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# On the stability of solutions of nonlinear parabolic differential-functional equations

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**Abstract.** We consider a nonlinear differential-functional parabolic boundary initial value problem

$$(1)\qquad \begin{cases} \mathcal{A}z + f(x, z(t, x), z(t, \cdot)) - \partial z/\partial t = 0 & \text{for } t > 0,\ x \in G, \\ z(t, x) = h(x) & \text{for } t > 0,\ x \in \partial G, \\ z(0, x) = \varphi_0(x) & \text{for } x \in G, \end{cases}$$

and the associated elliptic boundary value problem with Dirichlet condition

$$(2)\qquad \begin{cases} \mathcal{A}z + f(x, z(x), z(\cdot)) = 0 & \text{for } x \in G, \\ z(x) = h(x) & \text{for } x \in \partial G, \end{cases}$$

where $x = (x_1, \ldots, x_m) \in G \subset \mathbb{R}^m, G$ is an open and bounded domain with $C^{2+\alpha}$ $(0 < \alpha \le 1)$ boundary, the operator

$$\mathcal{A}z := \sum_{j,k=1}^{m} a_{jk}(x) \frac{\partial^2 z}{\partial x_j \partial x_k}$$

is uniformly elliptic in $\overline{G}$ and $z$ is a real $L^p(G)$ function.

The purpose of this paper is to give some conditions which will guarantee that the parabolic problem has a stable solution. Basing on the results obtained in [7] and [5, 6], we prove that the limit of the solution of the parabolic problem (1) as $t \to \infty$ is the solution of the associated elliptic problem (2), obtained by the monotone iterative method. The problem of stability of solutions of the parabolic differential equation has been studied by D. H. Sattinger [14, 15], H. Amann [3, 4], O. Diekmann and N. M. Temme [8], and J. Smoller [17]. Our results generalize these papers to encompass the case of differential-functional equations. Differential-functional equations arise frequently in applied mathematics. For example, equations of this type describe the heat transfer processes and the prediction of ground water level.

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**1. Notations, definitions and assumptions.** Let $D = (0,T) \times G$, where $T \le \infty$ and $G$ is an open and bounded domain, $G \subset \mathbb{R}^m$, with boundary $\partial G$; $\sigma := (0,T) \times \partial G$, $S_0 := \{(t,x) : t = 0, x \in \overline{G}\}$, $\Sigma := S_0 \cup \sigma$, $\overline{D} := D \cup \Sigma$.

We denote by $C(\overline{G}) := C(\overline{G}, \mathbb{R})$ the space of continuous functions with the norm

$$\|f\| := \max_{x \in \overline{G}} |f(x)|$$

and we denote by $C^{l+\alpha}(\overline{G}) := C^{l+\alpha}(\overline{G}, \mathbb{R})$, $C^{l+\alpha}(\overline{D}) := C^{(l+\alpha)/2, l+\alpha}(\overline{D}, \mathbb{R})$ ($l = 0, 1, 2, \ldots; 0 < \alpha \le 1$) the Hölder spaces and by $H^{l,p}(G)$ ($p \ge 1$) the Sobolev space with the respective norms $|f|_{l+\alpha}$, $|f|^{l+\alpha}$, $\|f\|_{l,p}$ (more information about these spaces can be found in [9], [10]).

We shall say that the operator $\mathcal{A}$ (see the abstract) is *uniformly elliptic in* $\overline{G}$ if there is a constant $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a_{jk}(x)\xi_j\xi_k \ge \mu \sum_{j=1}^{m} \xi_j^2$$

for all $x \in \overline{G}$ and $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$.

We say that the operator $\mathcal{F} = \partial/\partial t - \mathcal{A}$ is *uniformly parabolic in* $\overline{D}$ if $\mathcal{A}$ is uniformly elliptic in $\overline{D}$.

($\mathrm{H}_a$) We assume that $a_{jk} \in C^{0+\alpha}(\overline{G})$ and $a_{jk} = a_{kj}$ ($j, k = 1, \ldots, m$).

