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An example of a genuinely discontinuous generically chaotic transformation of the interval

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Abstract. It is proved that a piecewise monotone transformation of the unit interval (with a countable number of pieces) is generically chaotic. The Gauss map arising in connection with the continued fraction expansions of the reals is an example of such a transformation.

1. Generic chaos. The definition of generic chaos was orally suggested to me by Professor Andrzej Lasota and first came out in print soon after in paper [6]. It has since been studied by several authors ([3], [4], [10]). To define the notion, let us fix a metric space (X,ϱ) and a semigroup $\{S_t\}$ of transformations from X to X, where t runs over the set $\mathbb{N}$ of nonnegative integers or the set $\mathbb{R}_+$ of nonnegative reals. For the dynamical system so defined, we consider the set G of those pairs $(x,y)\in X^2$ for which

$$\liminf_{t\to\infty}\varrho(S_tx,S_ty)=0,\qquad \limsup_{t\to\infty}\varrho(S_tx,S_ty)>0.$$

We call the dynamical system $\{S_t\}$ *generically chaotic* iff the set G is residual in X^2, i.e., iff its complement is of the first category.

We call a single transformation $S:X\to X$ generically chaotic iff the semigroup $\{S^n\}_{n\ge 0}$ of its nonnegative iterates is generically chaotic.

Numerous examples of generically chaotic systems were given in [6]–[9]. In [8] a theorem was proved which states that under some consistency assumptions, weak mixing (in its ergodic-theory meaning) is generically chaotic. The theorem has, in turn, served to prove generic chaoticity of various systems on both finite- and infinite-dimensional spaces (see [8], [9]). We shall quote the exact statement of the theorem in question, even though we are going to prove generic chaos for a class of systems the theorem is not

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applicable to. This way we are able to see limitations of the theorem on the one hand and some advantages of the notion of generic chaos on the other.

THEOREM. *Let (X, ϱ) be a separable metric space with at least two non-isolated points. Let μ be a probability Borel measure on X, positive on nonempty open sets. If $S : X \to X$ is continuous, preserves the measure μ and is weakly mixing, then S is also generically chaotic.*

2. Piecewise monotone transformations. An example of a class of generically chaotic transformations is that of so-called piecewise monotone transformations of the unit interval I. We start with the definition of such a transformation.

Let $I = [0, 1] = A_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$, where the union is disjoint, $\# A_0 \le \aleph_0$, A_0 is closed and the Δ_i are open intervals with endpoints in A_0 (for $i = 1, 2, \ldots$). We consider a transformation $T : I \to I$ such that, for $i = 1, 2, \ldots$, $\varphi_i = T|_{\mathrm{cl}\, \Delta_i}$ is strictly monotone and continuous and $\varphi_i(\mathrm{cl}\, \Delta_i) = I$. We assume that the following condition (E) holds:

$$\text{(E)} \qquad \exists n_0 \in \mathbb{N}\ \exists q > 1\ \forall i = 1, 2, \ldots : \quad \inf\{|(\varphi_i^{(n_0)})'| : x \in \mathrm{dom}\, \varphi_i^{(n_0)}\} \ge q$$

(where the superscripts stand for iterates). Finally, let $A_{n+1} = A_n \cup T^{-1}(A_n)$ for $n = 0, 1, \ldots$ and assume that A_n is closed for $n = 0, 1, \ldots$ We will call a transformation with the above properties *piecewise monotone*.

Remark. Actually, φ_i is the continuous extension of $T|_{\Delta_i}$ onto $\mathrm{cl}\, \Delta_i$ for $i = 1, 2, \ldots$

THEOREM 1. *Every piecewise monotone transformation is generically chaotic.*

Proof. Let $A = \bigcup_{n \ge 0} A_n$. The set A is countable and thus of the first category. So, in the sequel, we may confine ourselves to considering the chaoticity of T on $I_0 = I \setminus A$.

Set

$$L_{n,\varepsilon} = \{(x, y) \in I_0^2 : \inf_{k \ge n} |T^k x - T^k y| < \varepsilon\}, \qquad \varepsilon > 0,\ n = 1, 2, \ldots;$$

$$U_n = \{(x, y) \in I_0^2 : \sup_{k \ge n} |T^k x - T^k y| > c\}, \qquad n = 1, 2, \ldots,$$

for some fixed $c \in (0, 1/2)$.

If we prove that the sets $L_{n,\varepsilon}$ and U_n are all open and dense, we conclude that T is generically chaotic since

$$G \supset \bigcap_{n=1}^{\infty} (L_{n,\varepsilon} \cap U_n),$$

where G is the generic set of the transformation T.

(i) The openness of the sets in question follows easily from the openness of $I \setminus A_n$ and the continuity of φ_i's and their iterates. One can actually repeat the proofs given in [6] (proof of Theorem 1) for A_0 finite.

