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Pseudo orbit tracing property and fixed points

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Abstract. If a continuous map f of a compact metric space has the pseudo orbit tracing property and is h-expansive then the set of all fixed points of f is totally disconnected.

The pseudo orbit tracing property is one of the most important notions in dynamical systems. The observation of periodic points is the first step to know the orbit structure of a system. In this paper we investigate the sets of all fixed points of continuous maps with the pseudo orbit tracing property. Our result gives a partial positive answer to the following question by Morimoto [8]:

Is the set of all fixed points of a C^1 diffeomorphism of a compact manifold totally disconnected if the diffeomorphism has the pseudo orbit tracing property?

We show in this paper that the set of all fixed points of a continuous map which has the pseudo orbit tracing property is totally disconnected under a supplementary condition of expansiveness with respect to entropy.

Let X be a compact metric space with metric d and $f : X \to X$ be a continuous map. For $\delta > 0$ a sequence $\{x_i\}$ of points in X is said to be a *δ-pseudo orbit* if $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, 2, \ldots$ We say that f has the *pseudo orbit tracing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}$ there is $y \in X$ satisfying $d(f^i(y), x_i) < \varepsilon$ for any $i \geq 0$. The y is said to be an *ε-tracing point* of $\{x_i\}$. For $\varepsilon > 0$ and $x \in X$ we denote by $\Phi_\varepsilon(x)$ the set $\bigcap_{n\geq 0} f^{-n}B_\varepsilon(f^n(x))$, where $B_\varepsilon(x) = \{y \in X : d(y, x) \leq \varepsilon\}$. We say that a subset E of X *(n, δ)-spans* a subset K if for each $y \in K$ there is $x \in E$ such that $d(f^i(x), f^i(y)) \leq \delta$ for all $i \in [0, n)$. Let $\gamma_n(\delta, K)$ be the minimum cardinality of sets which (n, δ)-span K. Since $\Phi_\varepsilon(x)$ is compact, $\gamma_n(\delta, \Phi_\varepsilon(x))$ is finite. Define

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$$\overline{\gamma}_f(\delta,\Phi_\varepsilon(x))=\limsup_{n\to\infty}\frac{1}{n}\log\gamma_n(\delta,\Phi_\varepsilon(x)),$$
$$h(f,\Phi_\varepsilon(x))=\lim_{\delta\to0}\overline{\gamma}_f(\delta,\Phi_\varepsilon(x)).$$

Then $\overline{\gamma}_f(\delta,\Phi_\varepsilon(x))$ increases as δ decreases. We denote by $\mathrm{Fix}(f)$ the set $\{x\in X : f(x)=x\}$. Our result is as follows.

THEOREM. *Let f have the pseudo orbit tracing property. If for any $x\in\mathrm{Fix}(f)$ there is $\varepsilon>0$ with $h(f,\Phi_\varepsilon(x))=0$, then $\mathrm{Fix}(f)$ is totally disconnected.*

A continuous map $f:X\to X$ is called *h-expansive* if there exists $\varepsilon>0$ such that $h(f,\Phi_\varepsilon(x))=0$ for any $x\in X$ (cf. [2]).

COROLLARY. *If an h-expansive map f has the pseudo orbit tracing property, then $\mathrm{Fix}(f)$ is totally disconnected.*

Misiurewicz constructed a C^1 diffeomorphism which is not h-expansive (cf. [7]). Thus Morimoto's question is still open for C^1 diffeomorphisms.

To prove the Theorem we need two lemmas. For $x,y\in X$, a δ-chain from x to y is a finite δ-pseudo orbit $\{x_0,x_1,\ldots,x_m\}$ with $x_0=x$, $x_m=y$. The number $m+1$ is called the *length* of the chain.

