \to \mathbb{R}^n$ using the representation of S as the image under a symplectic relation $L \subset \Omega$ of a subset Λ of the zero-section of $T^*\mathbb{R}^k$, i.e.

$$S = L(\Lambda) = \{p \in T^*\mathbb{R}^n : \exists_{\overline{p} \in \Lambda}\ (\overline{p}, p) \in L\}.$$

For practical purposes one seeks to classify germs of the projections $\pi_{\mathbb{R}^n}|_S$ and describe the structure of the corresponding variety of critical values. Assuming that L is generated by a smooth function $G : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ we easily find that this variety is defined as a generalized bifurcation diagram

$$Q_\Lambda(G) = \{q \in \mathbb{R}^n : G(\cdot, q) \text{ has a critical point belonging to } \Lambda\}.$$

In this paper we study the generalized bifurcation varieties of complex analytic families G using the technical tools of the theory of singularities

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of functions on varieties (cf. [3]). In Section 2 we provide the classification scheme of such varieties and introduce the notion of logarithmic stratification. In Section 3 we adapt to our Λ-bifurcation varieties the method for construction of logarithmic vector fields which is well known for the standard bifurcation and discriminant varieties (cf. [2, 14]). The specific algorithm explicitly calculating the tangent vector fields to $Q_\Lambda(G)$ and the representative examples of Λ-bifurcation varieties are discussed in Section 4.

2. Classification of generalized bifurcation varieties. Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. Let $(\Lambda, 0) \subset (\mathbb{C}^n, 0)$ be the germ of a reduced analytic subvariety of $\mathbb{C}^n$ at 0:

$$\Lambda = \{x \in \mathbb{C}^n : F(x) = 0\}, \qquad F \in \mathcal{O}_n.$$

The group of germs of diffeomorphisms $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which preserve Λ is denoted by $\mathcal{G}_\Lambda$. If J_Λ denotes the ideal in $\mathcal{O}_n$ consisting of germs of functions vanishing on Λ, then for $\phi \in \mathcal{G}_\Lambda$ the induced isomorphism $\phi^* : \mathcal{O}_n \to \mathcal{O}_n$ preserves J_Λ.

Two function-germs $g_1, g_2 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are called $\mathcal{G}_\Lambda$-*equivalent* if there is a diffeomorphism $\phi \in \mathcal{G}_\Lambda$ with $g_1 \circ \phi = g_2$ [3, 10].

We obtain elements of $\mathcal{G}_\Lambda$ by integrating vector fields tangent to Λ.

Definition 2.1. We denote by Ξ_Λ the $\mathcal{O}_n$-module of *logarithmic vector fields* for Λ, i.e. holomorphic vector fields on $(\mathbb{C}^n, 0)$, which, if considered as derivations, say $v : \mathcal{O}_n \to \mathcal{O}_n$, satisfy

$$v.h \in J_\Lambda \quad \text{for all } h \in J_\Lambda.$$

Modules of holomorphic vector fields of this type are discussed in [11].

A function-germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is k-$\mathcal{G}_\Lambda$-*determined* if for all $\widetilde{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with the same k-jet as g the germs g and $\widetilde{g}$ are $\mathcal{G}_\Lambda$-equivalent. Given a germ $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, a germ $G : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ is called a *deformation* of h if $G(x, 0) = h(x)$. Formally we look on a deformation of h as a pair (G, r). Given two deformations $(H, r), (G, q)$ of h, a *morphism* $(\Phi, l) : H \to G$ between them is defined as follows:

1. $\Phi : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}^n \times \mathbb{C}^q, 0)$ has the form $\Phi(x, u) = (\phi(x, u), u)$ with $\phi(\cdot, 0) = \mathrm{id}_{\mathbb{C}^n}$ and $\phi(\cdot, u) \in \mathcal{G}_\Lambda$ for all u near $0 \in \mathbb{C}^r$.
2. $l : (\mathbb{C}^r, 0) \to (\mathbb{C}^q, 0)$ is such that

$$G(\phi(x, u), l(u)) = H(x, u).$$

A deformation (G, q) of h is $\mathcal{G}_\Lambda$-*versal* if for any unfolding (H, r) of h there is a morphism $(\Phi, l) : H \to G$.