For instance, to prove that $L_{n,\varepsilon}$ is open, fix $(x_0, y_0) \in L_{n,\varepsilon}$. Then there exist $\eta \in (0, \varepsilon)$ and an integer $k \geq n$ such that $|T^k x_0 - T^k y_0| < \varepsilon - \eta$. There also exist neighbourhoods $\overline{M}$ of x_0 and $\overline{N}$ of y_0, each disjoint from A_k. Thus T^k is continuous on $\overline{M} \cup \overline{N}$ and there exist neighbourhoods $M \subset \overline{M}$ and $N \subset \overline{N}$ of the points x_0 and y_0 (respectively) such that

$$|T^k x - T^k x_0| < \eta/2 \quad \text{for } x \in M \quad \text{and} \quad |T^k y - T^k y_0| < \eta/2 \quad \text{for } y \in N.$$

Then, for $(x, y) \in M \times N$,

$$|T^k x - T^k y| \leq |T^k x - T^k x_0| + |T^k x_0 - T^k y_0| + |T^k y_0 - T^k y| < \varepsilon$$

and, consequently, $M \times N \subset L_{n,\varepsilon}$.

One proves the openness of U_n in a like manner.

(ii) Before proving the density of $L_{n,\varepsilon}$, we prove two lemmas based on condition (E).

LEMMA 1. *Let δ_n denote the least upper bound of the distances between two neighbouring points of the set A_n. Then $\delta_n \to 0$ as $n \to \infty$.*

Proof. First, observe that since φ_i are surjective, the sequence $(\delta_n)_n$ is nonincreasing. Now, for $n = 0, 1, \ldots,$ and for $i_0, i_1, \ldots, i_{n+1} = 1, 2, \ldots,$ let

$$\Sigma_{i_0 i_1 \ldots i_n i_{n+1}} = \varphi_{i_{n+1}}^{-1}(\Sigma_{i_0 i_1 \ldots i_n}),$$

where $\Sigma_i = \operatorname{cl} \Delta_i$ for $i = 1, 2, \ldots$ (We shall call an interval of the form $\Sigma_{i_0 \ldots i_n}$ an *interval of level n*.) Thus

$$\delta_n = \sup\{|\Sigma_{i_0 \ldots i_n}| : i_0, \ldots, i_n \in \mathbb{N}\}$$

for $n = 0, 1, \ldots$ We already know that $1 > \delta_0 \geq \delta_1 \geq \ldots$ We shall inductively prove that the subsequence $(\delta_{kn_0})_k$ tends to 0 as $k \to \infty$. To this end, we show that $\delta_{kn_0} \leq q^{-k}$. This is obviously true for $k = 0$. Assume it is true for some fixed positive integer k. Let Σ be any interval of level $(k+1)n_0$. Let a and b be its endpoints. Then $T^{n_0}(a)$ and $T^{n_0}(b)$ are the endpoints of some interval of level kn_0. Moreover, there is $i_0 \in \mathbb{N}$ such that $\Sigma \subset \Sigma_{i_0}$. Then

$$\begin{aligned}|T^{n_0}(a) - T^{n_0}(b)| &= |\varphi_{i_0}^{n_0}(a) - \varphi_{i_0}^{n_0}(b)| \\ &= |(\varphi_{i_0}^{n_0})'(\theta)| \cdot |a - b| \geq q|a - b|,\end{aligned}$$

for some $\theta \in \Sigma$. Further,

$$|a - b| \leq q^{-1}|T^{n_0}(a) - T^{n_0}(b)| \leq q^{-1} q^{-k} = q^{-(k+1)}.$$

Thus $\delta_{(k+1)n_0} \leq q^{-(k+1)}$. ∎

LEMMA 2. *For $x \in I_0$, the set $\bigcup_{n=0}^{\infty} T^{-n}(\{x\})$ is dense in I.*

Proof. First observe that if $x \in I_0$, then $\bigcup_{n=0}^{\infty} T^{-n}(\{x\}) \subset I_0$. Now fix $x \in I_0$, $y \in I$ and $\delta > 0$. According to Lemma 1, there is an integer n_0 such that $\delta_{n_0} < \delta/2$. Thus at least one interval of level n_0 lies in $(y-\delta, y+\delta)$. Since every φ_i is surjective, every such interval intersects $T^{-n_0}(\{x\})$, so that

$$(y-\delta, y+\delta) \cap \bigcup_{n=0}^{\infty} T^{-n}(\{x\}) \neq \emptyset.$$

Now we can easily prove that $L_{n,\varepsilon}$ is dense. Fix $(x_0, y_0) \in I^2$ and $\tau > 0$. From Lemma 2, it follows that there exist $\overline{x}, \overline{y} \in I_0$ such that

$$|x_0 - \overline{x}| < \tau, \qquad |y_0 - \overline{y}| < \tau,$$

and

$$T^i \overline{x} = T^i \overline{y} \quad \text{for sufficiently large } i \geq n$$

(choose both $\overline{x}$ and $\overline{y}$ from the counterimage of some point in I_0). Then $(\overline{x}, \overline{y}) \in L_{n,\varepsilon}$ for every $\varepsilon > 0$.