LEMMA 1. *If a closed subset C of X is connected, then for any $\alpha>0$ there is a positive integer m such that for any $x,y\in C$ there is an α-chain from x to y with length at most $m+1$.*

Proof. Let $U_{\alpha/2}(x)=\{y\in X : d(y,x)<\alpha/2\}$. Since C is compact, we can find $x_1,\ldots,x_k\in C$ such that $\bigcup_{i=1}^k U_{\alpha/2}(x_i)\supset C$. Put $m=k+1$. Since C is connected, for any $p,q\in\{x_1,\ldots,x_k\}$ we take $x_{i_0},\ldots,x_{i_l}\in\{x_1,\ldots,x_k\}$ with $x_{i_0}=p, x_{i_l}=q$ such that $U_{\alpha/2}(x_{i_j})\cap U_{\alpha/2}(x_{i_{j+1}})\neq\emptyset$ for $j=0,1,\ldots,l-1$. Thus $\{p,x_{i_1},\ldots,x_{i_{l-1}},q\}$ is an α-chain in C. We can assume that $x_{i_s}\neq x_{i_t}$ for $s\neq t$ (otherwise we replace the α-chain by one with smaller l). Hence the length of $\{p,x_{i_1},\ldots,x_{i_{l-1}},q\}$ is at most k. For any $x,y\in C$, we can find $p,q\in C$ such that $U_{\alpha/2}(p)\ni x$, $U_{\alpha/2}(q)\ni y$ and so the α-chain $\{x,p,x_{i_1},\ldots,x_{i_{l-1}},q,y\}$ has length at most $k+2=m+1$.

The following is an immediate fact from general topology.

LEMMA 2. *Let F be a closed subset of X. If for each $x\in F$ there is $\varepsilon>0$ such that $B_\varepsilon(x)\cap F$ is totally disconnected, then so is F.*

The proof is omitted. See [6] for the details.

Proof of Theorem. Put $F=\mathrm{Fix}(f)$. Since F is a closed subset of X, by Lemma 2 it is sufficient to show that for any $x\in F$ there is $\varepsilon>0$ such that $B_{\varepsilon/2}(x)\cap F$ is totally disconnected.

Let $x \in F$ and assume that $h(f, \Phi_\varepsilon(x)) = 0$ for some $\varepsilon > 0$. Let C denote the connected component of $B_{\varepsilon/2}(x) \cap F$ containing x and assume that C is not a one-point set. Then C is not finite and so for any positive integer a we can take $p_1, \ldots, p_a \in C$ ($p_i \neq p_j$ if $i \neq j$). Take ε_0 with $0 < \varepsilon_0 < \min\{d(p_i, p_j) : 1 \leq i \neq j \leq a\}$. Since f has the pseudo orbit tracing property, for any μ with $0 < \mu < \min\{\varepsilon_0/6, \varepsilon/2\}$ there is α with $0 < \alpha < \min\{\varepsilon_0/6, \varepsilon/2\}$ such that any α-pseudo orbit of f is μ-traced by a point $y \in X$. By Lemma 1 we can find a positive integer m such that for any $\overline{x}, \overline{y} \in C$ there is an α-chain $\{x_0, \ldots, x_m\} \subset C$ from $\overline{x}$ to $\overline{y}$ with length at most $m + 1$.

Take an n-tuple $(z_0, \ldots, z_{n-1})$ such that $z_j = p_{i_j} \in \{p_1, \ldots, p_a\}$ for $j = 0, \ldots, n-1$. Then for each consecutive pair z_i, z_{i+1} ($i = 0, \ldots, n-2$) we can find an α-chain in C from z_i to z_{i+1} with length at most $m + 1$. Since C is connected the α-chains may be assumed to have length $m + 1$ by adding some points of C if necessary (cf. [9]). Hence if $\{x_0^{(i)}, x_1^{(i)}, \ldots, x_m^{(i)}\}$ denotes the α-chain for the pair z_i, z_{i+1}, where $x_0^{(i)} = z_i$ and $x_m^{(i)} = z_{i+1}$, then $\{x_0^{(0)}, \ldots, x_{m-1}^{(0)}, x_0^{(1)}, x_1^{(1)}, \ldots, x_{m-1}^{(1)}, x_0^{(2)}, \ldots, x_{m-1}^{(n-2)}, x_m^{(n-2)}, \ldots\}$ is an α-pseudo orbit of f such that $x_m^{(k)} = x_0^{(k+1)}$ for $k = 0, 1, \ldots, n-3$. Since f has the pseudo orbit tracing property there is $u \in X$ such that

