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Approximation by nonlinear integral operators in some modular function spaces

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Abstract. Let G be a locally compact Hausdorff group with Haar measure, and let $L^0(G)$ be the space of extended real-valued measurable functions on G, finite a.e. Let ϱ and η be modulars on $L^0(G)$. The error of approximation $\varrho(a(Tf-f))$ of a function $f\in(L^0(G))_{\varrho+\eta}\cap\mathrm{Dom}\,T$ is estimated, where $(Tf)(s)=\int_G K(t-s,f(t))\,dt$ and K satisfies a generalized Lipschitz condition with respect to the second variable.

1. Let G be a locally compact Hausdorff group with neutral element θ and with the family $\mathcal{U}$ of open neighbourhoods of θ in G. For the sake of simplicity from now on we will assume G to be abelian. Let Σ be the Borel σ-field of G, let $|A|$ be the Haar measure of a measurable set $A\subset G$ and let $\int_G f(t)\,dt$ denote the Haar integral of f.

We shall denote by $M^0(G)$ the space of all extended real-valued measurable functions $f:G\to\overline{\mathbb{R}}=\mathbb{R}\cup\{\mp\infty\}$, and by $L^0(G)\subset M^0(G)$ its subspace of functions f finite almost everywhere (a.e.), both provided with equality a.e.

Let $\varrho:L^0(G)\to\overline{\mathbb{R}}_0^+$ and $\eta:L^0(G)\to\overline{\mathbb{R}}_0^+$ be two modulars in $L^0(G)$, and let $(L^0(G))_\varrho$ and $(L^0(G))_\eta$ be the respective modular spaces (for terminology, see e.g. [4]).

We make the following assumptions:

1° ϱ and η are *monotone*, i.e. if $f,g\in L^0(G)$ and $|f|\le|g|$, then $\varrho(f)\le\varrho(g)$ and $\eta(f)\le\eta(g)$;

2° ϱ is *$\mathcal{J}$-convex*, i.e. for any two measurable functions $p:G\to\mathbb{R}_0^+$ and

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$F : G \times G \to \mathbb{R}$ with $\int_G p(t)\,dt = 1$,

$$\varrho\Big(\int\limits_G p(t)|F(t,\cdot)|\,dt\Big) \le \int\limits_G p(t)\varrho(F(t,\cdot))\,dt;$$

3° η is *τ-bounded*, i.e. there are a number $c \ge 1$ and a measurable, bounded function $h : G \to \mathbb{R}_0^+$ such that $h(t) \to 0$ as $t \to \theta$ and

$$\eta(f(t+\cdot)) \le \eta(cf) + h(t), \qquad t \in G,$$

for all $f \in L^0(G)$ such that $\eta(f) < \infty$; we shall write $h_0 = \sup_{t\in G} h(t)$.

We may extend both modulars ϱ and η to $M^0(G)$, putting $\varrho(f) = \eta(f) = \infty$ for $f \in M^0(G) \setminus L^0(G)$.

Let $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be such that for all $t \in G$, the function $\psi(t,\cdot)$ is continuous and nondecreasing for $u \ge 0$, $\psi(t,0) = 0$, $\psi(t,u) > 0$ for $u > 0$, $\psi(t,u) \to \infty$ as $u \to \infty$, and such that for every $u \ge 0$, $\psi(t,u)$ is a measurable function of t.

The following connection between both modulars ϱ and η and the function ψ will be assumed:

(I) there is a set $G_0 \subset G$ with $|G \setminus G_0| = 0$ such that for every $\lambda \in \,]0,1[$ there exists a $C_\lambda \in \,]0,1[$ satisfying the inequality

$$\varrho[C_\lambda \psi(t, |F(\cdot)|)] \le \eta(\lambda F(\cdot))$$

for all $t \in G_0$ and $F \in L^0(G)$.

A condition of this type was introduced in special cases in [3].

Let us still remark that we may choose C_λ in such a manner that $C_\lambda \searrow 0$ as $\lambda \searrow 0$. Condition (I) implies immediately the following inequality:

$$\varrho[C_\lambda \psi(t, F_t(\cdot))] \le \eta(\lambda F_t(\cdot))$$

for every $t \in G_0$ and for any family $(F_t(\cdot))_{t\in G}$ of functions $F_t \in L^0(G)$.