Moreover, we assume that $h \in C^{2+\alpha}(\partial G)$, where $\partial G \in C^{2+\alpha}$ (therefore there is an $\widetilde{h} \in C^{2+\alpha}(\overline{G})$ such that $h(x) = \widetilde{h}(x)$ for all $x \in \partial G$), $\varphi_0 \in C^{2+\alpha}(\overline{G})$ and $\varphi_0(x) = h(x)$ for $x \in \partial G$.

We assume that the function $f : \overline{G} \times \mathbb{R} \times L^p(\overline{G}) \ni (x, y, s) \mapsto f(x, y, s) \in \mathbb{R}$ satisfies the following assumptions:

(L) $f$ satisfies the Lipschitz condition with respect to $y$ and $s$, i.e., for any $y$, $\widetilde{y}$, $s$, $\widetilde{s}$ we have

$$|f(x,y,s) - f(x,\widetilde{y},\widetilde{s})| \le L_1|y - \widetilde{y}| + L_2\|s - \widetilde{s}\| \quad \text{for } x \in \overline{G},$$

where $L_1$, $L_2$ are nonnegative constants;

($\mathrm{H}_f$) $f$ is Hölder continuous with exponent $\alpha$ ($0 < \alpha \le 1$), with respect to $x$ in $\overline{G}$;

(W) $f$ is increasing with respect to $y$ and $s$.

A function $z = z(t,x)$ will be called *regular in* $\overline{D}$ if it is continuous in $\overline{D}$ and has continuous derivatives $\partial z/\partial t$, $\partial z/\partial x_j$, $\partial^2 z/\partial x_j \partial x_k$ in $D$, so $z \in C(\overline{D}) \cap C^{1,2}(D)$.

Analogously, a function $z = z(x)$ will be called *regular in* $\overline{G}$ if $z \in C(\overline{G}) \cap C^2(G)$.

Functions $u = u(t,x)$ and $v = v(t,x)$ regular in $\overline{D}$ and satisfying the systems of inequalities

$$\begin{cases} \mathcal{F}[u] \le f(x, u(t,x), u(t,\cdot)) & \text{for } (t,x) \in D, \\ u(t,x) \le g(t,x) & \text{for } (t,x) \in \Sigma, \end{cases} \tag{3}$$

$$\begin{cases} \mathcal{F}[v] \ge f(x, v(t,x), v(t,\cdot)) & \text{for } (t,x) \in D, \\ v(t,x) \ge g(t,x) & \text{for } (t,x) \in \Sigma, \end{cases} \tag{4}$$

are called a *lower* and an *upper function for the parabolic problem* (1) *in* $\overline{D}$, respectively.

Analogously, functions $u = u(x)$ and $v = v(x)$ regular in $\overline{G}$ and satisfying the systems of inequalities

$$\begin{cases} \mathcal{A}u + f(x, u(x), u(\cdot)) \ge 0 & \text{for } x \in G, \\ u(x) \le h(x) & \text{for } x \in \partial G, \end{cases}$$

$$\begin{cases} \mathcal{A}v + f(x, v(x), v(\cdot)) \le 0 & \text{for } x \in G, \\ v(x) \ge h(x) & \text{for } x \in \partial G, \end{cases}$$

are called a *lower* and an *upper function for the elliptic problem* (2) *in* $\overline{G}$, respectively.

These functions are also called a lower and an upper solution [12], [14] or a sub- and a supersolution [4].

Assumption A. We assume that there exists at least one pair $u_0 = u_0(t,x)$, $v_0 = v_0(t,x)$ of a lower and an upper function for the parabolic problem (1) in $\overline{D}$.

Assumption A*. We assume that there exists at least one pair $u_0 = u_0(x)$, $v_0 = v_0(x)$ of a lower and an upper function for the elliptic problem (2) in $\overline{G}$ such that

$$u_0(x) \le v_0(x) \quad \text{for } x \in \overline{G}.$$

By a *weak solution of the elliptic problem* (2) *in* $\overline{G}$ we mean a function $u \in L^2(G)$ such that $\mathcal{A}u \in L^2(G)$ and

$$(\mathcal{A}u, \xi) + (\mathbf{F}u, \xi) = 0 \quad \text{for any test function } \xi \in C_0^\infty(G),$$

where $(\cdot,\cdot)$ is the inner product in $L^2(G)$, i.e., $(f,g) = \int_G fg\,dx$. The nonlinear *Nemytskiĭ operator* $\mathbf{F}$ is defined by

$$\mathbf{F}u(t,x) := f(x, u(t,x), u(t,\cdot)) \quad \text{and} \quad \mathbf{F}u(x) := f(x, u(x), u(\cdot)).$$

Analogously we define a *weak solution of the parabolic problem*.