Two deformations of h are *equivalent* if there exists a morphism between them which is an isomorphism.

Let $U \subset (\mathbb{C}^n, 0)$ be an open, sufficiently small subset of $\mathbb{C}^n$. We can also consider the sheaf $\mathcal{O}_U$ of holomorphic functions on U, and the sheaf Der_U of holomorphic vector fields on U, together with its subsheaf $\varXi_\varLambda$. Following [11] we introduce the logarithmic stratification of U determined by $\varXi_\varLambda$.

DEFINITION 2.2. Let $\{\varLambda_\alpha : \alpha \in I\}$ be a stratification of U with the following properties:

1. Each stratum $\varLambda_\alpha$ is a smooth connected immersed submanifold of U and $U = \bigcup_{\alpha \in I} \varLambda_\alpha$.
2. If $x \in \varLambda_\alpha$ then $T_x\varLambda_\alpha$ coincides with $\varXi_\varLambda(x)$.
3. If $\varLambda_\alpha$, $\varLambda_\beta$ are two distinct strata with $\varLambda_\alpha$ meeting the closure $\overline{\varLambda}_\beta$ of $\varLambda_\beta$, then $\varLambda_\alpha$ is contained in the boundary $\partial\varLambda_\beta$ of $\varLambda_\beta$.

Then $\{\varLambda_\alpha : \alpha \in I\}$ is called a *logarithmic stratification* of $\varLambda$ and $\varLambda_\alpha$ is a *logarithmic stratum.*

For any variety $\varLambda$ and sufficiently small U there always exists a unique logarithmic stratification of U.

The aim of this note is to construct the logarithmic stratification for generalized bifurcation varieties, and so to construct an appropriate module of logarithmic vector fields $\varXi_\varLambda$.

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $g \in \mathcal{O}_n$. We define the Jacobi ideal of g by

$$\varDelta_\varLambda(g) = \{v.g : v \in \varXi_\varLambda\}.$$

If $\varDelta_\varLambda(g) \supset \mathbf{m}_n^k$, then g is $(k+1)$-$\mathcal{G}_\varLambda$-determined, i.e. for all $\widetilde{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with the same $(k+1)$-jet as g the germs $g, \widetilde{g}$ are $\mathcal{G}_\varLambda$-equivalent. Here $\mathbf{m}_n$ is the maximal ideal of $\mathcal{O}_n$. As in the usual singularity theory setting [1] a deformation (G, r) of g is $\mathcal{G}_\varLambda$-versal if and only if

$$\frac{\partial G}{\partial u_1}(x, 0), \ldots, \frac{\partial G}{\partial u_r}(x, 0)$$

span $\mathcal{O}_n/\varDelta_\varLambda(g)$.

We know (cf. [2]) that if the set-germ

$$\{x \in \mathbb{C}^n : v.g(x) = 0 \text{ for all } v \in \varXi_\varLambda\}$$

at 0 is $\{0\}$ or empty then g has a $\mathcal{G}_\varLambda$-versal deformation. If the number $\mu = \dim_{\mathbb{C}} \mathcal{O}_n/\varDelta_\varLambda(g)$ is finite, then it is called the *multiplicity* of g on $\varLambda$ at 0, and is also denoted by $\mu_\varLambda(g)$.

Let (G, r) be a deformation of g.

DEFINITION 2.3. The analytic variety

$$Q_\varLambda(G) = \{u \in \mathbb{C}^r : G(\cdot, u) \text{ has a critical point on } \varLambda\}$$

is called the $\varLambda$-*bifurcation variety* of the family G.