A function $K : G \times \mathbb{R} \to \mathbb{R}$ will be called a *kernel function* if $K(t,0) = 0$ for $t \in G$ and $K(\cdot,u) \in L^1(G)$ for all $u \in \mathbb{R}$. Let $L : G \to \mathbb{R}_0^+, L \in L^1(G)$. We say that a kernel function K satisfies the (L,ψ)-*Lipschitz condition* if

$$|K(t,u) - K(t,v)| \le L(t)\psi(t, |u-v|)$$

for $t \in G, u, v \in \mathbb{R}$ (see [1], p. 10).

In the following we shall write $L = \int_G L(t)\,dt$, $p(t) = L(t)/L$.

Let us remark that if K is an (L,ψ)-Lipschitz kernel function and $f \in L^0(G)$, then the superposition $K(t, f(t+s))$ is a measurable function of $t \in G$ for all $s \in G$.

2. Following [1–4] we shall deal with nonlinear integral operators T of the form

$$(Tf)(s) = \int_G K(t-s, f(t))\,dt = \int_G K(t, f(t+s))\,dt.$$

We denote by $\operatorname{Dom} T$ the set of all functions $f \in L^0(G)$ such that $(Tf)(s)$ exists for a.e. $s \in G$ and Tf is a measurable function on G.

PROPOSITION 1. *Let* $f \in (L^0(G))_\eta \cap \operatorname{Dom} T$ *and let* $\lambda \in \,]0,1[$ *be so small that* $\eta(c\lambda f) < \infty$, *where* $c \geq 1$ *is the constant from* 3°. *Suppose that* K *is an* (L,ψ)*-Lipschitz kernel function and the condition* (I) *is satisfied. Then, for every* $\varepsilon > 0$, *there exists a* $U \in \mathcal{U}$ *such that*

$$\varrho\left(\frac{C_\lambda}{L} Tf\right) \leq \eta(c\lambda f) + h_0 \int_{G\setminus U} p(t)\,dt + \varepsilon.$$

Consequently, $\varrho((C_\lambda/L)Tf) < \infty$.

P r o o f. Applying monotonicity of ϱ, the (L,ψ)-Lipschitz condition, $\mathcal{J}$-convexity of ϱ and the condition (I), we obtain

$$\varrho\left(\frac{C_\lambda}{L} Tf\right) \leq \varrho\left(\frac{C_\lambda}{L} \int_G |K(t, f(t+\cdot)|)\,dt\right) \leq \varrho\left(\int_G p(t) C_\lambda \psi(t, |f(t+\cdot)|)\,dt\right)$$

$$\leq \int_G p(t)\varrho[C_\lambda \psi(t, |f(t+\cdot)|)]\,dt \leq \int_G p(t)\eta(\lambda |f(t+\cdot)|)\,dt.$$

Since $\eta(\lambda f) < \infty$, by τ-boundedness of η, we get $\eta(\lambda|f(t+\cdot)|) \leq \eta(c\lambda f) + h(t)$ for $t \in G$. Consequently, since $\int_G p(t)\,dt = 1$, we obtain

$$\varrho\left(\frac{C_\lambda}{L} Tf\right) \leq \eta(c\lambda f) + \int_G p(t)h(t)\,dt. \tag{1}$$

However, since $h(t) \to 0$ as $t \to \theta$, for any $\varepsilon > 0$ there is a $U \in \mathcal{U}$ such that $h(t) < \varepsilon$ for $t \in U$. Since $h(t) \leq h_0$ for $t \in G \setminus U$, the required inequality follows from (1).

3. The map $\omega_\eta : L^0(G) \times \mathcal{U} \to \overline{\mathbb{R}}_0^+ = [0,\infty]$ defined by

$$\omega_\eta(f, U) = \sup_{t\in U} \eta(f(t+\cdot) - f(\cdot))$$

for $f \in L^0(G), U \in \mathcal{U}$, is called the η-*modulus of continuity* (see [4], p. 85). We shall apply the following notation:

$$r_k = \sup_{1/k \leq |u| \leq k} \left| \frac{1}{u} \int_G K(t,u)\,dt - 1 \right|, \qquad A_k = \{t \in G : |f(t)| > k\},$$

$$B_k = \{t \in G : |f(t)| < 1/k\}, \quad C_k = G \setminus (A_k \cup B_k), \quad f \in L^0(G),$$

$$k = 1, 2, \ldots, \quad \text{and} \quad r = \sup_k r_k.$$

We shall give an estimate of the modular error of approximation $\varrho(a(Tf-f))$ for sufficiently small $a>0$.