We shall call a solution $\widehat{u} = \widehat{u}(x)$ of the elliptic problem (2) *asymptotically stable* if it is a stable solution of the parabolic problem (1) and

$$\lim_{t\to\infty} \|\widehat{u}(\cdot) - \widetilde{u}(t,\cdot)\| = 0,$$

where $\widetilde{u} = \widetilde{u}(t,x)$ is a solution of the parabolic problem (1).

**2. Preliminary remarks.** Let $u_0, v_0$ be a lower and an upper function for the elliptic problem (2) in $\overline{G}$ such that $u_0(x) \le v_0(x)$ for $x \in \overline{G}$. We define the set

$$K := \{(x, y, s) : x \in \overline{G},\ y \in [m_0, M_0],\ s \in \langle u_0, v_0\rangle\},$$

where

$$m_0 := \min_{x\in\overline{G}} u_0(x), \qquad M_0 := \max_{x\in\overline{G}} v_0(x)$$

and $\langle u_0, v_0\rangle$ is the segment

$$\langle u_0, v_0\rangle := \{s \in L^p(\overline{G}) : u_0(x) \le s(x) \le v_0(x) \text{ for } x \in \overline{G}\}.$$

Now, we assume that the function $f : \mathbb{R}^m \times \mathbb{R} \times L^p \ni (x, y, s) \mapsto f(x, y, s) \in \mathbb{R}$ satisfies in $K$ the following assumptions:

(a) $f(\cdot, y, s) \in C^{0+\alpha}(\overline{G})$ for $y \in [m_0, M_0]$, $s \in \langle u_0, v_0\rangle$;

(b) $f(x, \cdot, \cdot)$ is continuous for $x \in \overline{G}$;

(c) the derivative $\partial f/\partial y$ exists and is continuous, and $\left|\frac{\partial f}{\partial y}(x, y, s)\right| \le c_0$ in $K$, where $c_0 > 0$ is a constant;

(d) $f$ is increasing with respect to $s$.

Lemma 1. *Under the above assumptions, the elliptic problem* (2) *has at least one regular solution $z$ such that*

$$u_0(x) \le z(x) \le v_0(x) \quad \textit{for } x \in \overline{G},$$

*and the functions*

$$\underline{z}(x) = \lim_{n\to\infty} u_n(x) \quad \textit{for } x \in \overline{G}, \tag{5}$$

$$\overline{z}(x) = \lim_{n\to\infty} v_n(x) \quad \textit{for } x \in \overline{G}, \tag{6}$$

*are, respectively, the minimal and maximal solutions of the problem* (2) *in $K$, where $u_n$ and $v_n$ are defined as regular solutions in $G$ of the linear equations*

$$(\mathcal{A} - k\mathcal{I})u_n = -[f(x, u_{n-1}(x), u_{n-1}(\cdot)) + ku_{n-1}(x)],$$
$$(\mathcal{A} - k\mathcal{I})v_n = -[f(x, v_{n-1}(x), v_{n-1}(\cdot)) + kv_{n-1}(x)] \quad \textit{for } n = 1, 2, \ldots,$$

*with boundary condition from the problem* (2) *and $k > c_0$* (for the proof see [7]). ■

Remark 1. Assumption (c) can be weakened. Lemma 1 holds if we assume that the function $f(x, y, s)$ satisfies the Lipschitz condition with respect to $y$ in $K$.

**3. Stability of solutions.** Throughout this paper we keep all the assumptions of the first two sections. Moreover, we remark that without loss of generality we can consider the Dirichlet problem with homogeneous boundary condition

$$z(x) = 0 \quad \text{for } x \in \partial G,$$

i.e., when $h(x) \equiv 0$ on $\partial G$.