Define

$$\Sigma_\Lambda(G) = \left\{(x,u)\in \mathbb{C}^n\times\mathbb{C}^r : \frac{\partial G}{\partial x_i}(x,u) = 0,\ F(x) = 0\right\},$$

where $\Lambda = F^{-1}(0), F \in \mathcal{O}_n$. Then we see that

$$Q_\Lambda(G) = \pi_r(\Sigma_\Lambda(G)),$$

where $\pi_r : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^r$.

EXAMPLE 2.4. As a natural example we consider the simplest Λ-bifurcation varieties corresponding to singularities of functions on regular boundaries (cf. [1]). Let $\Lambda = \{(y,x) \in \mathbb{C}^{n+1} : y = 0\}$, $x = (x_1,\ldots,x_n)$. It is easy to check that for B_k and C_k singularities $Q_\Lambda(G)$ are smooth hypersurfaces. For the F_4 singularity

$$G(y,x,u) = y^2 + x^3 + u_1xy + u_2y + u_3x$$

the Λ-bifurcation variety $Q_\Lambda(G)$ is the Whitney cross cap

$$3u^2 + u_3u_1^2 = 0.$$

By straightforward calculations we prove that for unimodal, corank one boundary singularities of smallest codimension $\mu = 6$:

$$F_{1,0} : G(y,x,u) = x^3 + bx^2y + y^3 + u_1xy^2 + u_2xy + u_3y^2 + u_4x + u_5y,$$

$$K_{4,2} : G(y,x,u) = x^4 + ax^2y + y^2 + u_1x^2y + u_2x^2 + u_3yx + u_4x + u_5y,$$

the Λ-bifurcation varieties are:

1. The trivial extension of the Whitney cross cap variety in the case $F_{1,0}$.
2. The generalized Whitney cross cap (cf. [1], Section 9.6), given in the following parametric form:

$$u_1 = s,\quad u_2 = t,\quad u_3 = w,\quad u_4 = -4x^3 - 2tx,\quad u_5 = -(a+s)x^2 - wx.$$

For simplest unimodal, corank two boundary singularity of type L_6:

$$G(y,x,u) = x_1^2x_2 + x_2^3 + yx_1 + ayx_2 + u_1yx_2 + u_2x_1^2 + u_3x_1 + u_4x + u_5y,$$

the Λ-bifurcation variety $Q_\Lambda(G)$ is parametrized in the form

$$u_1 = s,\ u_2 = t,\ u_3 = -2x_1x_2 - 2x_1t,\ u_4 = -x_1^2 - 3x_2^2,\ u_5 = -x_1 - sx_2 - ax_2$$

and is an opening of the Σ^2-Boardmann singular mapping $\mathbb{C}^4 \to \mathbb{C}^4$.

3. Logarithmic vector fields. We denote by $\mathrm{Sing}(\Sigma_\Lambda(G))$ the singular part of $\Sigma_\Lambda(G)$. Then $\Sigma_\Lambda(G) - \mathrm{Sing}(\Sigma_\Lambda(G))$ decomposes into analytic strata $\Sigma_\Lambda^\alpha(G)$, $\alpha \in I$. We consider the family of mappings $\pi_r^\alpha = \pi_r|_{\Sigma_\Lambda^\alpha(G)}$. Critical points of these mappings are described by an extra n equations:

$$\mathrm{rank}\begin{pmatrix}\partial^2 G/\partial x_i\partial x_j\\ \partial F/\partial x_j\end{pmatrix}(x,u) < n.$$

We denote by $\Gamma_r^\alpha = \Gamma(\pi_r^\alpha)$ the set of critical values of the mapping π_r^α.