THEOREM 1. *Let* $f \in (L^0(G))_{\varrho+\eta} \cap \operatorname{Dom} T$. *Let* $\lambda \in]0,1[$ *and* $a \in]0, C_\lambda/(16L)[$ *be so small that* $\eta(2c\lambda f) < \infty$ *and* $\varrho(16af) < \infty$. *Then, for every* $U \in \mathcal{U}$, $k = 0,1,2,\ldots$ *and* $S \in \Sigma$, *we have*

$$\varrho(a(Tf-f)) \le \omega_\eta(\lambda f, U) + [2\eta(2c\lambda f) + h_0] \int\limits_{G\setminus U} p(t)\,dt + R_k, \tag{2}$$

where R_k *is given by*

$$\begin{aligned} R_0 &= \varrho(2arf),\\ R_k &= \eta(\lambda f \chi_{G\setminus S}) + \varrho(16af\chi_{G\setminus S}) + \eta(\lambda f \chi_{S\cap A_k})\\ &\quad + \varrho(16af\chi_{S\cap A_k}) + \eta(\lambda f\chi_{S\cap B_k}) + \varrho(16af\chi_{S\cap B_k})\\ &\quad + \varrho(8ar_k f), \qquad k = 1,2,\ldots \end{aligned}$$

Proof. We have $\varrho(a(Tf-f)) \le J_1 + J_2$, where

$$J_1 = \varrho\Big\{2a \int\limits_G |K(t, f(t+\cdot)) - K(t, f(\cdot))|\,dt\Big\},$$

$$J_2 = \varrho\Big\{2a\Big|\int\limits_G K(t, f(\cdot))\,dt - f(\cdot)\Big|\Big\}$$

(see [4], p. 88). By the (L,ψ)-Lipschitz condition, by $\mathcal{J}$-convexity of ϱ and taking into account the condition (I) we have

$$\begin{aligned} J_1 &\le \int\limits_G p(t)\varrho[C_\lambda \psi(t, |f(t+\cdot) - f(\cdot)|]\,dt\\ &\le \int\limits_U p(t)\eta(\lambda|f(t+\cdot) - f(\cdot)|)\,dt\\ &\quad + \int\limits_{G\setminus U} p(t)\eta(\lambda|f(t+\cdot)-f(\cdot)|)\,dt = J_1^1 + J_1^2. \end{aligned}$$

But

$$J_1^1 \le \int\limits_G p(t)\omega_\eta(\lambda f, U)\,dt \le \omega_\eta(\lambda f, U).$$

Now,

$$J_1^2 \le \int\limits_{G\setminus U} p(t)\eta(2\lambda f(t+\cdot))\,dt + \eta(2\lambda f)\int\limits_{G\setminus U} p(t)\,dt.$$

By τ-boundedness of η we obtain

$$\int\limits_{G\setminus U} p(t)\eta(2\lambda f(t+\cdot))\,dt \le [\eta(2c\lambda f) + h_0]\int\limits_{G\setminus U} p(t)\,dt$$

whence, by monotonicity of η,

$$J_1^2 \le [2\eta(2c\lambda f) + h_0] \int\limits_{G\setminus U} p(t)\,dt.$$

Consequently,

$$J_1 \le \omega_\eta(\lambda f, U) + [2\eta(2c\lambda f) + h_0] \int\limits_{G\setminus U} p(t)\,dt.$$

It remains to prove that $J_2 \le R_k$, $k = 0, 1, 2, \ldots$ For $k = 0$ this is obvious. Suppose $k > 0$. Then, taking any set $S \in \Sigma$, we have

$$\begin{aligned} J_2 \le\ & \varrho\Big\{8a\Big|\int\limits_G K(t, f(\cdot)\chi_{G\setminus S}(\cdot))\,dt - f(\cdot)\chi_{G\setminus S}(\cdot)\Big|\Big\} \\ & + \varrho\Big\{8a\Big|\int\limits_G K(t, f(\cdot)\chi_{S\cap A_k}(\cdot))\,dt - f(\cdot)\chi_{S\cap A_k}(\cdot)\Big|\Big\} \\ & + \varrho\Big\{8a\Big|\int\limits_G K(t, f(\cdot)\chi_{S\cap B_k}(\cdot))\,dt - f(\cdot)\chi_{S\cap B_k}(\cdot)\Big|\Big\} \\ & + \varrho\Big\{8a\Big|\int\limits_G K(t, f(\cdot)\chi_{S\cap C_k}(\cdot))\,dt - f(\cdot)\chi_{S\cap C_k}(\cdot)\Big|\Big\}. \end{aligned}$$