THEOREM 1. *Let $v_0 = v_0(x)$ be an upper function of the elliptic problem (2) in $\overline{G}$ and let $\widetilde{v} = \widetilde{v}(t,x)$ be a solution of the parabolic boundary initial value problem*

$$(7) \qquad \begin{cases} \mathcal{A}\widetilde{v} + f(x, \widetilde{v}(t,x), \widetilde{v}(t,\cdot)) - \partial\widetilde{v}/\partial t = 0 & \text{in } D, \\ \widetilde{v}(t,x) = 0 & \text{for } t > 0,\ x \in \partial G, \\ \widetilde{v}(0,x) = v_0(x) & \text{for } x \in G. \end{cases}$$

*Then*

$$\frac{\partial \widetilde{v}}{\partial t} \le 0 \quad \text{and} \quad \widetilde{v}(t,x) \le \widetilde{v}(0,x) = v_0(x) \quad \text{in } \overline{D}.$$

Proof. Under the above assumptions, from Theorem of [5], p. 39 (cf. also [6], p. 706) it follows that the parabolic problem (7) has a unique regular solution $\widetilde{v} = \widetilde{v}(t,x)$ in $\overline{D}$.

On the other hand, $v_0$ is an upper function for the elliptic problem (2). Since $\partial v_0/\partial t = 0$, $v_0$ is a solution of the problem

$$(8) \qquad \begin{cases} \mathcal{A}v_0 + f(x, v_0(x), v_0(\cdot)) - \partial v_0/\partial t = 0 & \text{in } D, \\ v_0(x) \ge 0 & \text{for } t > 0,\ x \in \partial G, \\ v_0(x) = v_0(x) & \text{for } t = 0,\ x \in G. \end{cases}$$

Applying J. Szarski's theorem on weak partial differential-functional inequalities (J. Szarski [18], Theorem 1, pp. 208–209) to systems (7), (8) we get

$$\widetilde{v}(t,x) \le v_0(x) \quad \text{in } \overline{D}.$$

Now we consider the function

$$\widetilde{v}_\kappa(t,x) := \widetilde{v}(t+\kappa, x) \quad \text{for } \kappa > 0.$$

This function satisfies the following parabolic problem

$$(9) \qquad \begin{cases} \mathcal{A}\widetilde{v}_\kappa + f(x, \widetilde{v}_\kappa(t,x), \widetilde{v}_\kappa(t,\cdot)) - \partial\widetilde{v}_\kappa/\partial t = 0 & \text{in } D, \\ \widetilde{v}_\kappa(t,x) = 0 & \text{for } t > 0,\ x \in \partial G, \\ \widetilde{v}_\kappa(0,x) = \widetilde{v}(0+\kappa, x) = \widetilde{v}(\kappa, x) \le v_0(x) & \text{for } x \in G. \end{cases}$$

Applying again the theorem on weak partial differential-functional inequalities to systems (7) and (9) we get

$$\widetilde{v}_\kappa(t,x) \le \widetilde{v}(t,x) \quad \text{in } \overline{D}.$$

The function $\widetilde{v}(t,x)$ is nonincreasing with respect to $t$. Indeed, let $t_1 < t_2$ and $\kappa = t_2 - t_1$. Then

$$\widetilde{v}(t_1,x) \ge \widetilde{v}_\kappa(t_1,x) = \widetilde{v}(t_1+\kappa, x) = \widetilde{v}(t_2,x).$$

This completes the proof. ∎

We can prove an analogous theorem for a lower function $u_0 = u_0(x)$ of the elliptic problem. In this case a solution $\widetilde{u}(t,x)$ of the parabolic problem with initial condition $u_0$ is nondecreasing with respect to $t$.