Now we assume that (G, r) is a $\mathcal{G}_\Lambda$-versal deformation of g. Let $g_0, \ldots, g_{\mu-1}$ be a basis of the quotient space $\mathcal{O}_n/\Delta_\Lambda(g)$ with $g_0 = 1$ and $g_i \in \mathbf{m}_n$. Then by the equivalence of deformations we get a miniversal deformation of $g \in \mathbf{m}_n^2$ (with minimal number of deformation parameters u), i.e.

$$G(x,u) = \sum_{i=1}^{\mu-1} u_i g_i(x) + g(x).$$

Now we have the following

PROPOSITION 3.1. *If* $\xi \in \Xi_{Q_\Lambda(G)}$ *then* ξ *is* π_r*-liftable*, *i.e. there exists a germ of a holomorphic vector field* $\widetilde{\xi}$ *on* $\mathbb{C}^n \times \mathbb{C}^r$ *which is tangent to* $\Sigma_\Lambda(G)$ *at* 0 *and*

$$\xi \circ \pi_r = d\pi_r \circ \widetilde{\xi}.$$

Proof. We see that ξ lifts by π_r at every point $u \in \mathbb{C}^r$ outside $\pi_r(\mathrm{Sing}(\Sigma_\Lambda(G))) \cup \bigcup_{\alpha\in I} \Gamma_r^\alpha$ to a holomorphic vector field $\widetilde{\xi}'$ on $\mathbb{C}^n \times \mathbb{C}^r$ tangent to $\Sigma_\Lambda(G)$ and defined off a set of codimension 2 in $\mathbb{C}^n \times \mathbb{C}^r$, namely

$$\mathbb{C}^n \times \pi_r(\mathrm{Sing}(\Sigma_\Lambda(G))) \cup \bigcup_{\alpha\in I} \Gamma_r^\alpha.$$

By Hartog's extension theorem [9], $\widetilde{\xi}'$ extends to a holomorphic vector field $\widetilde{\xi}$ tangent to $\Sigma_\Lambda(G)$. ∎

Now following the methods introduced in [3, 14] we give an algorithm for construction of the module $\Xi_{Q_\Lambda(G)}$ of vector fields for versal G. This algorithm is a generalization of a similar one constructed in [7] for vector fields tangent to the usual bifurcation varieties.

By Proposition 3.1, to obtain elements of $\Xi_{Q_\Lambda(G)}$ we have to construct all π_r-lowerable vector fields $\widetilde{\xi}$ tangent to $\Sigma_\Lambda(G)$.

Now we define the ideal

$$J_{\Sigma_\Lambda(G)} = \left\langle \frac{\partial G}{\partial x_1}(x,u), \ldots, \frac{\partial G}{\partial x_n}(x,u), F(x) \right\rangle \mathcal{O}_{n+r}.$$

Then the germ of the vector field

$$\widetilde{\xi} = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{r} \gamma_j \frac{\partial}{\partial u_j}, \qquad \beta_i, \gamma_j \in \mathcal{O}_{n+r},$$

at $0 \in \mathbb{C}^n \times \mathbb{C}^r$, which is tangent to $\Sigma_\Lambda(G)$, has the property

$$\widetilde{\xi}\left(\frac{\partial G}{\partial x_i}(x,u)\right) \in J_{\Sigma_\Lambda(G)}, \quad i = 1, \ldots, n, \tag{1}$$

$$\widetilde{\xi}(F(x)) \in J_{\Sigma_\Lambda(G)}. \tag{2}$$

Lemma 3.2. *Let*

$$\xi = \sum_{i=1}^{r} \alpha_i(u) \frac{\partial}{\partial u_i}, \qquad \xi \in \Xi_{Q_\Lambda(G)}.$$