By (L,ψ)-Lipschitz condition, monotonicity and $\mathcal{J}$-convexity of ϱ and by condition (I), for every $P \in \Sigma$ we get

$$\begin{aligned} & \varrho\Big\{8a\Big|\int\limits_G K(t, f(\cdot)\chi_P(\cdot))\,dt - f(\cdot)\chi_P(\cdot)\Big|\Big\} \\ & \qquad \le \varrho\Big\{16a\int\limits_G |K(t, f(\cdot)\chi_P(\cdot))|\,dt\Big\} + \varrho(16af\chi_P) \\ & \qquad \le \int\limits_G p(t)\varrho[C_\lambda\psi(t, f(\cdot)\chi_P(\cdot))]\,dt + \varrho(16af\chi_P) \\ & \qquad \le \eta(\lambda f\chi_P) + \varrho(16af\chi_P). \end{aligned}$$

Thus, for $P = G \setminus S, P = S \cap A_k, P = S \cap B_k$ and by the definition of r_k, we obtain

$$\begin{aligned} J_2 \le\ & \eta(\lambda f\chi_{G\setminus S}) + \varrho(16af\chi_{G\setminus S}) + \eta(\lambda f\chi_{S\cap A_k}) + \varrho(16af\chi_{S\cap A_k}) \\ & + \eta(\lambda f\chi_{S\cap B_k}) + \varrho(16af\chi_{S\cap B_k}) + \varrho(8ar_k f) \end{aligned}$$

and so the assertion follows.

4. Let $\mathcal{W}$ be a nonempty, abstract set of indices and let $\mathbf{W}$ be a filter of subsets of $\mathcal{W}$.

A family $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ of kernel functions will be called a *kernel*.

Let $\mathbb{L} = (L_w)_{w\in\mathcal{W}}$ be a family of nonnegative functions $L_w \in L^1(G)$. We say that the kernel $\mathbb{K}$ satisfies the $(\mathbb{L}, \psi)$-*Lipschitz condition* if the kernel functions K_w satisfy the (L_w, ψ)-Lipschitz condition, and $L = \sup_{\mathcal{W}} L_w = \sup_{\mathcal{W}} \int_G L_w(t)\,dt < \infty$. Set $p_w(t) = L_w(t)/L_w$ (see [1], pp. 12–13). The kernel $\mathbb{K}$ will be called *singular* if for every $U \in \mathcal{U}$,

$$\int\limits_{G\setminus U} p_w(t)\,dt \xrightarrow{\mathbf{W}} 0$$

and

$$r_k(w) = \sup_{1/k\le|u|\le k} \left| \frac{1}{u} \int\limits_G K_w(t,u)\,dt - 1 \right| \xrightarrow{\mathbf{W}} 0$$

for $k = 1, 2, \ldots$ If, moreover,

$$r(w) = \sup_{k=1,2,\ldots} r_k(w) \xrightarrow{\mathbf{W}} 0,$$

the kernel $\mathbb{K}$ will be called *strongly singular*.

Let us define a family $\mathbb{T} = (T_w)_{w\in\mathcal{W}}$ of operators by

$$(T_w f)(s) = \int\limits_G K_w(t-s, f(t))\,dt = \int\limits_G K_w(t, f(t+s))\,dt.$$

Set $\operatorname{Dom}\mathbb{T} = \bigcap_{w\in\mathcal{W}} \operatorname{Dom} T_w$. We shall deduce from Theorem 1 a theorem on convergence $\varrho(a(T_w f - f)) \xrightarrow{\mathbf{W}} 0$ for small $a > 0$. We need some additional notions, namely of absolute finiteness and absolute continuity of modulars (see [4], p. 84, [2], p. 4).

DEFINITION 1. A modular η on $L^0(G)$ is called *finite* if for every measurable set $A \subset G$ such that $|A| < \infty$ we have $\chi_A \in (L^0(G))_\eta$.