(iii) Finally, we prove that U_n is dense. Fix $(x_0, y_0) \in I^2$ and $\sigma > 0$. By Lemma 1, there exists an integer $p \geq n$ such that $\delta_p < \sigma$. Further, there exist intervals Σ' and Σ'' of level p such that $x_0 \in \Sigma'$ and $y_0 \in \Sigma''$. Since the restrictions of T^p to Σ' and Σ'' are monotone, surjective and continuous, one can find $\overline{x} \in (\operatorname{int} \Sigma') \setminus A$ and $\overline{y} \in (\operatorname{int} \Sigma'') \setminus A$ such that

$$T^p \overline{x} < \tfrac{1}{2}(1-c), \qquad T^p \overline{y} < \tfrac{1}{2}(1+c).$$

Then

$$|\overline{x} - x_0| < \sigma, \qquad |\overline{y} - y_0| < \sigma,$$

and, consequently, $(\overline{x}, \overline{y}) \in U_n$.

The theorem is proved. ∎

Remarks. 1. The above theorem generalizes Theorem 1 of [6] in two ways. First, the number of "pieces" Δ_i need not be finite, and, second, instead of assuming (as in [6]) that φ_i's are expansive, we only demand that some fixed iterate of φ_i's is expansive (i.e., it satisfies condition (E)).

2. In the case of a finite number of "pieces" one may hope for applying the theorem linking generic chaoticity of a continuous transformation to its weak mixing property, by considering the transformation as acting on the circle rather than on the interval. This is, however, not possible in the case of an infinite number of pieces; the set A_0 of the endpoints of the pieces is then infinite so it has to have at least one condensation point where the continuity has to be violated due to the surjectivity of the φ_i's.

3. The Gauss map. There is an old and interesting example of a transformation satisfying the assumptions of the theorem proved in Section 2.

Moreover, in this example we observe both features which make Theorem 1 of [6] and our new theorem different (see Remark 1 above). This is the so-called *Gauss map* given by

$$T(x) = \begin{cases} 1/x \pmod 1, & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$$

The Gauss map has played a crucial role in the theory of the continued fraction expansions of real numbers. The ergodic properties of the Gauss map have been studied by numerous authors, starting from Gauss himself in the early 19th century. Among contemporary authors, let us mention Cornfeld, Fomin and Sinai [2], Mañé [5] and Corless [1]. It is known that the Gauss map is exact, so it is also weakly mixing. (Cornfeld, Fomin and Sinai prove in [2] that under some additional assumptions every C^2 piecewise monotone transformation of the interval is exact.) Now we shall see the Gauss map satisfies the assumptions of Theorem 1, so it is generically chaotic.

THEOREM 2. *The Gauss map is generically chaotic.*

Proof. We use the notation introduced in Section 2. For the Gauss map T, we have $A_0 = \{0\} \cup \{1/n \mid n = 1, 2, \ldots\}$, so it is a closed countable set. We may arrange the intervals Δ_i so that $\Delta_i = (1/(i+1), 1/i)$, $i = 1, 2, \ldots$ Every φ_i is strictly decreasing, continuous and maps $\operatorname{cl} \Delta_i$ onto $I = [0,1]$. Condition (E) holds with $n_0 = 2$ and $q = 4$. Indeed, for $x \notin A_1$, there exist $k, l \in \mathbb{N}$ such that

$$T^2 x = \frac{1}{-kx + 1} - l,$$

so

$$(T^2)' x = \frac{1}{(1 - kx)^2}, \qquad x \in \left(\frac{1}{k+1}, \frac{1}{k}\right), \ k = 1, 2, \ldots$$

Thus

$$(T^2)' x \ge \left(1 - k \cdot \frac{1}{k+1}\right)^2 = (k+1)^2, \qquad x \in \left(\frac{1}{k+1}, \frac{1}{k}\right), \ k = 1, 2, \ldots$$

Finally, we have

$$(T^2)' x \ge 4, \qquad x \notin A_1.$$

It remains to prove that the sets A_n, $n = 1, 2, \ldots$, are closed. This can be done inductively by observing that the condensation points of A_{n+1} lie in A_n, for $n = 0, 1, \ldots$ The details depend on whether a sequence converging to a given condensation point of A_{n+1} lies in a finite or an infinite number of intervals of level $n+1$ and whether n is odd or even (i.e., whether the iterate of T is piecewise decreasing or increasing).

The proof is finished. ∎

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