The vector field $\widetilde{\xi} \in \Xi_{\Sigma_\Lambda(G)}$ *is a lifting of* ξ *if and only if for some* $\beta_i \in \mathcal{O}_{n+r}$ *and* $v_i \in \Xi_\Lambda$, $i = 1, \ldots, n$, *we have*

$$\sum_{j=1}^{n} \beta_j v_j \left(\frac{\partial G}{\partial x_i}(x,u) \right) + \sum_{j=1}^{\mu-1} \alpha_j(u) \frac{\partial g_j}{\partial x_i} \in J_{\Sigma_\Lambda(G)}, \tag{3}$$

where G *is* $\mathcal{G}_\Lambda$*-versal*,

$$G(x,u) = \sum_{i=1}^{\mu-1} u_i g_i(x) + g(x).$$

Proof. By straightforward check of the conditions (1) and (2). ∎

Now we use the arguments working for the bifurcation and discriminant sets. Consider the ideal

$$\widetilde{\Delta}_\Lambda(G) = \langle v_i.G \rangle \mathcal{O}_{n+r}$$

in $\mathcal{O}_{n+r}$, where v_i are generators of Ξ_Λ. Since G is $\mathcal{G}_\Lambda$-versal, by the preparation theorem the quotient module

$$A = \mathcal{O}_{n+r} / \widetilde{\Delta}_\Lambda(G)$$

is a free $\mathcal{O}_r$-module generated by $1, g_1, \ldots, g_{\mu-1}$. In fact, take $\pi(x,u) \to u$, and look on A as an $\mathcal{O}_{n+r}$-module. Then A is a finite $\mathcal{O}_r$-module if and only if $A/(\pi^* \mathbf{m}_r)A$ is finite over $\mathbb{C}$. We see that

$$\begin{aligned} A/(\pi^* \mathbf{m}_r)A &\cong \mathcal{O}_{n+r} / (\langle v_i.G \rangle + \mathbf{m}_r \mathcal{O}_{n+r}) \\ &\cong \mathcal{O}_n / \langle v_i.G(x,0) \rangle \mathcal{O}_n \cong \{1, g_1, \ldots, g_{\mu-1}\}_{\mathbb{C}}. \end{aligned}$$

Thus for any $h \in \mathcal{O}_{n+r}$ we can write

$$h(x,u) = \sum_{i=1}^{n} \beta_i(x,u)(v_i.G)(x,u) + \sum_{j=1}^{\mu-1} \alpha_j(u) g_j(x) + \alpha(u) \tag{4}$$

for some $\beta_i \in \mathcal{O}_{n+r}, \alpha_i \in \mathcal{O}_r$ and $\alpha \in \mathcal{O}_r$.

Now we have the basic result.

Theorem 3.3. *Let* $h \in \mathcal{O}_{n+r}$ *and suppose that*

$$\frac{\partial h}{\partial x_i}(x,u) \in J_{\Sigma_\Lambda(G)}, \qquad i = 1, \ldots, n.$$

Then the vector field

$$\xi = \sum_{i=1}^{r} \alpha_i(u) \frac{\partial}{\partial u_i},$$

where α_i, $1 \le i \le \mu - 1$, are defined in (4) and α_i, $i \ge \mu$, are arbitrary holomorphic functions from $\mathcal{O}_r$, is tangent to the Λ-bifurcation variety of the family G.

Proof. Take h in the form (4). For derivatives of h we have

$$\frac{\partial h}{\partial x_i}(x,u) = \sum_{j=1}^{n} \frac{\partial \beta_j}{\partial x_i}(v_j.G) + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_i}(v_j.G) + \sum_{j=1}^{\mu-1} \alpha_j(u) \frac{\partial g_j}{\partial x_i}(x)$$

and by assumptions this belongs to $J_{\Sigma_\Lambda(G)}$. We also have

$$\sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_i}(v_j.G) = \sum_{j=1}^{n} \beta_j v_j \left(\frac{\partial G}{\partial x_i}\right) \operatorname{mod}(J_{\Sigma_\Lambda(G)}).$$

So by Lemma 3.2 we obtain the lifting formula (3) for the vector field $\xi = \sum_{i=1}^{r} \alpha_i \partial/\partial u_i$, which is tangent to $Q_\Lambda(G)$. ■

One can also obtain the converse, which results immediately from the proof of Theorem 3.3.