DEFINITION 2. A modular η on $L^0(G)$ is called *absolutely finite* if it is finite and if for every $\varepsilon > 0$ and for every $\lambda_0 > 0$, there is a $\delta > 0$ such that $\eta(\lambda_0\chi_B) < \varepsilon$ for every measurable set $B \subset G$ of measure $|B| < \delta$.

DEFINITION 3. A modular η on $L^0(G)$ is called *absolutely continuous* (with respect to the measure in G) if there exists an $\alpha > 0$ such that for every $f \in L^0(G)$ with $\eta(f) < \infty$ the following two conditions are satisfied:

(α) for every $\varepsilon > 0$ there exists a measurable set $A \subset G$ such that $|A| < \infty$ and $\eta(\alpha f\chi_{G\setminus A}) < \varepsilon$;

(β) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\eta(\alpha f\chi_B) < \varepsilon$ for all measurable sets $B \subset G$ of measure $|B| < \delta$.

Let us remark that if η is monotone, τ-bounded, absolutely finite and absolutely continuous, then for every $f \in (L^0(G))_\eta$ there is a $\lambda_0 > 0$ such that for every $\varepsilon > 0$ there exists a $U_\varepsilon \in \mathcal{U}$ such that $\omega_\eta(\lambda_0 f, U_\varepsilon) < \varepsilon$ (see

[4], Theorem 1, p. 85; the condition (P) mentioned there is always satisfied, as was kindly shown to us by Prof. D. Candeloro).

THEOREM 2. *Let $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ be a singular kernel and let the modular ϱ be monotone and $\mathcal{J}$-convex, and the modular η-monotone, τ-bounded, absolutely finite and absolutely continuous. Let $f \in (L^0(G))_{\varrho+\eta} \cap \mathrm{Dom}\,\mathbb{T}$. Finally, let one of the following conditions hold*:

(a) $\mathbb{K}$ *is strongly singular*;

(b) ϱ *is finite and absolutely continuous.*

Then $\varrho(a(T_w f - f)) \xrightarrow{\mathbf{W}} 0$ for sufficiently small $a > 0$ (depending on f).

Proof. Choose an arbitrary $\varepsilon > 0$. Since η is monotone, τ-bounded, absolutely finite and absolutely continuous, there is a $U \in \mathcal{U}$ such that $\omega_\eta(\lambda_0 f, U) < \varepsilon/4$ for sufficiently small $\lambda_0 > 0$. Taking $\lambda_1 \in]0, \lambda_0[$ small enough, we get $\eta(2c\lambda_1 f) < \infty$. Due to singularity of $\mathbb{K}$, keeping the above $U \in \mathcal{U}$ fixed, we have $\int_{G\setminus U} p_w(t)\,dt \xrightarrow{\mathbf{W}} 0$. Hence there exists a $\mathcal{W}_1 \in \mathbf{W}$ such that

$$[2\eta(2c\lambda f) + h_0] \int\limits_{G\setminus U} p_w(t)\,dt < \varepsilon/4$$

for $\lambda \in]0, \lambda_1], w \in \mathcal{W}_1$ and the above $U \in \mathcal{U}$.

Thus, for a fixed $\lambda \in]0, \lambda_1[$ let C_λ be the corresponding constant in (I), and for $a \in]0, C_\lambda/(16L)[$, we have

$$\varrho(a(T_w f - f)) < \varepsilon/2 + R_k$$

for $w \in \mathcal{W}_1$ and $k = 0, 1, 2, \ldots$, where we have applied (2) with T_w and p_w in place of T and p.

Assuming that (a) holds, we apply (3) with $k = 0$, obtaining $R_0 = \varrho(2ar(w)f)$. However, since $f \in (L^0(G))_\varrho$, there is a $\mathcal{W}_2 \in \mathbf{W}$ such that $\varrho(2ar(w)f) < \varepsilon/2$ for $w \in \mathcal{W}_2$. This gives $\varrho(a(T_w f - f)) < \varepsilon$ for $w \in \mathcal{W}_1 \cap \mathcal{W}_2 \in \mathbf{W}$, which implies our assertion.