Therefore we have the inequalities

$$u_0(x) \le \widetilde{u}(t,x) \le \widetilde{v}(t,x) \le v_0(x) \quad \text{in } \overline{D}.$$

THEOREM 2. *If $u = u(t,x)$ is a regular uniformly bounded solution of the parabolic boundary initial value problem*

$$\begin{cases} \mathcal{A}u + f(x, u(t,x), u(t,\cdot)) - \partial u/\partial t = 0 & \text{for } (t,x) \in D, \\ u(t,x) = 0 & \text{for } t > 0, x \in \partial G, \\ u(0,x) = \varphi_0(x) & \text{for } x \in G \end{cases} \tag{10}$$

*and* $\lim_{t\to\infty} u(t,x) = \widehat{u}(x)$ *exists, then the function* $\widehat{u} = \widehat{u}(x)$ *is a regular solution of the elliptic boundary value problem*

$$\begin{cases} \mathcal{A}\widehat{u} + f(x, \widehat{u}(x), \widehat{u}(\cdot)) = 0 & \text{for } x \in G, \\ \widehat{u}(x) = 0 & \text{for } x \in \partial G. \end{cases} \tag{11}$$

P r o o f. The proof will be divided into two steps. First, we will prove that $\widehat{u}$ is a weak solution of the elliptic problem (11). Next we will show that $\widehat{u}$ is a regular solution of this problem.

To prove that $\widehat{u}$ is a weak solution of (11) we need to show that $\widehat{u} \in L^2(G)$, $\mathcal{A}\widehat{u} \in L^2(G)$ and

$$(\mathcal{A}\widehat{u}, \xi) + (\mathbf{F}\widehat{u}, \xi) = 0 \quad \text{for each test function } \xi \in C_0^\infty(G). \tag{12}$$

Integrating by parts the first term of (12) we can write this equation in the equivalent form

$$(\widehat{u}, \mathcal{A}^*\xi) + (\mathbf{F}\widehat{u}, \xi) = 0 \quad \text{for each } \xi \in C_0^\infty(G),$$

where the operator $\mathcal{A}^*$ is adjoint to $\mathcal{A}$,

$$\mathcal{A}^* u = \sum_{j,k=1}^{m} \frac{\partial^2}{\partial x_j \partial x_k}(a_{jk}(x)u(x)).$$

From the Theorem of [5], p. 39, it follows that the parabolic problem (10) has a unique regular solution. Multiplying the equation of (10) by any test function $\xi \in C_0^\infty(G)$ and integrating we get

$$\int_G \mathcal{A}u \cdot \xi \, dx + \int_G \mathbf{F}u \cdot \xi \, dx - \int_G u_t \cdot \xi \, dx = 0,$$

hence

$$(\mathcal{A}u, \xi) + (\mathbf{F}u, \xi) - (u_t, \xi) = 0 \quad \text{for each } \xi \in C_0^\infty(G) \text{ and every } t > 0.$$

Now integrating by parts the first term of the above equation we get

$$(u, \mathcal{A}^*\xi) + (\mathbf{F}u, \xi) - (u_t, \xi) = 0 \quad \text{for each } \xi \in C_0^\infty(G) \text{ and every } t > 0.$$

Choose any $T > 0$. Then, multiplying the above equality by $1/T$ and integrating with respect to $t$ on the interval $[0, T]$ we have

$$\frac{1}{T}\int_0^T (u, \mathcal{A}^*\xi)\,dt + \frac{1}{T}\int_0^T (\mathbf{F}u, \xi)\,dt - \frac{1}{T}\int_0^T (u_t, \xi)\,dt = 0.$$

From the assumption of the uniform boundedness of the solution $u(t,x)$ of the problem (10) and the Lebesgue dominated convergence theorem we get successively:

$$(\alpha) \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T (u, \mathcal{A}^*\xi)\,dt = \lim_{T\to\infty}\int_G \left(\mathcal{A}^*\xi \cdot \frac{1}{T}\int_0^T u(t,x)\,dt\right)dx$$
$$= \int_G \left(\mathcal{A}^*\xi \cdot \lim_{T\to\infty}\frac{1}{T}\int_0^T u(t,x)\,dt\right)dx = \int_G \mathcal{A}^*\xi \cdot \widehat{u}\,dx = (\widehat{u}, \mathcal{A}^*\xi),$$

because

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(t,x)\,dt - \widehat{u}(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T [u(t,x) - \widehat{u}(x)]\,dt$$
$$= \lim_{T\to\infty}\omega(T)/T = \lim_{T\to\infty}\omega'(T) = 0,$$

since $\omega'(T) = u(T,x) - \widehat{u}(x) \to 0$ as $T \to \infty$, where $\omega(T) = \int_0^T [u(t,x) - \widehat{u}(x)]\,dt$.