COROLLARY 3.4. *Let $\xi = \sum_{i=1}^{r} \alpha_i(u)\partial/\partial u_i$ be a tangent vector field to $Q_\Lambda(G)$. Then for some $h \in \mathcal{O}_{n+r}$,*

$$h = \sum_{i=1}^{n} \beta_i(v_i G) + \sum_{j=1}^{\mu-1} \alpha_j g_j + \alpha, \tag{5}$$

where $\beta_i \in \mathcal{O}_{n+r}$, $\alpha \in \mathcal{O}_r$ and $\partial h/\partial x_i \in J_{\Sigma_\Lambda(G)}$.

Proof. Take h in the form (5), where

$$\sum_{i=1}^{n} \beta_i v_i + \sum_{j=1}^{\mu-1} \alpha_j \frac{\partial}{\partial u_j} \in \Xi_{\Sigma_\Lambda(G)}.$$

Then by a simple check we find that $\partial h/\partial x_i \in J_{\Sigma_\Lambda(G)}$. ■

One can easily check that the space of germs $h \in \mathcal{O}_{n+r}$ such that $\partial h/\partial x_i(x,u) \in J_{\Sigma_\Lambda(G)}$, $i = 1, \ldots, n$, is an $\mathcal{O}_r$-module, which we denote by $\mathcal{H}_G$.

4. An algorithm. Now we present an algorithm which is useful in obtaining all tangent vector fields to $Q_\Lambda(G)$. We see that

$$\langle F \rangle J_{\Sigma_\Lambda(G)} + \widetilde{\Delta}_\Lambda^2(G) \subset \mathcal{H}_G.$$

Since $\Delta_\Lambda(g)$ contains some power of the maximal ideal $\mathbf{m}_n$, also the space

$$\frac{\mathcal{O}_n}{\Delta_\Lambda^2(g) + \langle F \rangle J_\Lambda(g)}, \qquad J_\Lambda(g) = \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}, F(x) \right\rangle,$$

is finite-dimensional with $\mathbb{C}$-basis, say, $\{f_1, \ldots, f_N\}$.

By the preparation theorem $\{f_i\}_{i=1}^N$ also generates

$$\frac{\mathcal{O}_{n+r}}{\widetilde{\Delta}_\Lambda^2(G)+\langle F\rangle J_{\Sigma_\Lambda(G)}}$$

as an $\mathcal{O}_r$-module.

Now any element $h\in\mathcal{H}_G$ can be written in the form

$$\begin{aligned}h(x,u)= &\sum_{i=1}^N \phi_i(u)f_i(x)+\sum_{i,j=1}^n \beta_{i,j}(x,u)\frac{\partial G}{\partial x_i}(x,u)\frac{\partial G}{\partial x_j}(x,u)\\ &+\sum_{i=1}^n \gamma_i(x,u)\frac{\partial G}{\partial x_i}(x,u)F(x)+\gamma_0(x,u)F(x)^2,\end{aligned}$$

where $\beta_{i,j},\gamma_i,\gamma_0\in\mathcal{O}_{n+r}$ and we seek elements $\phi_i\in\mathcal{O}_r$ such that

$$\sum_{i=1}^N \phi_i(u)\frac{\partial f_i}{\partial x_j}\in J_{\Sigma_\Lambda(G)},\qquad 1\le j\le n.$$

We show how to work with this approach and algorithm in several concrete cases.