Now suppose (b). We apply Theorem 1 with a given $S \in \Sigma$ with $|S| < \infty$. Since $A_1 \supset A_2 \supset \ldots$, we have $S \cap A_1 \supset S \cap A_2 \supset \ldots$, and $|S \cap A_1| < \infty$. Hence $\lim_{k\to\infty} |S\cap A_k| = |S \cap \bigcap_{k=1}^\infty A_k|$. But $f \in L^0(G)$ whence there exists a set $G_0 \subset G, G_0 \in \Sigma, |G_0| = 0$, such that $|f(t)| < \infty$ for $t \in G \setminus G_0$. From the inclusion $\bigcap_{k=1}^\infty A_k \subset G_0$, we deduce $\lim_{k\to\infty} |S \cap A_k| = 0$.

Now applying absolute continuity of η and ϱ, we may choose λ and a so small that

$$\eta(\lambda f \chi_{G\setminus S}) + \varrho(16af\chi_{G\setminus S}) < \varepsilon/12$$

for a suitable set $S \in \Sigma, |S| < \infty$.

Keeping S fixed, we may find an index k such that

$$\eta(\lambda f\chi_{S\cap A_k}) + \varrho(16af\chi_{S\cap A_k}) < \varepsilon/12.$$

Moreover, $\eta(\lambda f\chi_{S\cap B_k}) + \varrho(16af\chi_{S\cap B_k}) \le \eta((\lambda/k)\chi_S) + \varrho((16a/k)\chi_S)$ and since $\chi_S \in (L^0(G))_{\varrho+\eta}$ we may find k such that $\eta(\lambda f\chi_{S\cap B_k})+\varrho(16af\chi_{S\cap B_k}) < \varepsilon/12$, which gives $R_k < \varepsilon/4 + \varrho(8ar_k(w)f)$. Taking $w \in \mathcal{W}_1$ we obtain by (3), $\varrho(a(T_w f - f)) < 3\varepsilon/4 + \varrho(8ar_k(w)f)$. But $f \in (L^0(G))_\varrho$, whence there is a $\mathcal{W}_3 \in \mathbf{W}$ such that $\varrho(8ar_k(w)f) < \varepsilon/4$ for $w \in \mathcal{W}_3$. This gives $\varrho(a(T_w f - f)) < \varepsilon$ for $w \in \mathcal{W}_1 \cap \mathcal{W}_3 \in \mathbf{W}$, which implies our statement.

5. We give some examples of modulars ϱ and η satisfying the assumptions of Theorems 1 and 2.

Examples. 1. Let $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be such that $\Phi(0) = 0, \Phi(u) > 0$ for $u > 0$, Φ nondecreasing in $\mathbb{R}_0^+$, and $\Phi(u) \to \infty$ as $u \to \infty$. Then Φ generates a modular

$$\sigma(f) = I_\Phi(f) = \int_G \Phi(|f(t)|)\,dt$$

in $L^0(G)$, and the respective modular space $(L^0(G))_\sigma$ is the Orlicz space $L^\Phi(G)$.

The modular σ is monotone, absolutely finite, absolutely continuous and τ-bounded (with $c = 1, h(t) \equiv 0$). If Φ is convex on $\mathbb{R}_0^+$, then σ is $\mathcal{J}$-convex. Thus, I_Φ satisfies the assumption of Theorem 2, (b).

Finally, if we take two functions Φ_1 and Φ_2 and we put $\varrho = I_{\Phi_1}, \eta = I_{\Phi_2}$, then (I) is certainly satisfied with $\lambda = C_\lambda$ if we assume the concavity of the function ψ with respect to the second variable and that $(\Phi_1 \circ \psi)(u) \le \Phi_2(u)$ for $u \ge 0$.

2. Let $\mathcal{V}$ be a nonempty set of indices filtered by a set $\mathbf{W}$ of its subsets. Let $a_v : [a,b[\to \mathbb{R}_0^+, v \in \mathcal{V}$, be such that

1° $\int_a^b a_v(x)\,dx \le 1$ for all $v \in \mathcal{V}$;

2° if $g : [a,b[\to \mathbb{R}_0^+$ is such that $0 \le g(x) \nearrow s < \infty$ as $x \to b-$, then

$$\int_a^b a_v(x)g(x)\,dm \xrightarrow{\mathbf{W}} s;$$

3° for every Lebesgue measurable set $C \subset [a,b[$ of measure $m(C) > 0$ there exists a Lebesgue measurable subset C_1 of measure $m(C_1) > 0$ and an index $\overline{v} \in \mathcal{V}$ such that $a_{\overline{v}}(x) > 0$ m-almost everywhere in C_1.