Next,

$$(\beta) \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathbf{F}u, \xi)\,dt = \lim_{T\to\infty}\int_G \left(\xi \cdot \frac{1}{T}\int_0^T \mathbf{F}u\,dt\right)dx$$
$$= \lim_{T\to\infty}\int_G \left(\xi \cdot \frac{1}{T}\int_0^T f(x, u(t,x), u(t,\cdot))\,dt\right)dx.$$

Therefore

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (\mathbf{F}u, \xi)\,dt - (\mathbf{F}\widehat{u}, \xi)$$
$$= \lim_{T\to\infty}\int_G \xi \cdot \left[\frac{1}{T}\int_0^T (f(x, u(t,x), u(t,\cdot)) - f(x, \widehat{u}(x), \widehat{u}(\cdot)))\,dt\right]dx.$$

Since $u$ is uniformly bounded in $\overline{D}$, we can choose $C \geq 0$ such that

$|u(t,x)| \le C$ in $\overline{D}$. From this and the Lipschitz condition (L) it follows that

$$\begin{aligned}|f(x,u(t,x),u(t,\cdot)) - f(x,\widehat{u}(x),\widehat{u}(\cdot))| \\ &\le L_1|u(t,x)-\widehat{u}(x)| + L_2\|u(t,\cdot)-\widehat{u}(\cdot)\| \\ &\le (L_1+L_2)(C+\max_{x\in\overline{G}}|\widehat{u}|).\end{aligned}$$

Therefore, applying the Lebesgue dominated convergence theorem we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(\mathbf{F}u,\xi)\,dt = (\mathbf{F}\widehat{u},\xi).$$

Finally,

$$(\gamma)\qquad \begin{aligned}\frac{1}{T}\int_0^T(u_t,\xi)\,dt &= \frac{1}{T}\int_0^T\frac{\partial}{\partial t}(u,\xi)\,dt = \frac{(u(T,\cdot),\xi)-(u(0,\cdot),\xi)}{T}\\ &= \int_G\frac{u(T,x)-u(0,x)}{T}\cdot\xi(x)\,dx \to 0 \quad \text{as } T\to\infty,\end{aligned}$$

since

$$\left|\frac{u(T,x)-u(0,x)}{T}\cdot\xi(x)\right| \le \frac{1}{T}\cdot 2C\cdot\max_{x\in\overline{G}}|\xi| \to 0 \quad \text{as } T\to\infty.$$

Consequently, by $(\alpha)$, $(\beta)$ and $(\gamma)$ we have

$$(\widehat{u},\mathcal{A}^*\xi) + (\mathbf{F}\widehat{u},\xi) = 0 \quad \text{for each } \xi\in C_0^\infty(\overline{G}).$$

Hence $\widehat{u}$ is a weak solution of the elliptic problem (11).

Now we prove that $\widehat{u}$ is a regular solution of (11). Observe that $\widehat{u}$ is bounded in $G$, hence $\widehat{u}\in L^p(G)$. Thus, we have $\mathbf{F}\widehat{u}\in L^p(G)$ (see [19], p. 214, or [11], p. 31).

Consider the elliptic boundary value problem

$$(13)\qquad \begin{cases}\mathcal{A}\widehat{w} = -\mathbf{F}\widehat{u} & \text{in } G,\\ \widehat{w}|_{\partial G} = 0.\end{cases}$$

By virtue of the Agmon–Douglis–Nirenberg theorem on the existence and uniqueness in $L^p$ of the solution of the Dirichlet problem for a linear elliptic equation ([2], Theorem 15.2, p. 704, and D. Gilbarg and N. S. Trudinger [11], Theorems 9.13 and 9.15, pp. 239, 241) problem (13) has a unique weak solution $\widehat{w}\in H^{2,p}(G)$. Thus $\widehat{w}=\widehat{u}$ almost everywhere in $G$.

From the Rellich–Kondrachov theorem on compact imbeddings ([1], Theorem 6.2, p. 144, and [20], §28, Theorem 8, p. 262) we have $H^{2,p}\hookrightarrow C^{0+\alpha}$ when $p>m$. Therefore $\widehat{w}\in C^{0+\alpha}(\overline{G})$. Consequently, $\mathbf{F}\widehat{w}\in C^{0+\alpha}(\overline{G})$ (see [12], p. 214).