4.1. Let $\Lambda=\{(y,x)\in\mathbb{C}^{n+1}: y=0\}$, $x=(x_1,\ldots,x_n)$. Then for some $g\in\mathcal{O}_{n+1}$ and the versal unfolding G of g we have

$$\begin{aligned}\Delta_\Lambda(g)&=\left\langle y\frac{\partial g}{\partial y},\frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n}\right\rangle\mathcal{O}_{n+1},\\ \widetilde{\Delta}_\Lambda(G)&=\left\langle y\frac{\partial G}{\partial y},\frac{\partial G}{\partial x_1},\ldots,\frac{\partial G}{\partial x_n}\right\rangle\mathcal{O}_{n+1+r},\\ J_{\Sigma_\Lambda(G)}&=\left\langle \frac{\partial G}{\partial y},\frac{\partial G}{\partial x_1},\ldots,\frac{\partial G}{\partial x_n},y\right\rangle\mathcal{O}_{n+1+r}.\end{aligned}$$

As an example we take the simplest nontrivial case of type F_4 (cf. [7]):

$$g(y,x)=y^2+x^3.$$

Then $G(y,x,u)=y^2+x^3+u_1xy+u_2y+u_3x$ and

$$\begin{aligned}\widetilde{\Delta}_\Lambda^2(G)&=\langle 2y^2+yu_2+u_1xy,3x^2+u_1y+u_3\rangle,\\ J_{\Sigma_\Lambda(G)}&=\langle u_2+u_1x,3x^2+u_3,y\rangle,\end{aligned}$$

and also the quotient space

$$\frac{\mathcal{O}_{2+3}}{\widetilde{\Delta}_\Lambda^2(G)+\langle y\rangle J_{\Sigma_\Lambda(G)}}$$

is generated by $\{1,x,y,x^2,x^3,xy\}$ as an $\mathcal{O}_3$-module.

We see that the functions

$$h(y,x,u)=\alpha_1(u)+\alpha_2(u)x+\alpha_3(u)x^2+\alpha_4(u)x^3+\alpha_5(u)y+\alpha_6xy+\psi(y,x,u)$$

with

$$\alpha_5(u) + \alpha_6(u)x \in J_{\Sigma_\Lambda(G)},$$
$$\alpha_2(u) + 2\alpha_3(u)x + 3\alpha_4(u)x^2 \in J_{\Sigma_\Lambda(G)},$$
$$\psi \in \widetilde{\Delta}^2_\Lambda(G) + \langle y\rangle J_{\Sigma_\Lambda(G)}$$

form the space $\mathcal{H}_G$.

Now it is easy to calculate the basis of vector fields tangent to $Q_\Lambda(G)$ (cf. [7]):

$$V_1 = -u_1^2\frac{\partial}{\partial u_2} + 6u_2\frac{\partial}{\partial u_3},$$
$$V_2 = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2},$$
$$V_3 = -u_1\frac{\partial}{\partial u_1} + 2u_3\frac{\partial}{\partial u_3},$$
$$V_4 = 3u_2\frac{\partial}{\partial u_1} - u_1u_3\frac{\partial}{\partial u_2},$$

which satisfy the relation $-u_1V_4 + u_3V_1 - 3u_2V_3 = 0$.

4.2. In the case of Λ singular our algorithm leads to quite complicated calculations. We show only some steps of the procedure which make clear the differences with the nonsingular case.

Let $\Lambda = \{(x, y) \in \mathbb{C}^2 : F(x, y) = x^3 - y^2 = 0\}$. The module Ξ_Λ of vector fields tangent to Λ is generated by

$$\xi_1 = 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}, \qquad \xi_2 = 3x^2\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}.$$

We consider the simplest non-Morse function $g(x, y) = x^3 + y^2$. Its Jacobi ideal is

$$\Delta_\Lambda(g) = \langle x^2y, x^3 + y^2\rangle$$

and a versal deformation is

$$G(x, y, u) = x^3 + y^2 + u_1xy^2 + u_2xy + u_3x^2 + u_4y^2 + u_5x + u_6y.$$

The corresponding Λ-bifurcation variety $Q_\Lambda(G)$ is described by the equations

$$3x^2 + u_1y^2 + u_2y + 2u_3x + u_5 = 0,$$
$$2y + 2u_1xy + u_2x + 2u_4y + u_6 = 0$$

together with $x^3 - y^2 = 0$.