Let $\Phi : [a,b[\times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy

1) $\Phi(x,u)$ is a nondecreasing continuous function of $u \ge 0$, for every $x \in [a,b[$;

2) $\Phi(x,0) = 0, \Phi(x,u) > 0$ for $u > 0$, and $\Phi(x,u) \to \infty$ as $u \to \infty$, for every $x \in [a,b[$;

3) the limit $\lim_{x\to b-} \Phi(x,u) = \widetilde{\Phi}(u) < \infty$ exists for every $u \ge 0$;

4) $\Phi(x,u)$ is a Lebesgue measurable function of x in $[a,b[$, for every $u \ge 0$.

Moreover, suppose $\Phi(x,u)$ to be of monotone type in a subinterval $[c,b[\subset [a,b[$ and equicontinuous in $[a,b[$ at $u = 0$ (for these notions see [2], Section 4).

Let m be a measure on $[a,b[$ defined on all Lebesgue measurable subsets of $[a,b[$. Then

$$\mathcal{A}_\Phi(f) = \sup_{v\in\mathcal{V}} \int_a^b a_v(x)\mathcal{J}_\Phi(x,f)\,dm(x),$$

where

$$\mathcal{J}_\Phi(x,f) = \int_G \Phi(x,|f(t)|)\,dt,$$

is a modular in the subspace $L^0_m(G) \subset L^0(G)$ of functions f for which $\mathcal{J}_\Phi(x,f)$ is a Lebesgue measurable function on $[a,b[$.

In [2] sufficient conditions are obtained in order that $\mathcal{A}_\Phi$ be absolutely finite and absolutely continuous. Evidently $\mathcal{A}_\Phi$ is monotone. If $\Phi(x,u)$ is a convex function of $u \ge 0$ for all $x \in [a,b[$, then for all measurable functions $F : G \times G \to \mathbb{R}$ and $p : G \to \mathbb{R}^+_0$ with $\int_G p(t)\,dt = 1$ we have Jensen's inequality

$$\Phi\Big(x, \int_G p(t)F(t,s)\,dt\Big) \le \int_G p(t)\Phi(x,|F(t,s)|)\,dt$$

for $x \in [a,b[, s \in G$. Hence it follows that, in this case, $\mathcal{A}_\Phi$ is $\mathcal{J}$-convex. It is easily observed that $\mathcal{J}_\Phi(x, f(t+\cdot)) = \mathcal{J}_\Phi(x,f)$ for every $t \in G$, whence $\mathcal{A}_\Phi$ is τ-bounded with $c = 1, h(t) \equiv 0$.

The theory developed in [2] for the modular $\mathcal{A}_\Phi$ contains as a particular case the discrete modulars of the type

$$\mathcal{A}_\Phi(f) = \sup_{n\in\mathbb{N}} \sum_{i=1}^{\infty} a_{n,i} I_{\Phi_i}(f)$$

(for details see [2], Section 5).

Moreover, it is possible to prove that the more general modulars of type

$$\widetilde{\mathcal{A}}_\Phi(f) = \sup_{v\in\mathcal{V}} \int_a^b \mathcal{J}_\Phi(x,f)\,dm_w(x), \qquad f \in L^0_m(G),$$

where $\{m_w\}$ is a family of measures, satisfy the conditions of Theorem 1 and Theorem 2, under the assumptions of Section 6 of [2].

We remark that among these modulars there are those studied in [6], namely

$$I_\Phi(f) = \sup_{x\in [a,b[} \mathcal{J}_\Phi(x,f)$$

(see also Section 6 of [2]).

Hence we may state the following

COROLLARY 1. *Let* $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ *be a singular kernel and let* ϱ, η *be any of the modulars defined in Examples* 1 *or* 2, *satisfying* (I). *Suppose the function* Φ *generating* ϱ *is convex. In case any of the modulars is as in Example* 2, *suppose that the respective function* Φ *satisfies the assumptions of Theorem* 1 *of* [2]. *Then*, *for any function* $f \in (L^0(G))_{\varrho+\eta} \cap \mathrm{Dom}\, \mathbb{T}$,

$$\varrho(a(T_w f - f)) \xrightarrow{\mathbf{W}} 0$$

for sufficiently small $a > 0$.

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