By virtue of the Schauder theorem on the existence and uniqueness of solution of the Dirichlet problem for a linear elliptic equation (J. Schauder [16], see also A. Friedman [9], Theorem 18, p. 86, or O. A. Ladyzhenskaya and N. N. Ural'tseva [13], Theorem 1.3, p. 142 and Theorem 12.1, p. 277), the boundary value problem

$$\begin{cases} \mathcal{A}w = -\mathbf{F}\widehat{w} & \text{in } G, \\ w|_{\partial G} = 0, \end{cases}$$

has a unique regular solution $w \in C^{2+\alpha}(\overline{G})$. Therefore, it is easy to see that $w = \widehat{u}$. This completes the proof of the theorem. ■

THEOREM 3. *If $u = u(t,x)$ is a solution of the parabolic problem* (1) *in $\overline{D}$ with initial condition $\varphi_0$ such that*

$$\overline{z}(x) \le \varphi_0(x) \le v_0(x) \quad \text{in } \overline{G}, \tag{14}$$

*where $v_0$ is an upper function and $\overline{z}$ is the maximal solution of the elliptic problem* (2) *in $\overline{G}$ defined by* (6), *then $\lim_{t\to\infty} u(t,x) = \overline{z}(x)$, so $\overline{z}$ is asymptotically stable from above.*

*If*

$$u_0(x) \le \varphi_0(x) \le \underline{z}(x) \quad \text{in } \overline{G},$$

*where $u_0$ is a lower function and $\underline{z}$ is the minimal solution of the elliptic problem* (2) *in $\overline{G}$ defined by* (5), *then $\lim_{t\to\infty} u(t,x) = \underline{z}(x)$, so $\underline{z}$ is asymptotically stable from below.*

*If $\overline{z} = \underline{z} =: z$ (i.e., the elliptic problem* (2) *has the unique solution $z$) and*

$$u_0(x) \le \varphi_0(x) \le v_0(x) \quad \text{in } \overline{G},$$

*then $\lim_{t\to\infty} u(t,x) = z(x)$, so $z$ is asymptotically stable (i.e., both from above and from below).*

Proof. From the theorem on weak partial differential-functional inequalities and Theorem 1, each solution $u = u(t,x)$ of the parabolic problem (2) with the initial inequalities

$$u_0(x) \le \varphi_0(x) \le v_0(x) \quad \text{in } \overline{G}$$

satisfies

$$u_0(x) \le \widetilde{u}(t,x) \le u(t,x) \le \widetilde{v}(t,x) \le v_0(x) \quad \text{in } \overline{D}.$$

By assumption (14) the solution $\widetilde{v}$ satisfies

$$\overline{z}(x) \le \widetilde{v}(t,x) \le v_0(x) \quad \text{in } \overline{D}.$$

Hence $\widetilde{v}$ is bounded from below and by virtue of Theorem 1 the function $\widetilde{v}$ is nondecreasing with respect to $t$. Therefore $\lim_{t\to\infty} \widetilde{v}(t,x)$ exists. By Theorem 2 this limit is a solution of the elliptic problem (2) and $\overline{z} \le \lim_{t\to\infty} \widetilde{v}(t,x)$.

On the other hand, $z$ is the maximal solution of (2), so we get

$$\lim_{t\to\infty} \widetilde{v}(t,x) = \overline{z}(x). \tag{15}$$

Any solution $u = u(t,x)$ of the parabolic problem (1) in $\overline{D}$ with initial condition $\varphi_0$ satisfying the inequalities

$$\overline{z}(x) \le \varphi_0(x) \le v_0(x) \quad \text{in } \overline{G},$$

satisfies

$$\overline{z}(x) \le u(t,x) \le \widetilde{v}(t,x) \quad \text{in } \overline{D}.$$

Moreover, (15) holds and hence $\lim_{t\to\infty} u(t,x) = \overline{z}(x)$, so $\overline{z}$ is an asymptotically stable (from above) solution of the elliptic problem (2).

The rest of the proof runs analogously. ∎

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