The quotient space

$$\frac{\mathcal{O}_{2+6}}{\widetilde{\Delta}_\Lambda(G) + \langle x^3 - y^2\rangle J_{\Sigma_\Lambda(G)}}$$

is generated by $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^2y^2, xy^3, y^4\}$ as an $\mathcal{O}_6$-module. The functions

$$\begin{aligned}h(x, y, u) = \alpha_0(u) + \alpha_1(u)x + \alpha_2(u)y + \alpha_3(u)x^2 + \alpha_4(u)xy + \alpha_5(u)y^2 \\ + \alpha_6(u)x^3 + \alpha_7(u)x^2y + \alpha_8(u)xy^2 + \alpha_9(u)y^3 + \alpha_{10}(u)x^2y^2 \\ + \alpha_{11}(u)xy^3 + \alpha_{12}(u)y^4 + \psi(x, y, u)\end{aligned}$$

with

$$\begin{aligned}&\alpha_1(u) + 2\alpha_3(u)x + \alpha_4(u)y + 3\alpha_6(u)x^2 + 2\alpha_7(u)xy \\ &\qquad + \alpha_8(u)y^2 + 2\alpha_{10}(u)xy^2 + +\alpha_{11}(u)y^3 \in J_{\Sigma_\Lambda(G)}, \\ &\alpha_2(u) + \alpha_4(u)x + 2\alpha_5(u)y + \alpha_7(u)x^2 + 2\alpha_8(u)xy \\ &\qquad + 3\alpha_9(u)y^2 + 2\alpha_{10}(u)x^2y + 3\alpha_{11}(u)xy^2 + 4\alpha_{12}(u)y^3 \in J_{\Sigma_\Lambda(G)}, \\ &\qquad \psi \in \widetilde{\Delta}^2_\Lambda(G) + \langle x^3 - y^2\rangle J_{\Sigma_\Lambda(G)}\end{aligned}$$

form the space $\mathcal{H}_G$.

Remarks. 1. If g is a Morse singularity on Λ singular then the Λ-bifurcation variety $Q_\Lambda(G)$ is diffeomorphic to the product $\Lambda \times \mathbb{C}^k$ for some $k \in \mathbb{N} \cup \{0\}$.

2. Let $G(x, u)$ be a germ of a holomorphic family of functions. Let $\Lambda_0 \subset \mathbb{C}^n$ be a germ of a complex space. We consider a deformation of Λ_0, i.e. a family of varieties $\pi : \widetilde{\Lambda} \to \mathbb{C}^r$ with $\pi^{-1}(0) = \Lambda_0$. As a natural generalization of a Λ-bifurcation variety of G we have the $\widetilde{\Lambda}$-bifurcation variety of G defined by

$$Q_{\widetilde{\Lambda}}(G) = \left\{(x, u) \in \mathbb{C}^n \times \mathbb{C}^r : \frac{\partial G}{\partial x_i}(x, u) = 0,\ (x, u) \in \pi^{-1}(u),\ i = 1, \ldots, n\right\}.$$

If $\widetilde{\Lambda}$ is the versal deformation of Λ_0 (cf. [8, 5]) we may use the normal forms of $\widetilde{\Lambda}$ to consider the parametrized groups (deformations of groups) $u \to \mathcal{G}_{\widetilde{\Lambda}_u}$, $\widetilde{\Lambda}_u = \pi^{-1}(u)$ acting on families G. In case of families of hypersurfaces, $\widetilde{\Lambda}$ is given by the holomorphic function $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$, $\Lambda_u = \{x \in \mathbb{C}^n : F(\cdot, u) = 0\}$. So the classification problem of $\widetilde{\Lambda}$-varieties is reduced to the classification of map-germs $(F, G) : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}^2$ with right and modified left equivalences (cf. [13]).

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