$$d(f^{lm+j}(u), x_j^{(l)}) < \mu \tag{1}$$

for $l = 0, \ldots, n-2$ and $j = 0, \ldots, m$. Thus $u \in \Phi_\varepsilon(x)$ because $x_j^l \in C \subset B_{\varepsilon/2}(x)$ and $\mu < \varepsilon/2$. Take $0 < \delta < \varepsilon_0/6$ and an (mn, δ)-spanning set E for $\Phi_\varepsilon(x)$. Then we can find $\overline{u} \in E$ such that

$$d(f^k(\overline{u}), f^k(u)) \leq \delta \tag{2}$$

for $k = 0, 1, \ldots, mn-1$. If we take another n-tuple $(z'_0, \ldots, z'_{n-1})$ from the set $\{p_1, \ldots, p_a\}$, then there are $v \in \Phi_\varepsilon(x)$ and $\overline{v} \in E$ such that

$$d(f^{lm+j}(v), x_j'^{(l)}) < \mu \tag{3}$$

for $l = 0, \ldots, n-2$ and $j = 0, \ldots, m$, and

$$d(f^k(\overline{v}), f^k(v)) \leq \delta \tag{4}$$

for $k = 0, 1, \ldots, mn-1$. We have $u \neq v$. For if $(z_0, z_1, \ldots, z_{n-1}) \neq (z'_0, z'_1, \ldots \ldots, z'_{n-1})$, then $z_l \neq z'_l$ for some l. Thus

$$\begin{aligned}\varepsilon_0 < d(z_l, z'_l) &\leq d(z_l, f^{lm}(u)) + d(f^{lm}(u), f^{lm}(v)) + d(f^{lm}(v), z'_l) \\ &< \mu + d(f^{lm}(u), f^{lm}(v)) + \mu.\end{aligned}$$

Hence $d(f^{lm}(u), f^{lm}(v)) > \varepsilon_0 - 2\mu > 2\varepsilon_0/3 > 0$ and so $u \neq v$. We also have $\overline{u} \neq \overline{v}$. Indeed, if $z_l \neq z'_l$, then $d(f^{lm}(u), z_l) < \mu$ and $d(f^{lm}(v), z'_l) < \mu$ by

(1) and (3). Hence

$$\begin{aligned}\varepsilon_0 &< d(z_l, z_l') \\ &\le d(z_l, f^{lm}(u)) + d(f^{lm}(u), f^{lm}(\overline{u})) + d(f^{lm}(\overline{u}), f^{lm}(\overline{v})) \\ &\quad + d(f^{lm}(\overline{v}), f^{lm}(v)) + d(f^{lm}(v), z_l') \\ &\le \mu + \delta + d(f^{lm}(\overline{u}), f^{lm}(\overline{v})) + \delta + \mu \\ &\le 2(\mu + \delta) + d(f^{lm}(\overline{u}), f^{lm}(\overline{v})) \\ &\le 2\varepsilon_0/3 + d(f^{lm}(\overline{u}), f^{lm}(\overline{v}))\end{aligned}$$

and so $d(f^{lm}(\overline{u}), f^{lm}(\overline{v})) > \varepsilon_0/3 > 0$. Hence $\overline{u} \ne \overline{v}$. Therefore we have $\mathrm{Card}(E) \ge a^n$, where $\mathrm{Card}(E)$ denotes the cardinality of the set E. Thus $\overline{\gamma}_f(\delta, \Phi_\varepsilon(x)) \ge \frac{1}{m}\log a$ and so $h(f, \Phi_\varepsilon(x)) \ge \frac{1}{m}\log a > 0$, which is a contradiction. Thus C is a one-point set and $\mathrm{Fix}(f)$ is totally disconnected.

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